Simplicial surfaces in GAP

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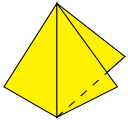
Basic data

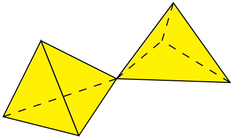
- Package name: `SimplicialSurfaces`
  - Not yet generally available
- Authors: Alice Niemeyer, Markus Baumeister
- Based on current research at Lehrstuhl B: Plesken, Strzelczyk and others
- Internally used packages:
  - `AttributeScheduler` by Gutsche
  - `Digraphs` by De Beule, Mitchell, Pfeiffer, Wilson et al.
  - `GAPDoc` by Lübeck
  - `AutoDoc` by Gutsche, Horn

Motivation

Goal: Investigate rigid paper folding in $\mathbb{R}^3$

- consider surfaces built from triangles (simplicial surfaces)
  - not closed under folding
  - allow more general structures:





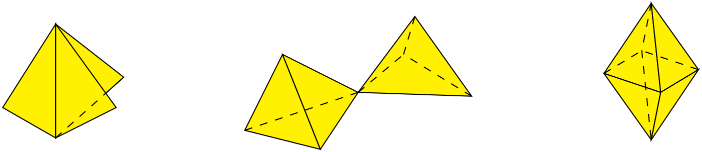
- embeddings are difficult to compute
  - some embeddings of asymmetric icosahedrons are infeasible to compute

$\rightsquigarrow$ focus on intrinsic properties

$\rightsquigarrow$ incidence geometry

Reasons for implementation in GAP

- can describe incidence geometry
  - allows flexible access to the incidence geometry (`AttributeScheduler`)
- can manage hierarchy of structures:



- works well with group–theoretic descriptions
- difference to `FinInG`–package by De Beule, Neunhöffer et al.
  - we only have two dimensions but can work with colourings and foldings

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2. Edge colouring and group properties
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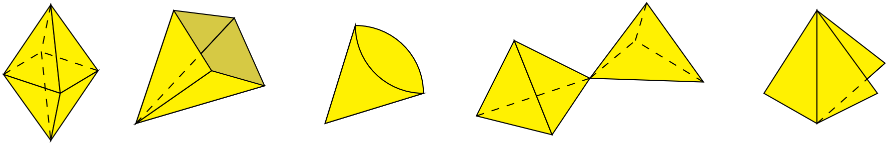
Triangular complexes

We want to describe different structures:

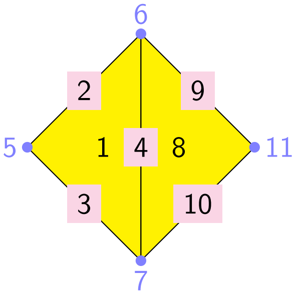


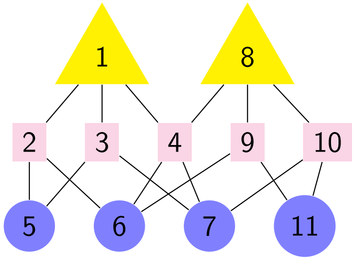
$\rightsquigarrow$ **triangular complexes**

- sets of vertices, edges and faces
- incidence relation between them
- every face is a triangle
- every vertex lies in an edge and every edge lies in a face

Isomorphism testing

Incidence structures can be interpreted as coloured graphs:





$\rightsquigarrow$ reduce to coloured graph isomorphism problem (with fixed colours)
$\rightsquigarrow$ solved efficiently in practice by `nauty/Traces` (McKay, Piperno)
In GAP: `NautyTracesInterface` (Gutsche, Niemeyer, Schweitzer)

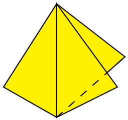
- calls C–functions directly without writing files
- also returns automorphism group

General properties

Some properties can be computed for all triangular complexes:

- Connectivity
- Euler–Characteristic

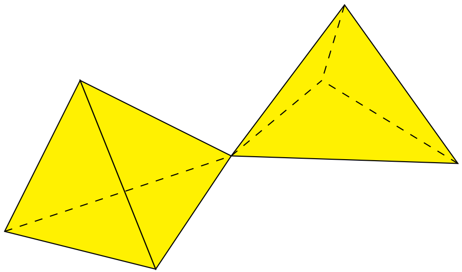
Orientability is **not** one of them. Counterexample:



$\Rightarrow$ every edge lies in at most two faces (for well–definedness)
$\rightsquigarrow$ **ramified simplicial surfaces**

Why ramified?

