

# Some commutator identities

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## 1 Introduction

Identities between commutators in a free group  $F$  play an important role in various areas of group theory. Some basic identities are

$$[xx', y] = {}^x[x', y][x, y] \quad (1)$$

$$[x, yy'] = [x, y]{}^y[x, y] \quad (2)$$

$$[x, x] = 1 \quad (3)$$

$${}^z[x, y] = [{}^zx, {}^zy] \quad (4)$$

for  $x, x', y, y', z \in F$ ,  $[x, y] = xyx^{-1}y^{-1}$ ,  ${}^xy = xyx^{-1}$ . Another basic relation that goes under the name of the *Hall-Witt identity* can be expressed as

$$[[x, y], {}^yz][[y, z], {}^zx][[z, x], {}^xy] = 1. \quad (5)$$

Online discussions [22, 18, 5, 21] and a subsequent paper [6] contain interesting questions and answers about analogues and generalizations of the Hall-Witt identity. In this note we suggest complementary analogues of the identity based on a well-known topological interpretation of the Hall-Witt identity which can be found for instance in [4] and [22]. Furthermore, we illustrate a machine assisted approach to realizing examples of these analogues.

First we must make precise what we mean by a ‘commutator identity’ and what we mean by saying that one such identity is a ‘consequence’ of others. Throughout the paper we use the language of free crossed modules (see for instance [2]) and crossed complexes (see for instance [1]) and nonabelian exterior products (see for instance [3]). Recall that for a group  $G$  and normal subgroup  $N \trianglelefteq G$  the *nonabelian exterior product*  $N \wedge G$  is the group generated by symbols  $x \wedge y$  ( $x \in N$ ,  $y \in G$ ) subject to the relations

$$xx' \wedge y = {}^x(x' \wedge y)(x, y) \quad (6)$$

$$x \wedge yy' = (x \wedge y)^y (x \wedge y) \quad (7)$$

$$x \wedge x = 1 \quad (8)$$

for  $x, x' \in N$ ,  $y, y' \in G$  where by definition  $^z(x \wedge y) = ^zx \wedge ^zy$  for  $z \in G$ . There is an action of  $G$  on  $N \wedge G$  which, together with the group homomorphism  $\partial: N \wedge G \rightarrow G$ ,  $x \wedge y \mapsto [x, y]$ , satisfy the axioms of a crossed module.

**Definition 1.1.** *For a free group  $F$  with normal subgroup  $R \trianglelefteq F$  we say that any element*

$$\sigma \in \ker(\partial: R \wedge F \rightarrow F)$$

*is an  $R$ -identity between commutators. We say that  $\sigma$  is a nontrivial  $R$ -identity if it is a non-trivial element of the kernel. We say that  $\sigma$  is a consequence of a collection  $\Sigma$  of  $R$ -identities if it lies in the (abelian) group generated by  $\Sigma$ .*

As a first example, consider the derived subgroup  $R = \gamma_2(F) = [F, F]$  of a free group. For any  $x, y, z \in F$  the element

$$([x, y] \wedge ^yz) ([y, z] \wedge ^zx) ([z, x] \wedge ^xy) \quad (9)$$

is a  $\gamma_2 F$ -identity between commutators. In Section 5 we illustrate how Brown and Loday's isomorphism [3]

$$H_3(F/R, \mathbb{Z}) \cong \ker(R \wedge F \rightarrow F) \quad (10)$$

in conjunction with the homological algebra package HAP [9] for the GAP [11] computer algebra system can be used to establish that (9) is a nontrivial  $\gamma_2 F$ -identity for any elements three  $x, y, z \in F$  that generate a free subgroup of rank 3. Furthermore, for a free group  $F$  let  $\Sigma_F$  denote the set of identities (9) where  $x, y, z$  range over a generating set for  $F$ . Then all  $\gamma_2 F$ -identities between commutators are consequences of  $\Sigma_F$ . This example extends routinely to produce a range of other identities between commutators.

It follows immediately from (10) that there are no nontrivial  $F$ -identities between commutators. In particular, (9) is trivial when considered as an  $F$ -identity between commutators. Also, there are no nontrivial  $\gamma_2 F$ -identities when  $F$  is free of rank  $\leq 2$  since in that case  $H_3(F/\gamma_2 F, \mathbb{Z}) = 0$ .

As a second example, consider the subgroup  $R = \gamma_3 F = [[F, F], F]$  of a free group  $F$ . The element

$$([x^{-1}, y][x, y] \wedge [y, x][y^{-1}, x]y^{-1})([y, x][y^{-1}, x] \wedge x^{-1}) \quad (11)$$

is a  $\gamma_3 F$ -identity between commutators for any  $x, y \in F$ . Isomorphism (10) and machine calculations again imply that (11) is a nontrivial  $\gamma_3 F$ -identity when  $x, y$  generate a free subgroup of rank 2. Furthermore, if  $F$  is free on the two generators  $x, y$  then all  $\gamma_3 F$ -identities between commutators are consequences of (11).

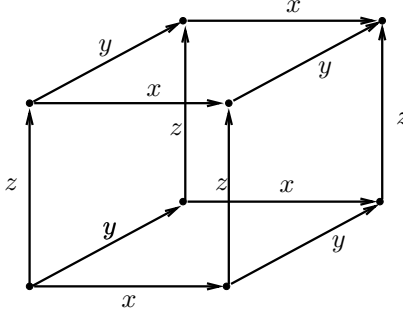


Figure 1: Homotopical syzygy for the Hall-Witt  $\gamma_2 F$ -identity

As a third example, consider the infinite cyclic group  $F = C_\infty$  generated by  $x$  and subgroup  $R = mC_\infty$  generated by  $x^m$  for some fixed positive integer  $m > 1$ . Then

$$x^m \wedge x \quad (12)$$

is a nontrivial  $mC_\infty$ -identity between commutators since it is a generator for  $H_3(C_m, \mathbb{Z}) \cong \ker(mC_\infty \wedge C_\infty \xrightarrow{0} C_\infty) \cong \mathbb{Z}_m$ .

