A short HAP tutorial

(A more comprehensive tutorial is available here
and
A related book is available here
and
The HAP home page is here)

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Contents

Chapter 1

Simplicial complexes & CW complexes

1.1 The Klein bottle as a simplicial complex

The following example constructs the Klein bottle as a simplicial complex K on 9 vertices, and then constructs the cellular chain complex $C_* = C_*(K)$ from which the integral homology groups $H_1(K,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}$, $H_2(K,\mathbb{Z}) = 0$ are computed. The chain complex $D_* = C_* \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is also constructed and used to compute the mod-2 homology vector spaces $H_1(K,\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_2(K,\mathbb{Z}) = \mathbb{Z}_2$. Finally, a presentation $\pi_1(K) = \langle x, y : yxy^{-1}x \rangle$ is computed for the fundamental group of K.

Example

```
gap> 2simplices:=
> [[1,2,5], [2,5,8], [2,3,8], [3,8,9], [1,3,9], [1,4,9],
>  [4,5,8], [4,6,8], [6,8,9], [6,7,9], [4,7,9], [4,5,7],
>  [1,4,6], [1,2,6], [2,6,7], [2,3,7], [3,5,7], [1,3,5]];;
gap> K:=SimplicialComplex(2simplices);
Simplicial complex of dimension 2.

gap> C:=ChainComplex(K);
Chain complex of length 2 in characteristic 0 .

gap> Homology(C,1);
[ 2, 0 ]
gap> Homology(C,2);
[  ]

gap> D:=TensorWithIntegersModP(C,2);
Chain complex of length 2 in characteristic 2 .

gap> Homology(D,1);
2
gap> Homology(D,2);
1

gap> G:=FundamentalGroup(K);
<fp group of size infinity on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f2*f1*f2^-1*f1 ]
```

1.2 The Quillen complex

Given a group G one can consider the partially ordered set $\mathscr{A}_p(G)$ of all non-trivial elementary abelian p-subgroups of G, the partial order being set inclusion. The order complex $\Delta\mathscr{A}_p(G)$ is a simplicial complex which is called the *Quillen complex* .

The following example constructs the Quillen complex $\Delta\mathscr{A}_2(S_7)$ for the symmetric group of degree 7 and $p = 2$. This simplicial complex involves 11291 simplices, of which 4410 are 2-simplices..

Example

```
gap> K:=QuillenComplex(SymmetricGroup(7),2);
Simplicial complex of dimension 2.

gap> Size(K);
11291

gap> K!.nrSimplices(2);
4410
```

1.3 The Quillen complex as a reduced CW-complex

Any simplicial complex K can be regarded as a regular CW complex. Different datatypes are used in HAP for these two notions. The following continuation of the above Quillen complex example constructs a regular CW complex Y isomorphic to (i.e. with the same face lattice as) $K = \Delta\mathscr{A}_2(S_7)$. An advantage to working in the category of CW complexes is that it may be possible to find a CW complex X homotopy equivalent to Y but with fewer cells than Y. The cellular chain complex $C_*(X)$ of such a CW complex X is computed by the following commands. From the number of free generators of $C_*(X)$, which correspond to the cells of X, we see that there is a single 0-cell and 160 2-cells. Thus the Quillen complex $$\Delta{\cal A}_2(S_7) \simeq \bigvee_{1\le i\le 160} S^2$$ has the homotopy type of a wedge of 160 2-spheres. This homotopy equivalence is given in [Kso00, (15.1)] where it was obtained by purely theoretical methods.

Example

```
gap> Y:=RegularCWComplex(K);
Regular CW-complex of dimension 2

gap> C:=ChainComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> C!.dimension(0);
1
gap> C!.dimension(1);
0
gap> C!.dimension(2);
160
```

Note that for regular CW complexes Y the function `ChainComplex(Y)` returns the cellular chain complex $C_*(X)$ of a (typically non-regular) CW complex X homotopy equivalent to Y. The cellular chain complex $C_*(Y)$ of Y itself can be obtained as follows.

Example
```
gap> CC:=ChainComplexOfRegularCWComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> CC!.dimension(0);
1316
gap> CC!.dimension(1);
5565
gap> CC!.dimension(2);
4410
```

1.4 Constructing a regular CW-complex from its face lattice

The following example begins by creating a 2-dimensional annulus A as a regular CW-complex, and testing that it has the correct integral homology $H_0(A,\mathbb{Z}) = \mathbb{Z}$, $H_1(A,\mathbb{Z}) = \mathbb{Z}$, $H_2(A,\mathbb{Z}) = 0$.