Typical example of a ramified simplicial surface:

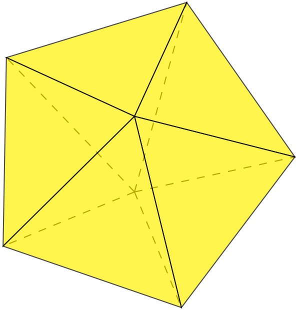


$\Rightarrow$ It is not a surface — there is a *ramification* at the central vertex
A **simplicial surface** does not have these ramifications.

Classification

Plesken/Strzelczyk classified all closed simplicial surfaces up to 20 triangles.

- building blocks are those without a 3–cycle of edges (not around face)
- e. g. exactly 87 non–isomorphic surfaces with 20 triangles
- e. g. only one surface with 10 triangles:



Progress report of triangular complexes

Already implemented:

- surface hierarchy
- elementary properties (e. g. connectivity, orientability [if defined])
- isomorphism testing
- classification of small surfaces (as a data base)

Not yet implemented:

- automorphism group
- advanced properties (any wishes?)

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Embedding questions

Given: A simplicial surface

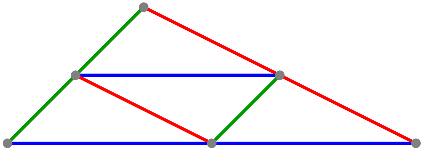
- For which edge lengths embeddable into $\mathbb{R}^3$? In how many ways?

Simplification: All triangles are isometric

Goal: Find an abstract structure that encodes this isometry condition

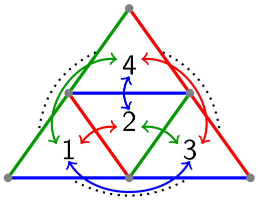
$\rightsquigarrow$ Consider a general triangle (all side lengths different)

$\rightsquigarrow$ Edge–colouring encodes different lengths



Colouring as permutation

Consider a tetrahedron with an edge colouring



simplicial surface $\Rightarrow$ at most two faces at each edge

$\rightsquigarrow$ every edge defines a transposition of incident faces

$\rightsquigarrow$ every colour class defines a permutation of the faces

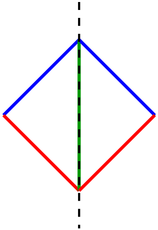
- (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)

$\rightsquigarrow$ group theoretic considerations

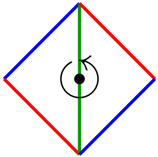
- The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces (fast computation for permutation groups)

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)



rotation (r)

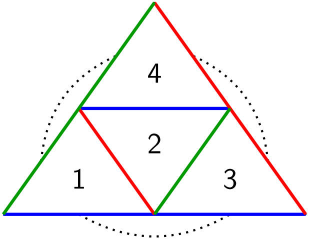
$\rightsquigarrow$ **mr–assignment** for the edges of the surface

Theorem

Permutations and mr–assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same mr–type
Example



has only r–edges.

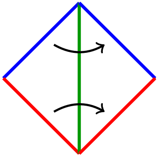
The mirror–case

If all edges are mirrors, the situation is simple

Theorem

A simplicial surface has only mirror–edges iff it covers a single triangle, i. e. there is a surjective incidence–preserving map to the simplicial surface consisting of exactly one face.