In Section 3 we explain how  $R$ -identities between commutators can be interpreted as (a possible ‘sum’ of) 2-spheres endowed with a CW-decomposition in which 1-cells are suitably labelled by generators of  $F$  and in which 2-cells are suitably labelled by relators in  $R$ . We follow Loday [15] in referring to such a labelled 2-sphere as a *homotopical syzygy*. A precise definition is given in [15]. Equivalently, we can define a homotopical syzygy to be a van Kampen diagram  $D$  associated to a presentation  $\langle \underline{x} \mid \underline{r} \rangle$  (see for instance [16]) with the property that the boundary of the diagram  $D$  spells the trivial word in the free group  $F$  on the set  $\underline{x}$ . Here  $\underline{r} \subset F$  is a set that normally generates  $R \triangleleft F$ . A homotopical syzygy representing the Hall-Witt identity is shown in Figure 1. Homotopical syzygies for identities (11) and (12) are shown in Figure 2 (a) and (b). Commutator identities arising from homotopical syzygies (or ‘sums’ of syzygies) are the analogues of the Hall-Witt identity that we pursue in this note. We describe and illustrate a machine assisted approach to computing examples of homotopical syzygies, and we explain how to convert ‘sums’ of syzygies into  $R$ -identities between commutators.

The syzygy of Figure 1 can also be viewed as a labelling of the boundary of a 3-cube specifying three homeomorphisms between pairs of faces. By using these three homeomorphisms to identify faces we obtain a closed compact orientable 3-manifold, namely the 3-torus  $S^1 \times S^1 \times S^1$  with canonical CW-structure involving one vertex, three edges, three 2-faces and one 3-cell. Some other examples of face pair identifications on the 3-cube yielding closed compact orientable 3-manifold quotients were studied by Henri Poincaré in his seminal paper *Analysis Situs* [17]. Poincaré’s third example is illustrated in Figure 3 (a) and has quotient

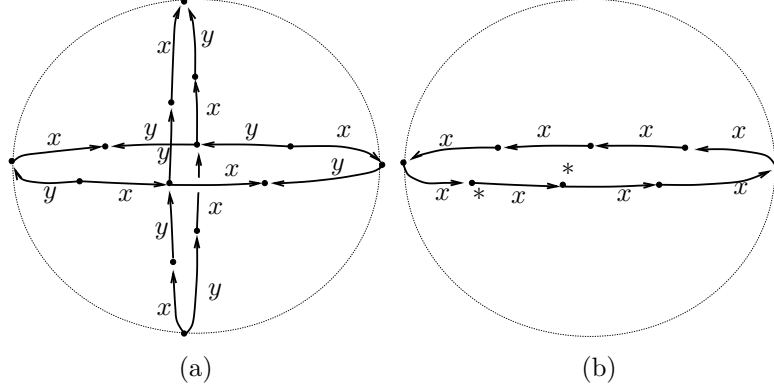


Figure 2: (a) Homotopical syzygy for the  $\gamma_3 F$ -identity (11). (b) Homotopical syzygy for the  $mC_\infty$ -identity (12) with  $m = 8$ , upper and lower hemisphere relator starting points indicated by  $*$ .

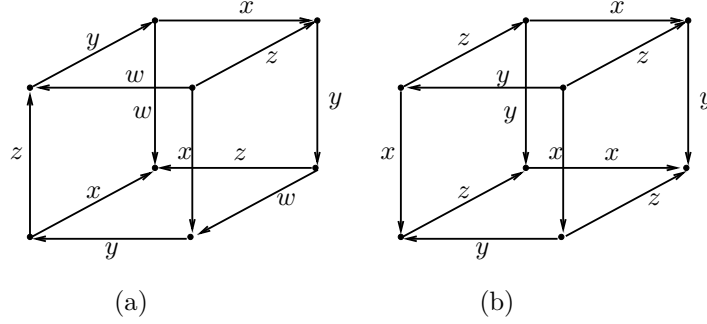


Figure 3: Homotopical syzygy for Poincaré's third example

manifold  $M$  with fundamental group  $\pi_1 M = Q_4$  equal to the quaternion group of order 8. Such manifolds will yield commutator identities. In the case of Figure 3 we take  $F$  to be the free group on  $\underline{x} = \{w, x, y, z\}$  and  $R \triangleleft F$  to be the normal subgroup normally generated by  $\underline{r} = \{xyzw^{-1}, wyxz^{-1}, zyxw^{-1}\}$ . Poincaré's third example in [17] then yields the following  $R$ -identity between commutators.

$$(xyzw^{-1} \wedge wy) (wyxz^{-1} \wedge zy) (zywx^{-1} \wedge xy) \quad (13)$$

The  $R$ -identity (13) is nontrivial because it represents a generator of  $H_3(\pi_1 M, \mathbb{Z}) \neq 0$ . Poincaré's fourth example is illustrated in Figure 3 (b) and yields the non-trivial  $R$ -identity

$$(y^z x \wedge {}^z y) ({}^z y x^{-1} \wedge {}^x y) ({}^x y y^{-1} \wedge yz) \quad (14)$$

where in this case  $F$  is free on  $x, y, z$  and  $R$  is normally generated by  $xyx^{-1}y^{-1}$ ,

$zyz^{-1}x^{-1}, yzxz^{-1}$ . (Poincaré's first example in [17] is the 3-torus and his second example is a quotient of the cube that is not a manifold.)

The paper is organized as follows. Section 2 contains some background homological algebra. Section 3 describes the general approach to constructing commutator identities. In Section 4 we illustrate the machine assisted approach to deriving commutator identities from compact closed orientable 3-manifolds. In Section 5 we illustrate the machine assisted approach to deriving commutator identities from the low dimensions of classifying spaces  $BG$  for groups  $G = F/R$ , with examples for  $G$  the free nilpotent group of classes 2 and 3 on three generators. These examples extend to other groups for which  $BG$  is implemented in [9]. In Section 6 we present some related calculations on the size of generating sets for the third integral homology of some free nilpotent groups. We also include some calculations on torsion in the fourth integral homology of such groups.

