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On the solvability for a *p*-*k*-Hessian inequality

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Abstract
In this paper, we discuss the solvability of a p-k-Hessian entire inequality. We prove that the inequality with sub-lower-critical exponent admits no negative solutions. Moreover, the exponent is sharp. The proof is based on choosing suitable test functions and integrating by parts.

Keywords Solvability · p-k-Hessian inequality · Integration by parts

Mathematics Subject Classification 35B08 · 35J60

1 Introduction

In this paper, we consider the following differential inequality

$$F_{k,p}[u] \geq (-u)^{\alpha} \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $F_{k,p}[u] = \left[D(|Du|^{p-2}Du)\right]_k$ is sum of k-th principal minors of the matrix $D(|Du|^{p-2}Du)$, $1 < p < +\infty, k = 1, 2, \cdots, n$. Let λ be the eigenvalues of $D(|Du|^{p-2}Du)$. Then $F_{k,p}[u] = \sigma_k(\lambda)$, where $\sigma_k(\lambda)$ is the k-th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

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We call $F_{k,p}[u]$ the p-k-Hessian operator. It is a generalization of Laplacian operator, p-Laplacian operator, Monge-Ampère operator and k-Hessian operator. And it is firstly introduced by Trudinger and Wang [18].

When $p = 2, k = 1$, (1.1) is the Laplacian inequality $\Delta u \geq (-u)^{\alpha}$. In the celebrated paper [4], Gidas and Spruck stated beautiful and deep results about the inequality. When $k = 1$, (1.1) coincides with the p-Laplacian inequality $\Delta_p u \geq (-u)^{\alpha}$. There are some splendid results about the p-Laplacian inequality by Serrin and Zou [16], and we first state one of them.

Theorem 1.1 *(*Serrin-Zou [16]*) Assume* $1 < p < n$*, then the differential inequality*

$$\Delta_p u \geq (-u)^{\alpha} \quad in\ \mathbb{R}^n$$

has a negative solution in $\mathbb{R}^n$ *if and only if* $\alpha > \frac{n(p-1)}{n-p}$.

When $p = 2$, 2-k-Hessian operator is the usual k-Hessian operator. Let Ω be a domain in $\mathbb{R}^n$. A function u is called k-admissible in Ω if $u \in C^2(\Omega)$ and $\lambda(D^2u) \in \Gamma_k$ in Ω, where

$$\Gamma_k := \{\lambda \in \mathbb{R}^n \mid \sigma_l(\lambda) \geq 0, l = 1, \cdots, k\}.$$

Let $\Phi^k(\Omega)$ be the class of k-admissible functions

$$\Phi^k(\Omega) := \left\{u \in C^2(\Omega) \middle| \lambda\big(D^2u(x)\big) \in \Gamma_k, \forall\, x \in \Omega\right\}.$$

When $2k < n$, denote $k^* := \frac{n(k+1)}{n-2k}$, $k* = \frac{(n+2)k}{n-2k}$ and $k_* := \frac{nk}{n-2k}$. k^* is the critical exponent for Sobolev embedding inequality established by Wang [19]. $k*$ is the critical exponent introduced by Tso [17] where the author considered the solvability of the Dirichlet problem $\sigma_k(D^2u) = (-u)^p$ in Ω and $u|_{\partial\Omega} = 0$. k_* is called lower critical exponent. For k-Hessian inequality $\sigma_k(D^2u) \geq (-u)^{\alpha}$, Phuc and Verbitsky [14, 15] proved that it has no negative solution for $\alpha \in (k, k_*]$ and showed that k_* is sharp. Ou [12] obtained the nonexistence result with sub-lower critical exponent via different approach.

Theorem 1.2 *(*Phuc-Verbitsky [14, 15], Ou [12]*) Assume* $2k < n$*, then the inequality*

$$\sigma_k(D^2u) \geq (-u)^{\alpha} \quad in\ \mathbb{R}^n$$

admits no negative entire solution in $\Phi^k(\mathbb{R}^n)$ *for any* $\alpha \in (-\infty, k_*]$. *Moreover,* k_* *is sharp.*

The paper is concerned with the p-k-Hessian inequality (1.1).

A function u is called p-k-admissible in Ω if $u \in C^1(\Omega)$, $|Du|^{p-2}Du \in C^2(\Omega)$ and $\lambda\big(D(|Du|^{p-2}Du)\big) \in \Gamma_k$ in Ω. Let $\Phi^{p,k}(\Omega)$ be the class of p-k-admissible functions. That is

$$\Phi^{p,k}(\Omega) := \left\{u \in C^1(\Omega) \middle| |Du|^{p-2}Du \in C^2(\Omega), \lambda\Big(D\big(|Du|^{p-2}Du(x)\big)\Big) \in \Gamma_k, \forall\, x \in \Omega\right\}.$$

If $pk < n$, denote by $k_{p,*} := \frac{n(p-1)k}{n-pk}$.

Now we state our main theorem as follows:

Theorem 1.3 *If $pk < n$, then (1.1) has no negative solution in $\Phi^{p,k}(\mathbb{R}^n)$ if and only if $\alpha \leq k_{p,*}$.*

We are going to prove Theorem 1.3 by using local integral estimates and their asymptotic behavior. To conduce these sharp estimates, we need to establish some iteration forms on the p-k-Hessian inequality. The approach is widely used in partial differential equation. For history of this idea and strategy, one can refer to Obata [10], Mitidieri and Pohozaev [8]. The iteration technique for k-Hessian inequality firstly arose in [2] by Chang, Gursky and Yang, see also González [5], Ou [12, 13].

Recently, Bao and Feng [1] gave the necessary and sufficient conditions on global solvability for another type p-k-Hessian inequalities

$$F_{k,p}[u] \geq f(u) \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

They proved that if $f(t)$ is monotonically non-decreasing in $(0, \infty)$, then (1.2) admits a positive p-k-admissible solution if and only if for any $a > 0$,

$$\int_a^\infty \left(\int_0^t f^k(s)\mathrm{d}s \right)^{-\frac{1}{(p-1)k+1}} \mathrm{d}t = \infty. \tag{1.3}$$

Condition (1.3) is a generalized Keller-Osserman condition. The original Keller-Osserman condition was proposed by Keller [7] and Osserman [11] when dealing with $\Delta u \geq f(u)$ in $\mathbb{R}^n$. Naito and Usami [9] extended it to the case of p-Laplacian. Ji and Bao [6] generalized it to the k-Hessian case. However, these are different issues from ours.