Consider



$\Rightarrow$ Unique map that preserves incidence

- Covering pulls back a mirror–colouring of the triangle
- Mirror–colouring defines a map to the triangle

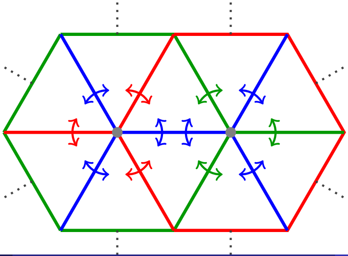
Construction from permutations

Start with three involutions σ_a, σ_b, σ_c in permutation representation (like generators of a finite group)

Theorem

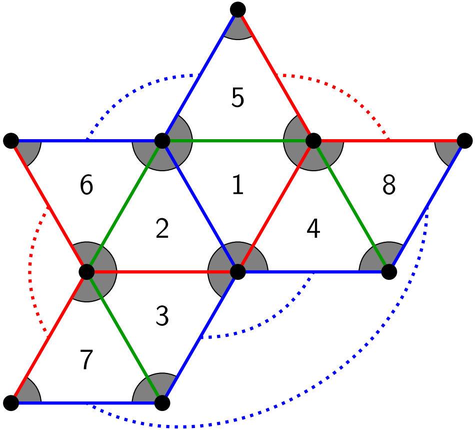
There exists a coloured surface with the given involutions where all edges are mirror edges.

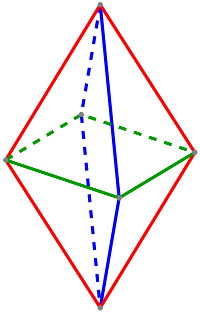
- The faces are the points moved by the involution triple
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)



Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$
$\sigma_b = (1,4)(2,3)(5,8)(6,7)$
$\sigma_c = (1,5)(2,6)(3,7)(4,8)$

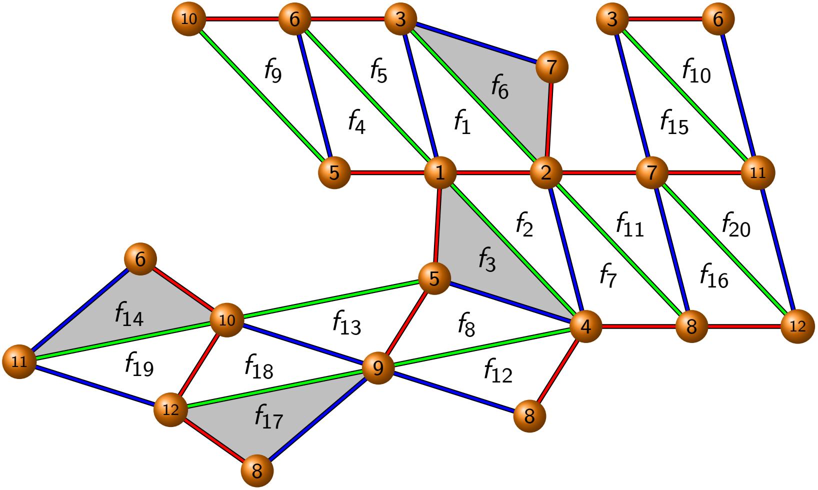




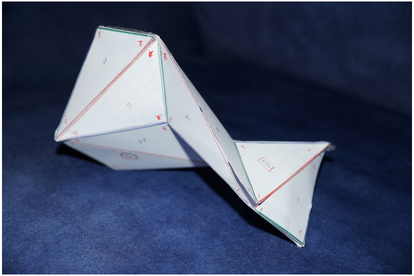
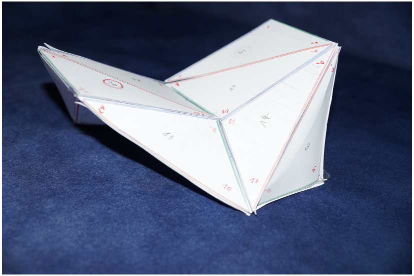
`iko`: coloured icosahedron

```
gap> DrawSurfaceToTikZ(iko,"NetIko.tex");
```

- Has to be manually untangled



Embedded icosahedron

Progress report of edge colouring

Implemented:

- computing all colourings of a given simplicial surface
- constructing all surfaces with given involutions
  1. up to (coloured) isomorphism
  2. with given mr–assignment
- drawing of simplicial surfaces
- constructing various coloured coverings

Still missing:

- for which lengths are the surfaces embeddable?
- does varying the lengths lead to another embedding?