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## 2 Some homological algebra

Recall [1] that a *crossed chain complex*  $\rho_*$  consists of a sequence of group homomorphisms

$$\cdots \xrightarrow{d_5} \rho_4 \xrightarrow{d_4} \rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1$$

and group actions  $\rho_1 \times \rho_n \rightarrow \rho_n, (x, y) \mapsto {}^x y$  for  $n \geq 1$ . The group  $\rho_1$  is required to act on itself by conjugation  ${}^x y = xyx^{-1}$ . The composite  $d_n d_{n+1} = 1$  is required to be the trivial homomorphism for  $n \geq 2$ . The homomorphisms  $d_n$  are required to preserve the action of  $\rho_1$  for  $n \geq 2$ . The groups  $\rho_n$  are required to be abelian for  $n \geq 3$ . The homomorphism  $d_2: \rho_2 \rightarrow \rho_1$  is required to satisfy the axiom of a crossed module, namely that  $yy'y^{-1} = {}^{d_2 y} y'$  for all  $y, y' \in \rho_2$ . We define

$$H_n(\rho_*) = \ker(\rho_n \xrightarrow{d_n} \rho_{n-1}) / \text{image}(\rho_{n+1} \xrightarrow{d_{n+1}} \rho_n) \quad (15)$$

for  $n \geq 1$  where  $d_1: \rho_1 \rightarrow 1$  is taken to be the trivial homomorphism. Note that  $H_1(\rho_*)$  can be nonabelian.

The crossed chain complex  $\rho_*$  is said to be *free* if  $\rho_1$  is a free group,  $d_2: \rho_2 \rightarrow \rho_1$  is a free crossed module, and  $\rho_n$  is a free  $\rho_1$ -module for  $n \geq 3$ . Set  $G = \rho_1/d_2(\rho_2)$ . The crossed chain complex  $\rho_*$  is said to be a *free crossed resolution* of  $G$  if  $\rho_*$  is free and  $H_n(\rho_*) = 1$  for all  $n \geq 2$ .

Every group  $G$  admits a free crossed resolution. For instance, one can always construct a CW-complex  $BG$  that is a classifying space for  $G$  in the sense that  $\pi_n(BG) = 0$  for  $1 \neq n \geq 0$ ,  $\pi_1(BG) \cong G$ . The space  $BG$  can be constructed to have a single 0-cell which is taken as the base-point in the definition of homotopy groups. Let  $X = BG$ . The fundamental crossed complex  $\rho_*(X)$  is constructed by taking  $\rho_1(X) = \pi_1 X$ ,  $\rho_n = \pi_n(X^n, X^{n-1})$  for  $n \geq 1$  and by taking the canonical actions and boundary homomorphisms. Because  $X$  is a classifying space,

$\rho_*X$  is a free crossed resolution of  $G = \pi_1 X$ . Furthermore, because  $X$  is a classifying space, we have Hurewicz isomorphisms  $\pi_n(X^n, X^{n-1}) \cong H^n(\tilde{X}^n, \tilde{X}^{n-1})$  for  $n \geq 3$  where  $\tilde{X}$  is the universal cover. The HAP [9] package for the GAP [11] computer algebra system implements a range of algorithms for constructing the first  $n$  terms in a CW-classifying space  $X = BG$  for several classes of groups  $G$ . The space is stored either as a regular CW-complex or as the cellular chain complex  $C_*(\tilde{X})$  of the universal cover, and in both cases mathematical information on the crossed resolution  $\rho_*(X)$  can be accessed. See [8] for mathematical and implementation details.

Given any free crossed resolution  $\rho_*$  we set  $G = \rho_1/d_2(\rho_2)$ ,  $F = \rho_1$ ,  $R = d_2(\rho_2)$ . We define the subgroup  $[\rho_n, F] = \langle xy y^{-1} : x \in F, y \in \rho_n \rangle$  of  $\rho_n$  and note that this subgroup is invariant under the action of  $F$ . From this definition we obtain a subcomplex  $[\rho_*, F]$  of  $\rho_*$ . We can form the quotient crossed complex  $\rho_* / [\rho_*, F]$ . The integral homology of  $G$  can be defined in terms of this quotient.

**Definition 2.1.** *For a free crossed resolution  $\rho_*$  of  $G$  we have*

$$H_n(G, \mathbb{Z}) = H_n(\rho_* / [\rho_*, F])$$

for  $n \geq 1$ .

The short exact sequence

$$[\rho_*, F] \twoheadrightarrow \rho_* \twoheadrightarrow \rho_* / [\rho_*, F] \quad (16)$$

of crossed complexes gives rise to a long exact sequence in homology, from which we derive the following.

**Lemma 2.2.** *There is an isomorphism*

$$H_{n+1}(G, \mathbb{Z}) \cong H_n([\rho_*, F])$$

for  $n \geq 2$  and isomorphism

$$H_2(G, \mathbb{Z}) \cong R \cap [F, F] / [R, F] .$$

We can construct a commutative diagram of group homomorphisms

$$\begin{array}{ccccc} H_3(G, \mathbb{Z}) & \twoheadrightarrow & R \wedge F & \twoheadrightarrow & [R, F] \\ \downarrow \psi & & \downarrow \phi & & \downarrow = \\ H_3(G, \mathbb{Z}) & \twoheadrightarrow & [\rho_2, F] / [d_3 \rho_3, F] & \twoheadrightarrow & [R, F] \end{array} \quad (17)$$

in which the top row is exact by (10) and the bottom row is exact by the case  $n = 2$  of Lemma 2.2. The vertical homomorphism  $\phi$  is induced by a well-defined canonical crossed pairing  $R \times F \rightarrow [\rho_2, F] / [d_3 \rho_3, F], (r, f) \mapsto [\tilde{r}, f] [d_3 \rho_3, F]$  where  $\tilde{r} \in \rho_2$  is any lift of  $r$ . The homomorphism  $\phi$  is clearly surjective. It

follows that the homomorphism  $\psi$  is surjective. Thus, if  $H_3(G, \mathbb{Z})$  is finitely generated then it follows immediately that  $\psi$  has to be an isomorphism. The homology group  $H_n(G, \mathbb{Z})$  is finitely generated if, for instance,  $G$  admits a free crossed resolution that is finitely generated in each degree. There is no need in this note to consider the case of groups with non-finitely generated third homology.