Now we outline the content of the paper. In Section 2, we state some facts concerning the elementary symmetric functions on non-symmetric matrices and p-k-Hessian operator. In Section 3, we prove Theorem 1.3.

2 Preliminary

For any $k = 1, \cdots, n$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$, the k-th elementary symmetric function on λ is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}. \tag{2.1}$$

Let $A = (a_{ij}) \in \mathbb{R}^n$ be an $n \times n$ matrix, we define $\sigma_k(A)$ by the sum of k-th principal minors of A. That is

$$\sigma_k(A) = \frac{1}{k!} \sum_{1 \leq i_1, \cdots, i_k, j_1, \cdots, j_k \leq n} \delta_{i_1 i_2 \cdots i_k}^{j_1 j_2 \cdots j_k} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_k j_k}, \tag{2.2}$$

where $\delta^{j_1 j_2 \cdots j_k}_{i_1 i_2 \cdots i_k}$ is the Kronecker symbol, which has the value $+1$ (respectively, -1) if $i_1, i_2, \cdots, i_k$ are distinct and $(j_1 j_2 \cdots j_k)$ is an even permutation (respectively, an odd permutation) of $(i_1 i_2 \cdots i_k)$, and has the value 0 in any other cases. We use the convention that $\sigma_0(A) = 1$. We denote the eigenvalues of A by $\lambda(A)$ and if $\lambda(A)$ are all real, it is clear that $\sigma_k(\lambda(A)) = \sigma_k(A)$, see [3]. Denote by $\sigma_k^{ij}(A) = \frac{\partial \sigma_k(A)}{\partial a_{ij}}$.It is easy to see that

$$\sigma_k(A) = \frac{1}{k} \sum_{i,j=1}^{n} \sigma_k^{ij}(A) a_{ij}, \tag{2.3}$$

and

$$\sum_{i=1}^{n} \sigma_k^{ii}(A) = (n - k + 1)\sigma_{k-1}(A). \tag{2.4}$$

For general (non-symmeric) matrices, Pietra, Gavitone and Xia proved the following result in [3]. From now on, we follow Einstein's summation convention for repeated indices from 1 to n.

Proposition 2.1 *(*Pietra, Gavitone and Xia [3]*) For any $n \times n$ matrix $A = (a_{ij})$, we have*

$$\sigma_k^{ij}(A) = \sigma_{k-1}(A)\delta_{ij} - \sigma_{k-1}^{il}(A) a_{jl}, \quad i, j, k = 1, 2, \cdots, n. \tag{2.5}$$

Proposition 2.2 *For any $n \times n$ matrix $A = (a_{ij})$, we have*

$$\sigma_k^{il}(A) a_{jl} = \sigma_k^{lj}(A) a_{li}, \quad i, j, k = 1, 2, \cdots, n. \tag{2.6}$$

Proof We prove it by induction. For $k = 1$, $\sigma_k^{il} = \delta_{il}$, (2.6) holds naturally. Suppose it holds for $k - 1$, then

$$\begin{aligned} \sigma_k^{il}(A) a_{jl} &= (\sigma_{k-1}(A)\delta_{il} - \sigma_{k-1}^{is}(A) a_{ls}) a_{jl} \\ &= \sigma_{k-1}(A) a_{ji} - \sigma_{k-1}^{is}(A) a_{ls} a_{jl} \\ &= \sigma_{k-1}(A) a_{ji} - \sigma_{k-1}^{sl}(A) a_{si} a_{jl}. \end{aligned} \tag{2.7}$$

While

$$\begin{aligned} \sigma_k^{lj}(A) a_{li} &= (\sigma_{k-1}(A)\delta_{lj} - \sigma_{k-1}^{lm}(A) a_{jm}) a_{li} \\ &= \sigma_{k-1}(A) a_{ji} - \sigma_{k-1}^{lm}(A) a_{jm} a_{li}. \end{aligned} \tag{2.8}$$

Hence (2.6) holds. □

Proposition 2.3 *Suppose $X = (X_1, \cdots, X_n)$ is a C^2 vector field, $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} = \partial_j X_i := X_{i,j}$. Then*

$$\partial_j \sigma_k^{ij}(A) = 0, \quad i, k = 1, 2, \cdots, n. \tag{2.9}$$

Proof We prove it also by induction. For $k = 1$, $\sigma_1^{ij}(A) = \delta_{ij}$, then $\partial_j \sigma_1^{ij}(A) = \partial_j \delta_{ij} = 0$. Suppose (2.9) holds for $k-1$, that is

$$\partial_j \sigma_{k-1}^{ij}(A) = 0. \tag{2.10}$$

Then

$$\begin{aligned}\partial_j \sigma_k^{ij}(A) &= \partial_j(\sigma_{k-1}\delta_{ij} - \sigma_{k-1}^{il} a_{jl}) = \partial_j(\sigma_{k-1}\delta_{ij} - \sigma_{k-1}^{lj} a_{li}) \\ &= \partial_i \sigma_{k-1}(A) - \partial_j \sigma_{k-1}^{lj}(A) a_{li} - \sigma_{k-1}^{lj} \partial_j a_{li} \\ &= \partial_i \sigma_{k-1}(A) - \sigma_{k-1}^{lj} \partial_i a_{lj} = 0,\end{aligned} \tag{2.11}$$

where we use proposition 2.2 in the second equality, we use (2.10) and the fact $\partial_j a_{li} = \partial_j(\partial_i X_l) = \partial_i(\partial_j X_l) = \partial_i a_{lj}$ in the fourth equality. Thus (2.9) holds. □

Proposition 2.4 *Suppose $u \in \Phi^{p,k}(\Omega)$. Then in $\Omega \setminus \{x : |Du(x)| = 0\}$, there holds*

$$\begin{aligned}&\sigma_s^{ij}(D(|Du|^{p-2}Du)) \le \sum_{l=1}^{n} \sigma_s^{ll}(D(|Du|^{p-2}Du)), \\ &s = 1, 2, \cdots, k, \ i, j = 1, 2, \cdots, n.\end{aligned} \tag{2.12}$$