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What kind of folding?

- Folding of surface in $\mathbb{R}^3$
- Fold only at given edges (no introduction of new folding edges)
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

Embeddings are very hard

- At every point in time the folding process has to be embedded
- We can only show foldability for specific small examples
  - Usually using regularity (like crystallographic symmetry)
  - No general method
- It is very hard to define iterated foldings in an embedding

Folding without embedding

Central idea:

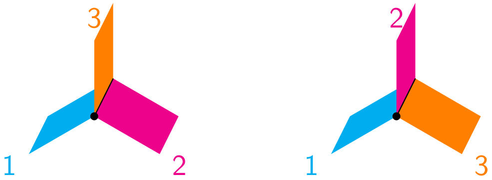
- Don't model the folding process (needs embedding)
- Describe starting and final folding state
  - Only consider changes in the topology (like identification of faces)
  - allows abstraction from embedding

$\rightsquigarrow$ Incidence geometry (triangular complex/simplicial surface)

- Captures some folding restrictions (rigidity of tetrahedron)
- Has to be refined

- Concept should allow reversible folding
- We need an ordering of the faces:

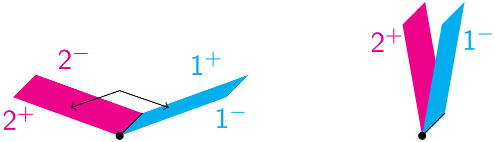
- We also need a cyclic order of the faces around an edge:



$\rightsquigarrow$ **folding complex**

How to describe folding?

Needs specification of two face sides:

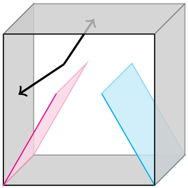


$\Rightarrow$ Characterize folding by specifying the two face sides that touch
$\rightsquigarrow$ **folding plan**

How does folding plan work?

Folding of two faces can induce folding of other faces:

- Can apply to arbitrarily many faces
- The induced folding is not unique



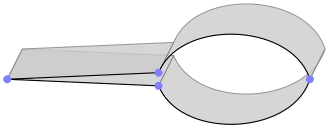
How does folding plan work?

Folding of two faces can induce folding of other faces:

- Can apply to arbitrarily many faces
- The induced folding is not unique

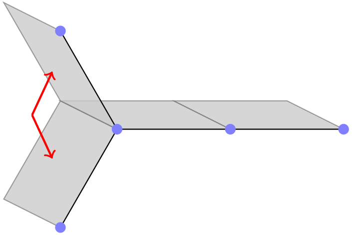
$\Rightarrow$ Identify only two faces at a time (non–uniqueness becomes choice)

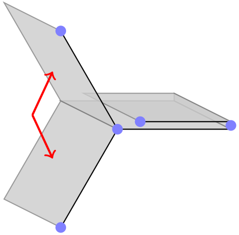
- $\rightsquigarrow$ Relax the rigidity–constraint:
  - Allow non–rigid configurations as transitional states



Structure of multiple foldings

With folding plans we can perform the same folding in different folding complexes

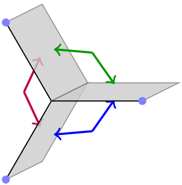


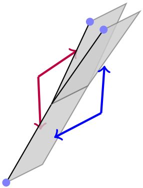


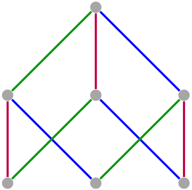
$\rightsquigarrow$ more structure on the set of possible foldings