### 3 A procedure for generating nontrivial identities between commutators

To construct a nontrivial identity between commutators we choose a group  $G$  that has nontrivial homology  $H_3(G, \mathbb{Z}) \neq 0$  in degree 3. The group  $G$  should be one for which we can construct the first four dimensions of a CW-classifying complex  $X = BG$ . The first three dimensions suffice if we don't require a proof of the nontriviality of the identity. We assume that there are only finitely many cells in dimensions  $\leq 4$ . The constructed CW-complexes should have universal covers with *regular* 2-skeleta  $\tilde{X}^2$ . This means that the attaching maps of 1-cells and 2-cells in  $\tilde{X}$  should be homeomorphisms. We can then easily read off a free presentation  $G \cong \langle \underline{x} \mid \underline{r} \rangle$ . We store  $X$  by storing the cellular chain complex  $C_*(\tilde{X})$  as a chain complex of finitely generated  $\mathbb{Z}G$ -modules.

There is an isomorphism  $\rho_n(X) = \pi_n(X^n, X^{n-1}) \cong H^n(\tilde{X}^n, \tilde{X}^{n-1}) = C_n(\tilde{X})$  for  $n \geq 3$  and a surjection  $\rho_2(X) = \pi_2(X^2, X^1) \twoheadrightarrow H_2(\tilde{X}^2, \tilde{X}^1) = C_2(\tilde{X})$ . Moreover, there is a natural isomorphism  $C_2(\tilde{X}) \cong \rho_2(X)^{ab}$  of  $\mathbb{Z}G$ -modules and of kernels

$$\ker(\rho_2 \xrightarrow{d_2} \rho_1) \cong \ker(C_2(\tilde{X}) \xrightarrow{d_2} C_1(X)). \quad (18)$$

Thus relevant information about the free crossed resolution  $\rho_*(X)$  can be derived from an explicit description of the free resolution  $C_*(\tilde{X})$ . See [2, 1] for an account of these isomorphisms.

We can use the chain complex  $C_*(X) = C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z}$  to find a set  $\mathcal{B}$  of vectors in the finitely generated free abelian group  $C_3(X)$  corresponding to a generating set for  $H_3(G, \mathbb{Z}) \cong \ker(C_3(X) \xrightarrow{d_3} C_2(X)) / \text{image}(C_4(X) \xrightarrow{d_4} C_3(X))$ .

The long exact homology sequence arising from the short exact sequence of chain complexes

$$[C_*(\tilde{X}), G] \hookrightarrow C_*(\tilde{X}) \twoheadrightarrow C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z} \quad (19)$$

yields an isomorphism

$$H_3(G, \mathbb{Z}) \cong H_2([C_*(\tilde{X}), G]). \quad (20)$$

Thus  $\mathcal{B}$  corresponds to a set of elements  $\mathcal{B}' \subset [C_2(\tilde{X}), G] \cong [\rho_2(X)^{ab}, G]$  that generate  $H_3(G, \mathbb{Z})$  via isomorphism (20). The subset  $\mathcal{B}'$  consists of  $\mathbb{Z}G$ -linear combinations of boundaries of the free generators of  $C_3(\tilde{X})$  and can be read off directly from the cellular chain complex  $C_*(\tilde{X})$  and the vectors in  $\mathcal{B}$ . The boundary of a free  $\mathbb{Z}G$ -generator of  $C_3(\tilde{X})$  corresponds to the boundary of a

3-cell in  $\tilde{X}$  and can be interpreted as a homotopical syzygy if the 3-skeleton  $\tilde{X}^3$  is regular. If  $\tilde{X}^3$  is not regular then the boundary of a 3-cell may be a quotient of a syzygy, but this seems not to cause problems in practice. The vertices and edges of a syzygy determine an embedding of a planar graph into the 2-sphere. Rather than using a formal algorithm to obtain this embedding we simply use Graphviz [10] software which, in practice, provides a sufficient visualization of the required embedding. When  $\tilde{X}^3$  is not regular the visualization may be that of a quotient of an embedded graph. Thus, a set of homotopical syzygies, representing the boundaries of the finite set of free  $\mathbb{Z}G$ -generators of  $C_3(\tilde{X})$ , can be readily visualized using a computer.

There exists a subset  $\mathcal{B}'' \subset [\rho_2(X), F]$  that maps bijectively to  $\mathcal{B}'$  under the surjection  $\rho_2(X) \twoheadrightarrow C_2(\tilde{X})$ . Each element in  $\mathcal{B}''$  is represented by a ‘sum’ of the generating homotopical syzygies where, formally, a homotopical syzygy is an element of  $\pi_2(X^2) = \ker(\rho_2(X) = \pi_2(X^2, X^1) \rightarrow \rho_1(X) = F)$  and the ‘sum’ is an  $F$ -combination of elements in the free crossed module  $\rho_2(X)$ .

Let  $\mathcal{B}''' \subset [\rho_2(X), F]/[d_3(\rho_3(X)), F]$  be the image of  $\mathcal{B}''$ . The pre-image of  $\mathcal{B}'''$  under the isomorphism  $\phi$  of diagram (17) will be a set of nontrivial  $R$ -identities between commutators that generate  $\ker(R \wedge F \rightarrow F)$ . In practice, these  $R$ -identities can be read directly from the homotopical syzygies and the set  $\mathcal{B}$ .