Proof For any point x_0 in $\Omega \setminus \{x : |Du(x)| = 0\}$, we claim that $u \in C^1$ and $|Du|^{p-2}Du \in C^1$ if and only if $u \in C^2$. The sufficiency is obvious. Now we prove the necessity. For any $i, j \in \{1, \cdots, n\}$, since $(|Du|^{p-2}u_i)_j$ exists at x_0, we have

$$\begin{aligned}&\lim_{t \to 0} \frac{|Du(x_0 + te_j)|^{p-2} u_i(x_0 + te_j) - |Du(x_0)|^{p-2} u_i(x_0)}{t} \\ &= |Du(x_0)|^{p-2}(\delta_{il} - (p-2)\frac{u_i(x_0)u_l(x_0)}{|Du(x_0)|^2}) \lim_{t \to 0} \frac{u_l(x_0 + te_j) - u_l(x_0)}{t}\end{aligned} \tag{2.13}$$

Let

$$B_{ij} := |Du|^{p-2}(\delta_{ij} + (p-2)\frac{u_i u_j}{|Du|^2}). \tag{2.14}$$

Since B_{ij} is positive definite provided $p > 1$, B is invertible. Thus we prove the sufficiency.

In the following, we do the following computations at x_0. Note that

$$(|Du|^{p-2}u_i)_j = |Du|^{p-2}u_{ij} + (p-2)|Du|^{p-4}u_m u_{mj} u_i = B_{im} u_{mj}. \tag{2.15}$$

If we take $B^{\frac{1}{2}}$ and $B^{-\frac{1}{2}}$ positive definite matrices,

$$B^{\frac{1}{2}}_{ij} = |Du|^{\frac{p-2}{2}}(\delta_{ij} + (\sqrt{p-1} - 1)|Du|^{-2}u_i u_j), \tag{2.16}$$

and

$$B^{-\frac{1}{2}}_{ij} = |Du|^{\frac{2-p}{2}}(\delta_{ij} - \frac{\sqrt{p-1} - 1}{\sqrt{p-1}} \frac{u_i u_j}{|Du|^2}). \tag{2.17}$$

It is easy to check that the matrices defined in (2.16) and (2.17) are the square root matrices of B and B^{-1} respectively. Then

$$\frac{\partial (B^{\frac{1}{2}} D^2 u B^{\frac{1}{2}})_{ml}}{\partial (BD^2u)_{ij}} = B^{-\frac{1}{2}}_{mi} B^{\frac{1}{2}}_{jl}. \tag{2.18}$$

Since the eigenvalues of $D(|Du|^{p-2}Du)$ and $B^{\frac{1}{2}} D^2 u B^{\frac{1}{2}}$ are the same, we obtain,

$$F_{k,p}[u] = \sigma_k(D(|Du|^{p-2}Du)) = \sigma_k(B^{\frac{1}{2}} D^2 u B^{\frac{1}{2}}).$$

If we denote by $\sigma_k^{ij} := \frac{\partial \sigma_k(BD^2u)}{\partial (BD^2u)_{ij}}$ and $\tilde{\sigma}_k^{ij} := \frac{\partial \sigma_k(B^{\frac{1}{2}} D^2 u B^{\frac{1}{2}})}{\partial (B^{\frac{1}{2}} D^2 u B^{\frac{1}{2}})_{ij}}$. It follows from $\lambda(D(|Du|^{p-2} Du)) \in \Gamma_k$ that $\tilde{\sigma}_k^{ij}$ is positive definite. So

$$|\tilde{\sigma}_k^{ij}| \leq \sum_{l=1}^{n} \tilde{\sigma}_k^{ll}. \tag{2.19}$$

By (2.18), we get

$$\sigma_k^{ij} = \tilde{\sigma}_k^{ml} B^{-\frac{1}{2}}_{im} B^{\frac{1}{2}}_{jl}. \tag{2.20}$$

By (2.16), (2.17) and (2.19), we obtain

$$|\sigma_k^{ij}| \leq \sum_{m=1}^{n} \tilde{\sigma}_k^{mm} = \sum_{m=1}^{n} \sigma_k^{mm}. \tag{2.21}$$

□

3 Proof of Theorem 1.3

In this section, we first prove that (1.1) admits negative admissible solution when $\alpha > k_{p,*}$.

Let $a=\frac{p}{p-1}>1$, $b=\frac{\alpha(p-1)}{\alpha-(p-1)k}>0$ and $w=-C_*\big(A+|x|^a\big)^{-\frac{k(p-1)}{\alpha-(p-1)k}}$, $A>0$ is a constant and $C_*>0$ is to be determined later. Then $w\in C^1(\mathbb{R}^n)$,

$$|Dw|^{p-2}Dw=\tilde{C}C_*^{p-1}\big(A+|x|^a\big)^{-b}x,$$

where $\tilde{C}=\big(\frac{kp}{\alpha-(p-1)k}\big)^{p-1}$. For $i,j,m=1,2,\cdots,n$,

$$\begin{aligned}(|Dw|^{p-2}w_i)_j &= \tilde{C}C_*^{p-1}\big(A+|x|^a\big)^{-b}\delta_{ij}-\tilde{C}C_*^{p-1}ab\big(A+|x|^a\big)^{-b-1}|x|^{a-2}x_ix_j\\ &= \tilde{C}C_*^{p-1}\big(A+|x|^a\big)^{-b}\bigg(\delta_{ij}-ab\frac{|x|^{a-2}x_ix_j}{A+|x|^a}\bigg),\end{aligned}$$

and

$$\begin{aligned}\big(|Dw|^{p-2}w_i\big)_{jm} &= \tilde{C}C_*^{p-1}a^2b(b+1)\big(A+|x|^a\big)^{-b-2}|x|^{2a-4}x_ix_jx_m\\ &-\tilde{C}C_*^{p-1}ab\big(A+|x|^a\big)^{-b-1}\big[(a-2)|x|^{a-4}x_ix_jx_m\\ &+|x|^{a-2}\delta_{im}x_j+|x|^{a-2}\delta_{jm}x_i+|x|^{a-2}\delta_{ij}x_m\big].\end{aligned}$$

We have $|Dw|^{p-2}Dw\in C^2(\mathbb{R}^n)$ and $\forall\, l=1,2,\cdots,k$,

$$\begin{aligned}F_{l,p}[w] &= \tilde{C}^lC_*^{l(p-1)}\big(A+|x|^a\big)^{-bl}\big[C_{n-1}^l+C_{n-1}^{l-1}\big(1-\frac{ab|x|^a}{A+|x|^a}\big)\big]\\ &= \tilde{C}^lC_*^{l(p-1)}C_n^l\big(A+|x|^a\big)^{-bl}\bigg(1-\frac{abl}{n}\frac{|x|^a}{A+|x|^a}\bigg).\end{aligned}$$