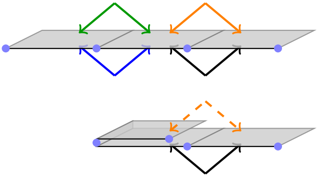
Folding graph

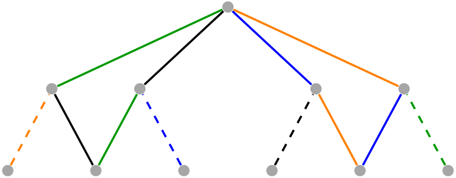
- Vertices are folding complexes (modelling folding states)
- Edges are folding plans connecting two folding complexes



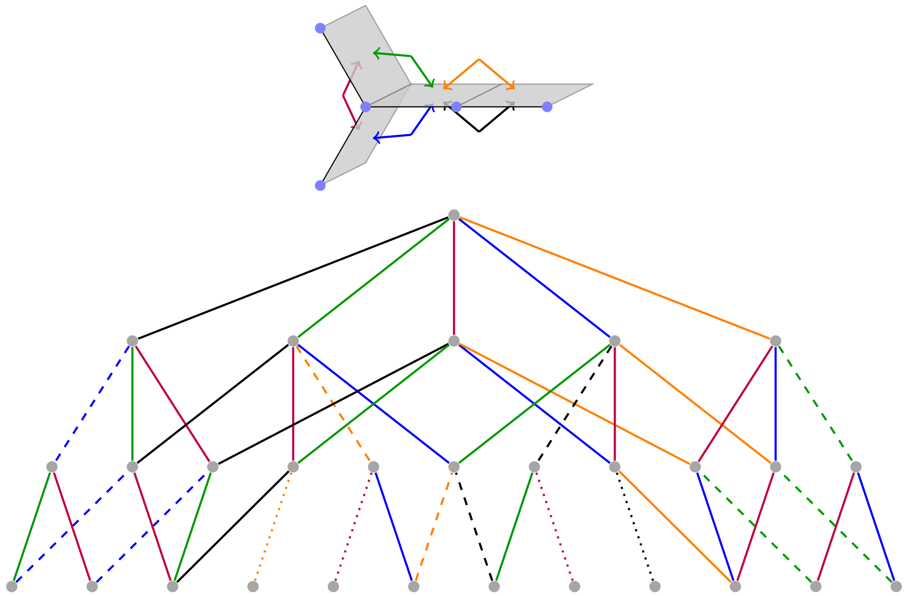






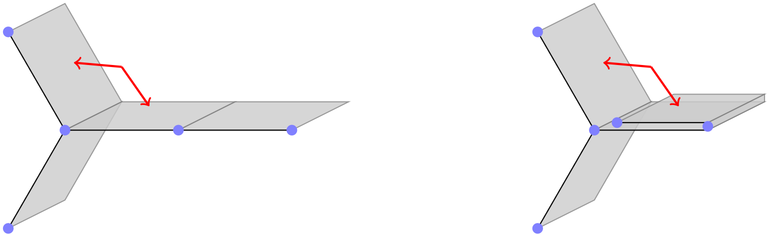


Larger graph



Drawback of folding plans

Some foldings that "should" be the same, aren't:



⇒ If you know the folding structure of a small complex, you can't easily find the folding structure of an extended complex

⇝ Folding plans are not optimal to model folding

Progress report of abstract folding

In development:

- folding complex
- folding plans
- folding graph

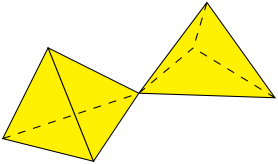
Missing:

- better folding description
- extension of folding graphs
- properties of folding graphs

Summary: SimplicialSurfaces

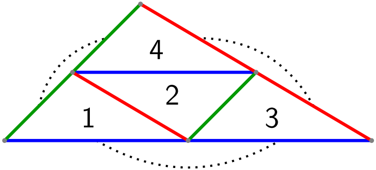
Triangular complexes

- mostly complete



Edge colouring

- current theory implemented
- a lot of theory still to discover



Abstract folding

- framework exists
- needs proper implementation