## 4 Commutator identities from 3-manifolds

The above procedure for producing commutator identities is particularly straightforward for certain groups  $G$  arising as the fundamental group of a compact connected closed orientable 3-manifold  $M$ . Orientability means  $H_3(M, \mathbb{Z}) = \mathbb{Z}$ . For prime  $M$  : (i) if  $G = \pi_1 M$  is infinite and not cyclic then  $M = BG$  is a classifying space; (ii) if  $G$  is infinite cyclic then  $M = S^1 \times S^2$ ; (iii) if  $G$  is finite then  $M = X^3$  is the 3-skeleton of a classifying space  $X = BG$  with  $H_3(G, \mathbb{Z})$  a quotient of  $H_3(M, \mathbb{Z})$ . Cases (i) and (iii) reduce the task of constructing a CW-classifying space to that of constructing a 3-manifold  $M$  with CW-structure.

Let the boundary  $S^2$  of the 3-ball  $B^3$  be given some regular CW-structure for which the 2-cells can be paired into combinatorially isomorphic pairs (i.e. having the same number of 1-cells in their closures). For each pair we can isomorphically identify the two faces in some way so that the resulting quotient space  $B^3/\sim$  is naturally a (not necessarily regular) CW-complex. By keeping the face pair identifications unchanged, but barycentrically subdividing the original CW-decomposition of  $B^3$ , we can ensure that the quotient  $B^3/\sim$  is a regular CW-complex. We can then use the simplification procedure of [8, Algorithm 1.4.2] to obtain a potentially smaller regular CW-structure on  $B^3/\sim$ . Every compact closed 3-manifold can in principle be realized in this way, as can many non-manifolds. The regular CW-complex  $B/\sim$  is a manifold precisely when, in its barycentric subdivision, the link of each vertex is a 2-sphere. This condition is straightforward to check. Thus, in principle, given any regular



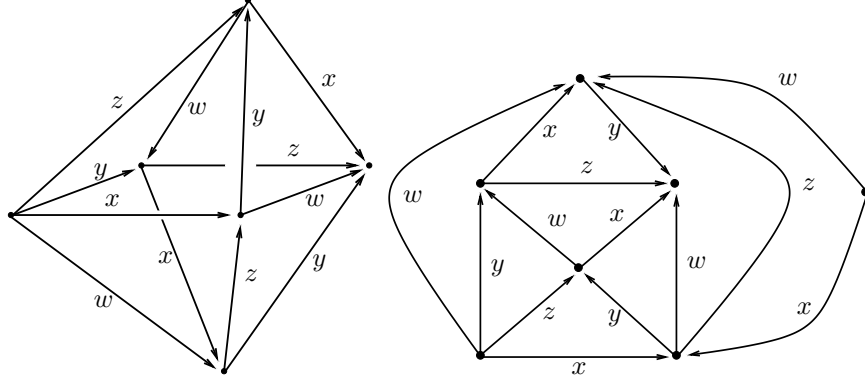


Figure 4: Syzygy for a finite 3-manifold with corresponding van Kampen diagram

CW-decomposition of  $S^2$  we can decide which face pairings are manifolds.

This construction of manifolds can be found in Poincaré's *Analysis situs* [17] where it is used to construct several manifolds from face pairings on the 3-cube. Following [23] we call such quotients *cube manifolds*. A list of all orientable cube manifolds is given, with some repetitions in [20]. There are in fact 18 such manifolds up to homeomorphism and these are listed in the online manual for HAP [9]. Closed compact orientable 3-manifolds can also be obtained by identifying pairwise faces of other polytopes, and functions are implemented in HAP for investigating such identifications in the case of the octahedron, dodecahedron,  $n$ -prism and  $n$ -bipyramid. For instance, as explained in the HAP manual, one can check on the computer that the face identifications for the octahedron shown in Figure 4 yield an orientable 3-manifold  $M$  with finite fundamental group  $\pi_1 M = SL_3(\mathbb{Z}_3)$ . Let  $F$  be the free group on  $\underline{x} = \{w, x, y, z\}$  and define the relators  $A = xw^{-1}y$ ,  $B = wx^{-1}z$ ,  $C = z^{-1}yw^{-1}$ ,  $D = y^{-1}x^{-1}z$ . Let  $R$  be the normal subgroup of  $F$  generated by these relators. A van Kampen diagram can always be represented as a product of conjugates of relators. The van Kampen diagram of Figure 4 is represented by the product

$$A (y^{-1}x^{-1}yzx^{-1}A^{-1}) D (z^{-1}yzw^{-1}yzD^{-1}) C ({}^wzC^{-1}) B ({}^xw^{-1}B^{-1}) \quad (21)$$

which can be regarded as an element of the free crossed module  $\rho_2(M)$ . The corresponding  $R$ -identity between commutators is the element

$$(A \wedge y^{-1}x^{-1}yzx^{-1})(D \wedge z^{-1}yzw^{-1}yz)(C \wedge {}^wz)(B \wedge {}^xw^{-1}) . \quad (22)$$

Note that the orientability of  $M$  ensures that the relators occur in oppositely oriented pairs in the defining syzygy for  $M$ . Hence the element (21) in  $\rho_2(M)$

can be expressed as a commutator and lifted to  $R \wedge F$ . Since the unique 3-cell of  $M$  generates  $H_3(M, \mathbb{Z})$  the element (22) is a generator for  $H_3(\pi_1 M, \mathbb{Z})$  where  $\pi_1 M = F/R$ .

## 5 Commutator identities from other classifying spaces

The HAP package implements procedures for computing free  $\mathbb{Z}G$ -resolutions  $R_* = C_*(\tilde{X})$  for various classes of groups where  $X$  is a CW-classifying space for  $G$ . It includes procedures for finite groups, finitely generated nilpotent groups, crystallographic groups, spherical Artin groups, Coxeter groups and certain arithmetic groups. In this section we illustrate how to derive some commutator identities from free nilpotent groups of class  $\leq 2$ . The illustration can be adapted to other classes of groups. Throughout we use the notation  $\gamma_1 G = G$  and  $\gamma_n G = [\gamma_{n-1} G, G]$  for any group  $G$ .