Since $1-\frac{abl}{n}\geq 1-\frac{abk}{n}=1-\frac{k}{n}\frac{\alpha p}{\alpha-(p-1)k}>0$, then $F_{l,p}[w]\geq 0$, $l=1,2,\cdots,k$. Hence for $\alpha>k_{p,*}$, $w\in\Phi^{p,k}(\mathbb{R}^n)$ and $F_{k,p}[w]\geq(-w)^\alpha$ after we choose $C_*=\big(\frac{(n-1)!}{k!(n-k)!}\frac{(pk)^{(p-1)k}(\alpha(n-pk)-n(p-1)k)}{(\alpha-(p-1)k)^{(p-1)k+1}}\big)^{\frac{1}{\alpha-(p-1)k}}>0$.

Now we prove that there is no negative admissible solution to (1.1) when $\alpha\leq k_{p,*}$. Assume $u<0$ be a solution of (1.1) in Γ_k. In the following, we use σ_k and σ_k^{ij} instead of $\sigma_k(D(|Du|^{p-2}Du))$ and $\frac{\partial\sigma_k(D(|Du|^{p-2}Du))}{\partial(|Du|^{p-2}u_i)_j}$ respectively for convenience.

Let η be a C^2 cut-off function satisfying:

$$\begin{cases}\eta\equiv 1 & \text{in } B_R,\\ 0\leq\eta\leq 1 & \text{in } B_{2R},\\ \eta\equiv 0 & \text{in } \mathbb{R}^n\backslash B_{2R},\\ |D\eta|\leq\frac{C}{R} & \text{in } \mathbb{R}^n,\end{cases}\tag{3.1}$$

where and in the following C is a constant independent of R and u.

For $s=1,\cdots,k$, denote

$$
\begin{aligned}
B_s &:= \int_{\mathbb{R}^n} \sigma_{k-s}|Du|^{sp}(-u)^{-\delta-s}\eta^{\theta},\\
M_s &:= \int_{\mathbb{R}^n} \sigma_{k-s+1}^{ij}|Du|^{sp-2}u_iu_j(-u)^{-\delta-s}\eta^{\theta},\\
E_s &:= \int_{\mathbb{R}^n} \sigma_{k-s+1}^{ij}|Du|^{sp-2}u_i\eta_j(-u)^{-\delta-s+1}\eta^{\theta-1},\\
b_s &:= \frac{1}{s!}(1-\frac{1}{p})^{s-1}(k-\frac{k-s}{p})\prod_{j=0}^{s-1}(\delta+j),
\end{aligned}
\tag{3.2}
$$

and

$$
e_1:=\theta,\ e_s:=\theta\frac{1}{(s-1)!}(1-\frac{1}{p})^{s-1}\prod_{j=0}^{s-2}(\delta+j)\quad s=2,\cdots,k,
\tag{3.3}
$$

where δ and θ are two constants. We have the following expansion.

Lemma 3.1

$$
k\int_{\mathbb{R}^n}\sigma_k(-u)^{-\delta}\eta^{\theta} = -\sum_{s=1}^{k} b_sB_s - \sum_{s=1}^{k} e_sE_s.
\tag{3.4}
$$

Proof Let

$$
g=(-u)^{-\delta-s}\eta^{\theta}.
\tag{3.5}
$$

By proposition 2.1, we have

$$
\begin{aligned}
M_s &= \int_{\mathbb{R}^n} \sigma_{k-s+1}^{ij}|Du|^{sp-2}u_iu_jg\\
&= \int_{\mathbb{R}^n} \big(\sigma_{k-s}\delta_{ij}-\sigma_{k-s}^{il}(|Du|^{p-2}u_j)_l\big)|Du|^{sp-2}u_iu_jg\\
&= \int_{\mathbb{R}^n} \sigma_{k-s}|Du|^{sp}g - \int_{\mathbb{R}^n}\sigma_{k-s}^{il}(|Du|^{p-2}u_j)_l|Du|^{sp-2}u_iu_jg.
\end{aligned}
\tag{3.6}
$$

Note that

$$
\begin{aligned}
&\sigma_{k-s}^{il}(|Du|^{p-2}u_j)_l|Du|^{sp-2}u_iu_jg\\
&= \sigma_{k-s}^{il}(|Du|^{p-2}u_j)_l|Du|^{p-2}u_j|Du|^{(s-1)p}u_ig\\
&= \sigma_{k-s}^{il}\frac{1}{2}(|Du|^{2p-2})_l|Du|^{(s-1)p}u_ig\\
&= \sigma_{k-s}^{il}\frac{1}{2}(|Du|^{2p-2})_l|Du|^{(s-2)p+2}|Du|^{p-2}u_ig
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma_{k-s}^{il}\frac{1}{2}\frac{2(p-1)}{sp}(|Du|^{sp})_l|Du|^{p-2}u_i g\\
&= \frac{p-1}{sp}\sigma_{k-s}^{il}|Du|^{p-2}u_i(|Du|^{sp})_l g\\
&= \frac{p-1}{sp}(\sigma_{k-s}^{il}|Du|^{(s+1)p-2}u_i g)_l - \frac{p-1}{sp}\sigma_{k-s}^{il}(|Du|^{p-2}u_i)_l|Du|^{sp}g\\
&\quad -\frac{p-1}{sp}\sigma_{k-s}^{il}|Du|^{(s+1)p-2}u_i g_l.
\end{aligned}\tag{3.7}
$$

Putting (3.5) and (3.7) into (3.6), we get

$$
\begin{aligned}
M_s &= \frac{1}{s}(k-\frac{k-s}{p})\int_{\mathbb{R}^n}\sigma_{k-s}|Du|^{sp}(-u)^{-\delta-s}\eta^\theta\\
&+(\delta+s)\frac{1}{s}(1-\frac{1}{p})\int_{\mathbb{R}^n}\sigma_{k-s}^{ij}|Du|^{(s+1)p-2}u_iu_j(-u)^{-\delta-s-1}\eta^\theta\\
&+\theta\frac{1}{s}(1-\frac{1}{p})\int_{\mathbb{R}^n}\sigma_{k-s}^{ij}|Du|^{(s+1)p-2}u_i\eta_j(-u)^{-\delta-s}\eta^{\theta-1}.
\end{aligned}\tag{3.8}
$$