### 5.1 Free abelian

Let us first consider the free group  $F = F(x, y, z)$  of rank 3 and normal subgroup  $R = [F, F]$ . The quotient  $G = F/R$  is free abelian of rank 3. A free  $\mathbb{Z}G$ -resolution  $R_* = C(\tilde{X})$  is constructed in the GAP session shown in Table 1. The  $\mathbb{Z}G$ -ranks of the free  $\mathbb{Z}G$ -modules  $R_n$  are seen to be 1, 3, 3, 1 in dimensions  $n = 0, 1, 2, 3$  respectively. This is the minimum possible number of free generators because, as the commands in the session confirm, we have the well-known isomorphisms  $H_n(G, \mathbb{Z}) \cong \mathbb{Z}$  for  $n = 1, 3$  and  $H_n(G, \mathbb{Z}) \cong \mathbb{Z}^3$  for  $n = 1, 2$ . The session computes the presentation  $G \cong \langle x, y, z \mid [y, x] = 1, [z, x] = 1, [z, y] = 1 \rangle$ . The final command computes a visualization of the homotopical syzygy displayed in Figure 1 corresponding to the boundary of the free generator of  $C_3(\tilde{X}) \cong \mathbb{Z}G$ .

Let  $\underline{r} = \{A, B, C\}$  be a set of three abstract symbols and let  $\delta: \underline{r} \rightarrow F$  be the function defined by  $\delta(A) = [x, y]$ ,  $\delta(B) = [z, x]$ ,  $\delta(C) = [y, z]$ . Let  $\partial: C(\underline{r}) \rightarrow F$  be the free crossed module on  $\delta$ . This free crossed module coincides with the low-dimensional part of  $\rho_*(X)$ . The syzygy of Figure (1) can be expressed as the element

$$A (yzy^{-1} A^{-1}) C (zzz^{-1} C^{-1}) B (xyx^{-1} B^{-1}) \quad (23)$$

in  $C(\underline{r}) \cong \rho_2(X)$ . The preimage under  $\psi$  of this element is the identity (9). Hence (9) is a nontrivial  $\gamma_2 F$ -identity between commutators representing a generator of  $H_3(G, \mathbb{Z}) \cong \mathbb{Z}$ .

**Proposition 5.1.** *Let  $F$  be a finitely generated free group. Then under the isomorphism (10) the homology group  $H_3(F/\gamma_2 F, \mathbb{Z})$  is generated by the elements (9) as  $x, y, z$  range over some fixed generating set of  $F$ .*

```

gap> n:=3;;c:=1;;
gap> G:=Image(NqEpimorphismNilpotentQuotient(FreeGroup(n),c));;
gap> R:=ResolutionNilpotentGroup(G,4);;
gap> List([0..4],R!.dimension);
[ 1, 3, 3, 1, 0 ]
gap> Homology(TensorWithIntegers(R),0);
[ 0 ]
gap> Homology(TensorWithIntegers(R),1);
[ 0, 0, 0 ]
gap> Homology(TensorWithIntegers(R),2);
[ 0, 0, 0 ]
gap> Homology(TensorWithIntegers(R),3);
[ 0 ]
gap> IsZero(BoundaryMatrix(TensorWithIntegers(R),3));
true
gap> P:=PresentationOfResolution(R);;
gap> P.freeGroup;
<free group on the generators [ x, y, z ]>
gap> P.relators;
[ y^-1*x^-1*y*x, z^-1*x^-1*z*x, z^-1*y^-1*z*y ]
gap> IdentityAmongRelatorsDisplay(R,1); #See Figure 1 for display

```

Table 1: Computing the homotopical syzygy of Figure 1

*Proof.* The cohomology ring  $H^*(F/\gamma_2 F, \mathbb{Z})$  is generated by elements in degree 1. In particular  $H^3(F/\gamma_2 F, \mathbb{Z})$  is generated by cup products  $x \cup y \cup z$  with  $x, y, z$  free generators of  $H^1(F/\gamma_2 F, \mathbb{Z}) \cong F/\gamma_2 F$ . Under the canonical isomorphism  $H^3(F/\gamma_2 F, \mathbb{Z}) \cong H_3(F/\gamma_2 F, \mathbb{Z})$  the product  $x \cup y \cup z$  corresponds to the element (9). The proposition follows.  $\square$

## 5.2 Free nilpotent of class 2

Running the commands of Table 2 with  $n = 2$  and  $c = 2$  produces a free resolution for the free nilpotent group of class 2 on two generators. This resolution does not have the minimal possible number of generators in each degree. However, by including the extra command `R:=ContractedComplex(R)` the resolution is converted to one with minimal possible number of free generators. This minor variant of Table 2 produces a visualization of the homotopical syzygy in Figure 2(a) from which the nontrivial identity (11) follows.

Let us now consider the free group  $F$  on  $n$  generators  $x_1, \dots, x_n$ . The free nilpotent group  $G = F/\gamma_3 F$  of class 2 admits a presentation on the generating set  $\underline{x} = \{x_i, c_{ij} : 1 \leq i < j \leq n\}$  which we order by specifying  $x_i < x_j$  for  $i < j$ ,  $c_{ij} < c_{k\ell}$  for  $i < k$  or  $i = k, j < \ell$ ,  $x_i < c_{k\ell}$ . The set  $\underline{x}$  has size  $|\underline{x}| = n + (n(n-1)/2)$ . The presentation relators are  $\underline{r} = \{x_i x_j c_{ij} x_i^{-1} x_j^{-1}, uvu^{-1}v^{-1} : i < j, u < v \in \underline{x}\}$ . Repeating the commands of Table 2 for  $c = 3$  and low values of  $n$  (for example  $n \leq 7$ ) yields a free  $\mathbb{Z}G$ -resolution with  $R_i$  a free  $\mathbb{Z}G$ -module of rank  $\binom{|\underline{x}|}{i}$ . The resolution is based on the perturbation technique of wall [24]. While the rank of the modules  $R_i$  is determined by a simple formula the implementation makes some non-canonical choices in constructing the boundary homomorphisms  $d_i: R_i \rightarrow R_{i-1}$ . The choices are made recursively, the choice for  $d_i$  made to ensure that  $d_{i-1}d_i = 0$ .