That is

$$
M_s = \frac{1}{s}(k-\frac{k-s}{p})B_s + (\delta+s)\frac{1}{s}(1-\frac{1}{p})M_{s+1} + \theta\frac{1}{s}(1-\frac{1}{p})E_{s+1}. \tag{3.9}
$$

On the other hand, by (2.3), we obtain

$$
\begin{aligned}
&k\int_{\mathbb{R}^n}\sigma_k(-u)^{-\delta}\eta^\theta = \int_{\mathbb{R}^n}\sigma_k^{ij}(|Du|^{p-2}u_i)_j(-u)^{-\delta}\eta^\theta\\
&= -\delta\int_{\mathbb{R}^n}\sigma_k^{ij}|Du|^{p-2}(-u)^{-\delta-1}u_iu_j\eta^\theta - \theta\int_{\mathbb{R}^n}\sigma_k^{ij}|Du|^{p-2}u_i\eta_j(-u)^{-\delta}\eta^{\theta-1}\\
&= -\delta M_1 - \theta E_1.
\end{aligned}\tag{3.10}
$$

Substituting (3.9) into (3.10) iteratively, we get (3.4). This finish the proof. □

Now we prove Theorem 1.3.

Proof Multiply both sides of (1.1) by $k(-u)^{-\delta}\eta^\theta$, we have

$$
k\int_{\mathbb{R}^n}(-u)^{\alpha-\delta}\eta^\theta \le k\int_{\mathbb{R}^n}\sigma_k(-u)^{-\delta}\eta^\theta. \tag{3.11}
$$

Now we estimate the error term E_s. Let $U = B_{2R}\backslash\overline{B_R}$, then $\mathrm{Supp}(D\eta)\subseteq\overline{U}$. By (2.4), proposition 2.4 and $|D\eta|\le\frac{C}{R}$, we have

$$
|E_s| \le C\int_{\mathbb{R}^n}\sigma_{k-s}|Du|^{sp-1}|D\eta|(-u)^{-\delta-s+1}\eta^{\theta-1}
$$

$$\leq \frac{C}{R}\int_U \sigma_{k-s}|Du|^{sp-1}(-u)^{-\delta-s+1}\eta^{\theta-1}. \tag{3.12}$$

Using Young's inequality with exponent pair $(\frac{ps}{ps-1}, ps)$, $\forall\varepsilon > 0$, the last inequaliy turns into

$$\begin{aligned}|E_s| \leq \varepsilon \int_U \sigma_{k-s}|Du|^{sp}(-u)^{-\delta-s}\eta^{\theta} \\ +\frac{C_\varepsilon}{R^{sp}}\int_U \sigma_{k-s}(-u)^{-\delta-(1-p)s}\eta^{\theta-sp}.\end{aligned} \tag{3.13}$$

Now we deal with the last term in (3.13). Note that

$$\begin{aligned}
&\frac{1}{R^{sp}}\int_U \sigma_{k-s}(-u)^{-\delta-(1-p)s}\eta^{\theta-sp}\\
&\cong \frac{1}{R^{sp}}\int_U \sigma_{k-s}^{ij}(|Du|^{p-2}u_i)_j(-u)^{-\delta-(1-p)s}\eta^{\theta-sp}\\
&= -\frac{\delta+s-sp}{R^{sp}}\int_U \sigma_{k-s}^{ij}|Du|^{p-2}u_iu_j(-u)^{-\delta-s-1+sp}\eta^{\theta-sp}\\
&\quad -\frac{\theta-sp}{R^{sp}}\int_U \sigma_{k-s}^{ij}|Du|^{p-2}u_i\eta_j(-u)^{-\delta-s+sp}\eta^{\theta-sp-1}\\
&\lesssim \frac{1}{R^{sp}}\int_U \sigma_{k-s-1}|Du|^{p}(-u)^{-\delta-s-1+sp}\eta^{\theta-sp}\\
&\quad +\frac{1}{R^{sp+1}}\int_U \sigma_{k-s-1}|Du|^{p-1}(-u)^{-\delta-s+sp}\eta^{\theta-sp-1}\\
&\leq \varepsilon\int_U \sigma_{k-s-1}|Du|^{p(s+1)}(-u)^{-\delta-s-1}\eta^{\theta}\\
&\quad +\frac{C_\varepsilon}{R^{(s+1)p}}\int_U \sigma_{k-s-1}(-u)^{-\delta+(p-1)(s+1)}\eta^{\theta-(s+1)p},
\end{aligned} \tag{3.14}$$

where we apply Young's inequality with exponent pairs $(s+1, \frac{s+1}{s})$ and $(\frac{p}{p-1}(s+1), \frac{p(s+1)}{ps+1})$ respectively in the last inequality, and we use "$\lesssim$" and "$\cong$" to drop out some positive constants independent of R and u. Repeating the same process, we get

$$\begin{aligned}
&\frac{1}{R^{sp}}\int_U \sigma_{k-s}(-u)^{-\delta-(1-p)s}\eta^{\theta-sp}\\
&\leq \varepsilon\int_U \sigma_{k-s-1}|Du|^{p(s+1)}(-u)^{-\delta-s-1}\eta^{\theta}\\
&\quad +\frac{C_\varepsilon}{R^{(s+1)p}}\int_U \sigma_{k-s-1}(-u)^{-\delta+(p-1)(s+1)}\eta^{\theta-(s+1)p}\\
&\leq \varepsilon\int_U \sigma_{k-s-1}|Du|^{p(s+1)}(-u)^{-\delta-s-1}\eta^{\theta}+\varepsilon\int_U \sigma_{k-s-2}|Du|^{p(s+2)}(-u)^{-\delta-s-2}\eta^{\theta}\\
&\quad +\frac{C_\varepsilon}{R^{(s+2)p}}\int_U \sigma_{k-s-2}(-u)^{-\delta+(p-1)(s+2)}\eta^{\theta-(s+2)p}
\end{aligned}$$

$$\leq \varepsilon \sum_{i=s+1}^{k} B_i + \frac{C_\varepsilon}{R^{pk}} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp}. \tag{3.15}$$

Putting (3.4), (3.13) and (3.15) together, we obtain

$$k \int_{\mathbb{R}^n} \sigma_k(-u)^{-\delta} \eta^\theta + \sum_{s=1}^{k} (b_s - \varepsilon) B_s \lesssim \frac{1}{R^{kp}} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp}. \tag{3.16}$$

Now for $\alpha \in (-\infty, k_{p,*}]$ we divide the proof into four cases.

(a) $\alpha = (p-1)k$,
(b) $\alpha \in (-\infty, (p-1)k)$,
(c) $\alpha \in ((p-1)k, k_{p,*})$,
(d) $\alpha = k_{p,*}$.

In all cases of (a)-(c), $b_s > 0$ for all $s = 1, \cdots, k$. For the **case (a)**, let $\delta = \alpha$, $\theta > n$. It follows from (3.11) and (3.16) that

$$\int_{\mathbb{R}^n} \eta^\theta \leq \int_{\mathbb{R}^n} \sigma_k(-u)^{-\delta} \eta^\theta \lesssim \frac{1}{R^{pk}} \int_{\mathbb{R}^n} \eta^{\theta-kp} \lesssim \varepsilon \int_{\mathbb{R}^n} \eta^\theta + R^{n-\theta}, \tag{3.17}$$

where we use Young's inequality with exponent pair $(\frac{\theta}{kp}, \frac{\theta}{\theta-kp})$ in the last inequality. Now we choose ε small enough and let $R \to \infty$, we get a contradiction.

For **cases (b) and (c)**, let $\delta > \frac{n-kp}{kp}(k_{p,*}-\alpha)$ and $0 < \delta < \frac{n-kp}{kp}(k_{p,*}-\alpha)$ respectively, then we have $\frac{\alpha-\delta}{(p-1)k-\delta} > 1$. By Young's inequality with $(\frac{\alpha-\delta}{(p-1)k-\delta}, \frac{\alpha-\delta}{\alpha-(p-1)k})$, we obtain

$$\frac{1}{R^{pk}} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp} \leq \varepsilon \int_U (-u)^{\alpha-\delta} \eta^\theta + \frac{C_\varepsilon}{R^{\frac{kp(\alpha-\delta)}{\alpha-(p-1)k}}} \int_U \eta^{\theta-\frac{kp(\alpha-\delta)}{\alpha-(p-1)k}}. \tag{3.18}$$

Put (3.11), (3.18) into (3.16), and let $R \to \infty$, we get a contradiction after choosing $\theta > \frac{kp(\alpha-\delta)}{\alpha-(p-1)k}$.

For the **case (d)**, it follows from (3.10) for choosing $\delta = 0$ that

$$k \int_{\mathbb{R}^n} \sigma_k \eta^\theta = -\theta E_1. \tag{3.19}$$

By (2.12), (3.1), (3.2) and Young's inequality with exponent $(\frac{p}{p-1}, p)$, we have

$$\begin{aligned} R^\beta |E_1| \leq\ & R^{\beta-1} \int_U \sigma_{k-1} |Du|^{p-1} \eta^{\theta-1} \\ \leq\ & \int_U \sigma_{k-1} |Du|^p (-u)^{-\delta-1} \eta^\theta + R^{(\beta-1)p} \int_U \sigma_{k-1} (-u)^{(p-1)(\delta+1)} \eta^{\theta-p} \\ \leq\ & B_1 + V_1, \end{aligned} \tag{3.20}$$

where $\delta \in (0, \min\{p-1, \frac{k^2 p(p-1)}{(n-pk)(kp-1)}\})$, $\beta = \frac{n\delta}{\alpha}$ and $V_1 := R^{(\beta-1)p} \int_U \sigma_{k-1} (-u)^{(p-1)(\delta+1)} \eta^{\theta-p}$. Let

$$V_s := R^{sp(\beta-1)} \int_U \sigma_{k-s}(-u)^{-\delta-s+sp(\delta+1)} \eta^{\theta-ps}, \tag{3.21}$$

and

$$W_s := R^{\beta-sp} \int_U \sigma_{k-s}(-u)^{(p-1)s} \eta^{\theta-sp}. \tag{3.22}$$

We can prove the following two inequalities:

$$V_s \lesssim B_{s+1} + V_{s+1} + W_{s+1}, \tag{3.23}$$

and

$$W_s \lesssim B_{s+1} + V_{s+1} + W_{s+1}. \tag{3.24}$$

By (2.3), divergence theorem, (3.1) and Proposition 2.4, we obtain

$$\begin{aligned} V_s &= R^{sp(\beta-1)} \int_U \sigma_{k-s}(-u)^{-\delta-s+sp(\delta+1)} \eta^{\theta-ps} \\ &\cong R^{sp(\beta-1)} \int_U \sigma_{k-s}^{ij} (|Du|^{p-2} u_i)_j (-u)^{-\delta-s+sp(\delta+1)} \eta^{\theta-ps} \\ &\cong R^{sp(\beta-1)} \int_U \sigma_{k-s}^{ij} |Du|^{p-2} u_i u_j (-u)^{-\delta-s-1+sp(\delta+1)} \eta^{\theta-ps} \\ &\quad + R^{sp(\beta-1)} \int_U \sigma_{k-s}^{ij} |Du|^{p-2} u_i \eta_j (-u)^{-\delta-s+sp(\delta+1)} \eta^{\theta-ps-1} \\ &\lesssim R^{sp(\beta-1)} \int_U \sigma_{k-s-1} |Du|^p (-u)^{-\delta-s-1+sp(\delta+1)} \eta^{\theta-ps} \\ &\quad + R^{sp(\beta-1)-1} \int_U \sigma_{k-s-1} |Du|^{p-1} (-u)^{-\delta-s+sp(\delta+1)} \eta^{\theta-ps-1}. \end{aligned} \tag{3.25}$$

Applying Young's inequality with exponent pair $(s+1, \frac{s+1}{s})$ to the first term of last line in (3.25), we derive that

$$\begin{aligned} & R^{sp(\beta-1)} \int_U \sigma_{k-s-1} |Du|^p (-u)^{-\delta-s-1+sp(\delta+1)} \eta^{\theta-ps} \\ &\lesssim \int_U \sigma_{k-s-1} (-u)^{-\delta-s-1} |Du|^{(s+1)p} \eta^{\theta} \\ &\quad + R^{(s+1)p(\beta-1)} \int_U \sigma_{k-s-1} (-u)^{-\delta-s-1+(s+1)p(\delta+1)} \eta^{\theta-(s+1)p} \\ &\leq B_{s+1} + V_{s+1}. \end{aligned} \tag{3.26}$$