For  $n = 3$  one can check that the presentation of  $G$  associated to the  $R_*$  is the one just described. To ease notation we set  $x := x_1, y := x_2, z := x_3, a := c_{1,2}, b := c_{1,3}, c := c_{2,3}$ . There are 20 free generators of  $R_3$  which give rise to the syzygies four of which are pictured in Figure 5. Using the notation of Figure 5 the 20 syzygies are:  $s_1(x, y, z)$ ,  $s_2(x, y, a)$ ,  $s_2(x, z, b)$ ,  $s_2(y, z, c)$ ,  $s_3(x, y, b)$ ,  $s_3(x, y, c)$ ,  $s_3(x, z, a)$ ,  $s_3(x, z, c)$ ,  $s_3(y, z, a)$ ,  $s_3(y, z, b)$ ,  $s_4(t, u, v)$  for  $t \in \underline{x}$ ,  $u, v \in \{a, b, c\}$  where  $t < u < v$ .

Now  $H_3(R_* \otimes_{\mathbb{Z}G} \mathbb{Z}) \cong \mathbb{Z}^{12}$  and  $Z_3 = \ker(R_3 \otimes_{\mathbb{Z}G} \mathbb{Z} \rightarrow R_2 \otimes_{\mathbb{Z}G} \mathbb{Z}) \cong \mathbb{Z}^{16}$ . A basis  $\mathcal{B}$  for the kernel  $Z_3$  can be represented by a set  $\mathcal{B}''$  of 16 homotopical syzygies. A computer calculation of  $\mathcal{B}$  shows that we can take  $\mathcal{B}''$  to consist of the ten syzygies of the form  $s_4$  plus the three syzygies of the form  $s_2$  plus the three combined syzygies  $s_3(x, y, b)^{f_1} s_3(x, z, a)$ ,  $s_3(x, z, c)^{f_2} s_3(y, z, b)$ ,  $s_3(y, z, a)^{f_3} s_3(x, z, c)$  involving conjugating elements  $f_i \in F$ . A computer calculation show that by removing a certain four syzygies of the form  $s_4$  from  $\mathcal{B}''$  we obtain a set of 12 syzygies corresponding to 12 basis elements of  $H_3(G, \mathbb{Z})$ . To be precise, the four syzygies that we remove are:  $s_4(x, a, c)$ ,  $s_4(x, b, c)$ ,  $s_4(y, b, c)$ ,  $s_4(a, b, c)$ . The twelve syzygies represent nontrivial  $\gamma_3 F$ -identities between commutators. Six of these identities are of the form (24), while three are of the form

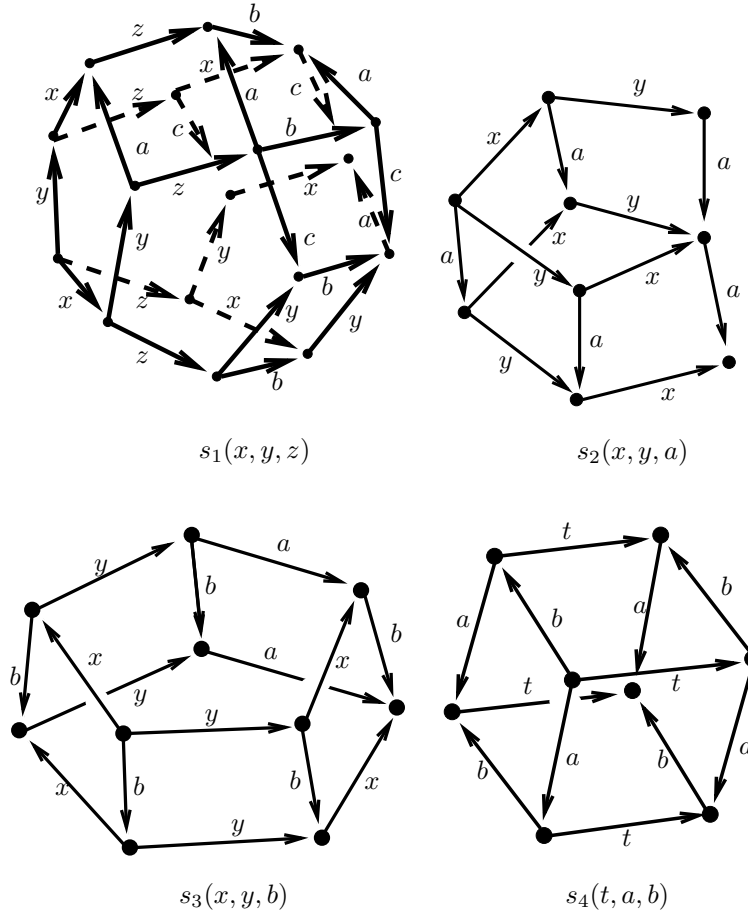


Figure 5: Syzygies for the free nilpotent group of class 2 on free generators  $x, y, z$  with  $a = [y^{-1}, x^{-1}]$ ,  $b = [z^{-1}, x^{-1}]$ ,  $c = [z^{-1}, y^{-1}]$  and  $t$  equal to any of these six generators.

(25) and three are of the form (26).

$$([a, b] \wedge {}^b t) ([b, t] \wedge {}^t a) ([t, a] \wedge {}^a b) \quad (24)$$

$$([y, a] \wedge x)^{-1} {}^{[y, a]}([x, a] \wedge {}^a y) (yxa^{-1}y^{-1}x^{-1} \wedge {}^{xy}a) \quad (25)$$

$$\begin{aligned} & x^{-1}y^{-1}xyb^{-1}y^{-1}x^{-1} \{ (yxa^{-1}y^{-1}x^{-1} \wedge {}^{xy}b)^{-1} ([y, b] \wedge {}^b x) ([b, x] \wedge {}^{xy}byb^{-1}) \} \\ & x^{-1}z^{-1}xza^{-1}z^{-1}x^{-1} \{ (zxb^{-1}z^{-1}x^{-1} \wedge {}^{xz}a)^{-1} ([z, a] \wedge {}^a x) ([a, x] \wedge {}^{xz}aza^{-1}) \} \end{aligned} \quad (26)$$

Let  $F_3$  denote the free group on three generators  $x, y, z$  and set  $a := [y^{-1}, x^{-1}]$ ,  $b := [z^{-1}, x^{-1}]$ ,  $t \in \{a, b\}$ . Then identities (24), (25), (26) can be expressed as the  $\gamma_3 F_3$ -identities (27) – (29). By permuting  $x, y, z$  we obtain a set of  $\gamma_3 F_3$ -identities between commutators that generate  $H_3(F_3/\gamma_3 F_3, \mathbb{Z}) \cong \mathbb{Z}^{12}$ . A non-redundant generating set can be obtained by appropriately choosing six identities of the form (27), and three each of the forms (28) and (29).

$$\begin{aligned} & ([y^{-1}, x^{-1}], [z^{-1}, x^{-1}]) \wedge {}^{[z^{-1}, x^{-1}]}t ([z^{-1}, x^{-1}], t) \wedge {}^t[y^{-1}, x^{-1}] \\ & ([t, [y^{-1}, x^{-1}]] \wedge {}^{[y^{-1}, x^{-1}]}[z^{-1}, x^{-1}]) \end{aligned} \quad (27)$$

$$([x^{-1}, y][x^{-1}, y^{-1}] \wedge x)^{-1} {}^{[x^{-1}, y][x^{-1}, y^{-1}]}([x, [y^{-1}, x^{-1}]] \wedge {}^{[y^{-1}, x^{-1}]}y) \quad (28)$$

$$([y, b] \wedge {}^b x) ([b, x] \wedge {}^{xy}byb^{-1}) {}^{xy[b, a]z^{-1}x^{-1}}\{([z, a] \wedge {}^a x) ([a, x] \wedge {}^{xz}aza^{-1})\} \quad (29)$$

Computer calculations show that only syzygies of the form  $s_1, s_2, s_3, s_4$  arise as the boundaries of degree 3 generators in the resolution produced by HAP for  $F_n/\gamma_3 F_n$  for  $n = 4, 5$ . It should be straightforward to extend this observation to a result for all  $n$  using a theoretical analysis of the low-dimensional terms in the resolution for free nilpotent groups furnished by Wall's technique [24].

We note that in [7] the language of *multiplicative lie rings* is used to provide an analogous result about  $\gamma_3 F_n$ -identities.

## 6 Some calculations on the homology of free nilpotent groups

The homology group  $H_3(F/\gamma_{c+1}F, \mathbb{Z})$  of a free nilpotent group of class  $c$  is known to be free abelian of finite rank [12]. The generators of this homology group correspond to a set of commutator identities from which all  $\gamma_{c+1}F$ -identities between commutators can be derived. The following data on the number of such generators was computed using [9].

$c \backslash n$	1	2	3	4	5	6	7	8	9	10
1	0	0	1	4	10	20	35	56	84	120
2	0	1	12	56	176	441	952	1848	3312	5577
3	0	3	70	552						
4	0	13	436							
5	0	35	2616							
6	0	112								
7	0	345								

Table 2: Rank of  $H_3(F/\gamma_{c+1}F, \mathbb{Z})$  for the free group  $F$  on  $n$  generators

**Proposition 6.1.** *Let  $F/\gamma_{c+1}F$  be the free nilpotent group of class  $c$  on  $n$  generators. The rank of the free abelian group  $H_3(F/\gamma_{c+1}F, \mathbb{Z})$  is given, for some low values of  $c$  and  $n$ , in Table 2.*

The homology computations in Table 2 were not obtained from a free resolution of a free nilpotent group  $G$  of class  $c$  on  $n$  generators. Instead they were obtained directly from the Chevalley-Eilenberg complex of the free Lie ring  $L$  of class  $c$  on  $n$  generators using the isomorphism

$$H_k(G, \mathbb{Z}) \cong H_k(L, \mathbb{Z}), \quad (30)$$

proved in [12] for  $k \leq 3$ . This isomorphism is proved in [13] for  $k \geq 0$  when  $G$  is free nilpotent of class  $c = 2$ . A free resolution for an arbitrary finitely generated nilpotent group is implemented in [9] using a recursive formula involving central extensions of groups. By contrast, the Chevalley-Eilenberg complex is described by an explicit formula and is more efficient when applicable.

Our methods for computing homology of groups extend, in principle, to higher dimensions. The method for finitely generated nilpotent groups was one of the first to be implemented in [9] and, as an aside, we mention that it can be used to establish results on torsion such as the following.

**Proposition 6.2.** *Let  $F_n$  denote the free group on  $n$  generators. We have:*

$$H_4(F_2/\gamma_6(F_2), \mathbb{Z}) \cong \mathbb{Z}_7 \oplus \mathbb{Z}^{85}, \quad (31)$$

$$H_4(F_3/\gamma_4(F_3), \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{171}, \quad (32)$$

$$H_4(F_4/\gamma_3(F_4), \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^{84}. \quad (33)$$

Isomorphism (6.2) was originally computed in [14]. An analogue of isomorphism (31) was computed in [19] for the free nilpotent Lie ring of class 5 on two generators. Note that isomorphism (30) for degree  $k = 4$  and  $c = 5$  is not covered by the results in [12, 13].

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