Similarly, applying Young's inequality with exponent pair $(\frac{(s+1)p}{p-1}, \frac{(s+1)p}{sp+1})$ to the last term of last line in (3.25), we derive that

$$\begin{aligned}
&R^{sp(\beta-1)-1}\int_U \sigma_{k-s-1}|Du|^{p-1}(-u)^{-\delta-s+sp(\delta+1)}\eta^{\theta-ps-1}\\
&\lesssim \int_U \sigma_{k-s-1}(-u)^{-\delta-s-1}|Du|^{(s+1)p}\eta^{\theta}\\
&\quad +R^{\frac{\beta s(s+1)p^2}{sp+1}-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{(p-1)(s+1)+(\frac{s(s+1)p^2}{sp+1}-1)\delta}\eta^{\theta-(s+1)p}\\
&\lesssim \int_U \sigma_{k-s-1}(-u)^{-\delta-s-1}|Du|^{(s+1)p}\eta^{\theta}\\
&\quad +R^{(s+1)p(\beta-1)}\int_U \sigma_{k-s-1}(-u)^{-\delta-s-1+(s+1)p(\delta+1)}\eta^{\theta-(s+1)p}\\
&\quad +R^{\beta-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{(s+1)(p-1)}\eta^{\theta-(s+1)p}\\
&\le B_{s+1}+V_{s+1}+W_{s+1},
\end{aligned}\tag{3.27}$$

where the last inequality follows from using Young's inequality with exponent $(\frac{s(s+1)p^2+p-1}{s(s+1)p^2-(sp+1)}, \frac{s(s+1)p^2+p-1}{(s+1)p})$. Substituting (3.26) and (3.27) into (3.25), we arrive at (3.23).

By (2.3), divergence theorem, (3.1) and Proposition 2.4, we obtain

$$\begin{aligned}
W_s &= R^{\beta-sp}\int_U \sigma_{k-s}(-u)^{(p-1)s}\eta^{\theta-sp}\\
&\cong R^{\beta-sp}\int_U \sigma_{k-s}^{ij}(|Du|^{p-2}u_i)_j(-u)^{(p-1)s}\eta^{\theta-sp}\\
&\cong R^{\beta-sp}\int_U \sigma_{k-s}^{ij}|Du|^{p-2}u_iu_j(-u)^{(p-1)s-1}\eta^{\theta-sp}\\
&\qquad +R^{\beta-sp}\int_U \sigma_{k-s}^{ij}|Du|^{p-2}u_i\eta_j(-u)^{(p-1)s}\eta^{\theta-sp-1}\\
&\lesssim R^{\beta-sp}\int_U \sigma_{k-s-1}|Du|^{p}(-u)^{(p-1)s-1}\eta^{\theta-sp}\\
&\qquad +R^{\beta-sp-1}\int_U \sigma_{k-s-1}|Du|^{p-1}(-u)^{(p-1)s}\eta^{\theta-sp-1}.
\end{aligned}\tag{3.28}$$

Applying Young's inequality with exponent pair $(s+1, \frac{s+1}{s})$ to the first term of last line in (3.28), we derive that

$$\begin{aligned}
&R^{\beta-sp}\int_U \sigma_{k-s-1}|Du|^{p}(-u)^{(p-1)s-1}\eta^{\theta-sp}\\
&\lesssim \int_U \sigma_{k-s-1}|Du|^{(s+1)p}(-u)^{-\delta-s-1}\eta^{\theta}
\end{aligned}$$

$$
\begin{aligned}
&+R^{\frac{(s+1)\beta}{s}-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{\frac{\delta}{s}+(s+1)(p-1)}\eta^{\theta-(s+1)p}\\
\lesssim\ &\int_U \sigma_{k-s-1}|Du|^{(s+1)p}(-u)^{-\delta-s-1}\eta^{\theta}\\
&+R^{\beta-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{(s+1)(p-1)}\eta^{\theta-(s+1)p}\\
&+R^{(s+1)p(\beta-1)}\int_U \sigma_{k-s-1}(-u)^{-\delta-s-1+(s+1)p(\delta+1)}\eta^{\theta-(s+1)p}\\
\le\ & B_{s+1}+W_{s+1}+V_{s+1},
\end{aligned}
\tag{3.29}
$$

where the last inequality follows from using Young's inequality with exponent pair $(s(s+1)p-s, \frac{s(s+1)p-s}{(sp-1)(s+1)})$ for $R^{\frac{\beta}{s}}(-u)^{\frac{\delta}{s}}\cdot 1$. Then applying Young's inequality with exponent pair $(\frac{(s+1)p}{p-1}, \frac{(s+1)p}{sp+1})$ to $|Du|^{p-1}\cdot R^{\beta-(sp+1)}(-u)^{sp+1+\delta}\eta^{-(sp+1)}$, we obtain

$$
\begin{aligned}
&R^{\beta-sp-1}\int_U \sigma_{k-s-1}|Du|^{p-1}(-u)^{(p-1)s}\eta^{\theta-sp-1}\\
&\quad\lesssim \int_U \sigma_{k-s-1}|Du|^{(s+1)p}(-u)^{-\delta-s-1}\eta^{\theta}\\
&\qquad+R^{\frac{(s+1)p\beta}{sp+1}-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{(\frac{(s+1)p}{sp+1}-1)\delta+(s+1)(p-1)}\eta^{\theta-(s+1)p}\\
&\quad\lesssim \int_U \sigma_{k-s-1}|Du|^{(s+1)p}(-u)^{-\delta-s-1}\eta^{\theta}\\
&\qquad+R^{\beta-(s+1)p}\int_U \sigma_{k-s-1}(-u)^{(s+1)(p-1)}\eta^{\theta-(s+1)p}\\
&\qquad+R^{(s+1)p(\beta-1)}\int_U \sigma_{k-s-1}(-u)^{-\delta-s-1+(s+1)p(\delta+1)}\eta^{\theta-(s+1)p}\\
&\quad\le B_{s+1}+W_{s+1}+V_{s+1},
\end{aligned}
\tag{3.30}
$$

where the last inequality follows from using Young's inequality with exponent pair $(\frac{(sp+p-1)(sp+1)}{sp^2(s+1)}, \frac{(sp+p-1)(sp+1)}{p-1})$. Substituting (3.29) and (3.30) into (3.28), we arrive at (3.24).

Using (3.23) and (3.24), we obtain

$$
V_s \lesssim \sum_{i=s+1}^{k} B_i + V_k + W_k. \tag{3.31}
$$

Putting (3.31) into (3.20), we arrive at

$$
R^{\beta}|E_1| \lesssim \sum_{i=1}^{k} B_i + W_k + V_k. \tag{3.32}
$$

Note that for fixed $\delta \in (0, \min\{p-1, \frac{k^2 p(p-1)}{(n-pk)(kp-1)}\})$, $b_s > 0$ and (3.16) still holds. So we get

$$\sum_{i=1}^{k} B_i \lesssim \frac{1}{R^{kp}} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp}. \tag{3.33}$$

Substituting (3.21), (3.22) and (3.33) into (3.32), we have

$$\begin{aligned} |E_1| \lesssim\ & R^{-\beta-kp} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp} \\ & + R^{-kp} \int_U (-u)^{k(p-1)} \eta^{\theta-kp} \\ & + R^{(kp-1)\beta-kp} \int_U (-u)^{-\delta-k+kp(\delta+1)} \eta^{\theta-kp}. \end{aligned} \tag{3.34}$$

Since $\alpha = k_{p,*} = \frac{n(p-1)k}{n-kp} > (p-1)k$, we have $a_1 := \frac{\alpha}{(p-1)k} > 1$. Let b_1 satisfiy $\frac{1}{a_1} + \frac{1}{b_1} = 1$, $\theta > b_1 kp = \frac{\alpha}{(\alpha-(p-1)k)} pk = n$, and $\lambda_1 = \frac{\theta}{b_1} - kp$. After using Hölder inequality, we derive that

$$\begin{aligned} & R^{-kp} \int_U (-u)^{(p-1)k} \eta^{\theta-pk} \\ & \leq R^{-kp} \left(\int_U \left((-u)^{(p-1)k} \eta^{\theta-kp-\lambda_1} \right)^{a_1} \right)^{\frac{1}{a_1}} \left(\int_U \eta^{\lambda_1 b_1} \right)^{\frac{1}{b_1}} \\ & \lesssim \left(\int_U (-u)^{\alpha} \eta^{\theta} \right)^{\frac{(p-1)k}{\alpha}}, \end{aligned} \tag{3.35}$$

where $\left(\int_U \eta^{\lambda_1 b_1} \right)^{\frac{1}{b_1}} \lesssim R^{\frac{n}{b_1}} = R^{kp}$, $(p-1)ka_1 = \alpha$, and $(\theta - kp - \lambda_1)a_1 = \theta$. Similarly, we have $a_2 := \frac{\alpha}{(p-1)k-\delta} > 1$. Let b_2 satisfy $\frac{1}{a_2} + \frac{1}{b_2} = 1$, $\theta > b_2 kp = \frac{kp\alpha}{\alpha+\delta-(p-1)k}$, $\lambda_2 = \frac{\theta}{b_2} - kp$. Using Hölder inequality again, we have

$$\begin{aligned} & R^{-\beta-kp} \int_U (-u)^{-\delta+(p-1)k} \eta^{\theta-kp} \\ & \leq R^{-\beta-kp} \left(\int_U \left((-u)^{-\delta+(p-1)k} \eta^{\theta-kp-\lambda_2} \right)^{a_2} \right)^{\frac{1}{a_2}} \left(\int_U \eta^{\lambda_2 b_2} \right)^{\frac{1}{b_2}} \\ & \lesssim \left(\int_U (-u)^{\alpha} \eta^{\theta} \right)^{\frac{(p-1)k-\delta}{\alpha}}. \end{aligned} \tag{3.36}$$

Since $\delta < \frac{k^2 p(p-1)}{(n-pk)(kp-1)}$, we have that $a_3 := \frac{\alpha}{(kp-1)\delta + k(p-1)}$. Let b_3 satisfy $\frac{1}{a_3} + \frac{1}{b_3} = 1$, $\theta > b_3 kp = \frac{kp\alpha}{\alpha - (kp-1)\delta - k(p-1)}$, $\lambda_3 = \frac{\theta}{b_3}$. By Hölder inequality, we get

$$\begin{aligned} & R^{(kp-1)\beta - kp} \int_U (-u)^{-\delta - k + kp(\delta+1)} \eta^{\theta - kp} \\ \leq\ & R^{(kp-1)\beta - kp} \left(\int_U \left((-u)^{-\delta - k + kp(\delta+1)} \eta^{\theta - kp - \lambda_3} \right)^{a_3} \right)^{\frac{1}{a_3}} \left(\int_U \eta^{\lambda_3 b_3} \right)^{\frac{1}{b_3}} \\ \lesssim\ & \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(kp-1)\delta + k(p-1)}{\alpha}}. \end{aligned} \tag{3.37}$$

Substituting (3.35), (3.36), (3.37) into (3.34), we obtain

$$\begin{aligned} |E_1| \lesssim & \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(p-1)k}{\alpha}} + \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(p-1)k - \delta}{\alpha}} \\ & + \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(kp-1)\delta + k(p-1)}{\alpha}}. \end{aligned} \tag{3.38}$$

It follows from (3.19) and (3.38) that

$$\begin{aligned} & k \int_{\mathbb{R}^n} \sigma_k \eta^\theta \lesssim \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(p-1)k}{\alpha}} \\ & + \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(p-1)k - \delta}{\alpha}} + \left(\int_U (-u)^\alpha \eta^\theta \right)^{\frac{(kp-1)\delta + k(p-1)}{\alpha}}. \end{aligned} \tag{3.39}$$

Since $0 < \frac{(p-1)k}{\alpha}, \frac{(p-1)k-\delta}{\alpha}, \frac{(kp-1)\delta + k(p-1)}{\alpha} < 1$, from (3.39), we have

$$\int_{\mathbb{R}^n} (-u)^\alpha \eta^\theta \leq constant < \infty. \tag{3.40}$$

This implies

$$\int_U (-u)^\alpha \eta^\theta \to 0 \quad \text{as} \quad R \to +\infty. \tag{3.41}$$

Hence, from (3.39) again, as $R \to +\infty$, we have

$$\int_{\mathbb{R}^n} (-u)^\alpha \eta^\theta \leq 0. \tag{3.42}$$

This contradicts with u being a negative solution. □

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Declarations

Conflicts of Interest The authors have no conflict of interest.

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