Example
```
gap> FL:=[];; #The face lattice
gap> FL[1]:=[[1,0],[1,0],[1,0],[1,0]];;
gap> FL[2]:=[[2,1,2],[2,3,4],[2,1,4],[2,2,3],[2,1,4],[2,2,3]];;
gap> FL[3]:=[[4,1,2,3,4],[4,1,2,5,6]];;
gap> FL[4]:=[];;
gap> A:=RegularCWComplex(FL);
Regular CW-complex of dimension 2

gap> Homology(A,0);
[ 0 ]
gap> Homology(A,1);
[ 0 ]
gap> Homology(A,2);
[  ]
```

Next we construct the direct product $Y = A\times A\times A\times A\times A$ of five copies of the annulus. This is a 10-dimensional CW complex involving 248832 cells. It will be homotopy equivalent $Y \simeq X$ to a CW complex X involving fewer cells. The CW complex X may be non-regular. We compute the cochain complex $D_* = \mathrm{Hom}_{\mathbb{Z}}(C_*(X),\mathbb{Z})$ from which the cohomology groups
$H^0(Y,\mathbb{Z}) = \mathbb{Z}$,
$H^1(Y,\mathbb{Z}) = \mathbb{Z}^5$,
$H^2(Y,\mathbb{Z}) = \mathbb{Z}^{10}$,
$H^3(Y,\mathbb{Z}) = \mathbb{Z}^{10}$,
$H^4(Y,\mathbb{Z}) = \mathbb{Z}^5$,
$H^5(Y,\mathbb{Z}) = \mathbb{Z}$,
$H^6(Y,\mathbb{Z}) = 0$
are obtained.

Example
```
gap> Y:=DirectProduct(A,A,A,A,A);
Regular CW-complex of dimension 10
```

```
gap> Size(Y);
248832
gap> C:=ChainComplex(Y);
Chain complex of length 10 in characteristic 0 .

gap> D:=HomToIntegers(C);
Cochain complex of length 10 in characteristic 0 .

gap> Cohomology(D,0);
[ 0 ]
gap> Cohomology(D,1);
[ 0, 0, 0, 0, 0 ]
gap> Cohomology(D,2);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(D,3);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(D,4);
[ 0, 0, 0, 0, 0 ]
gap> Cohomology(D,5);
[ 0 ]
gap> Cohomology(D,6);
[  ]
```

1.5 Cup products

Continuing with the previous example, we consider the first and fifth generators $g_1^1, g_5^1 \in H^1(W,\mathbb{Z}) = \mathbb{Z}^5$ and establish that their cup product $g_1^1 \cup g_5^1 = -g_7^2 \in H^2(W,\mathbb{Z}) = \mathbb{Z}^{10}$ is equal to minus the seventh generator of $H^2(W,\mathbb{Z})$. We also verify that $g_5^1 \cup g_1^1 = -g_1^1 \cup g_5^1$.

Example

```
gap> cup11:=CupProduct(FundamentalGroup(Y));
function( a, b ) ... end

gap> cup11([1,0,0,0,0],[0,0,0,0,1]);
[ 0, 0, 0, 0, 0, 0, -1, 0, 0, 0 ]

gap> cup11([0,0,0,0,1],[1,0,0,0,0]);
[ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ]
```

This computation of low-dimensional cup products is achieved using group-theoretic methods to approximate the diagonal map $\Delta\colon Y \to Y \times Y$ in dimensions ≤ 2. In order to construct cup products in higher degrees HAP requires a cellular inclusion $\overline{Y} \hookrightarrow Y \times Y$ with projection $p\colon \overline{Y} \twoheadrightarrow Y$ that induces isomorphisms on integral homology. The function `DiagonalApproximation(Y)` constructs a candidate inclusion, but the projection $p\colon \overline{Y} \twoheadrightarrow Y$ needs to be tested for homology equivalence. If the candidate inclusion passes this test then the function `CupProduct(Y)`, involving the candidate space, can be used for cup products.

The following example calculates $g_3^3 \cup g_3^1 = g_1^4$ where $W = S \times S \times S \times S$ is the direct product of four circles, and where g_k^n denotes the k-th generator of $H^n(W,\mathbb{Z})$.

Example

```
gap> S:=SimplicialComplex([[1,2],[2,3],[1,3]]);;
gap> S:=RegularCWComplex(S);;
gap> W:=DirectProduct(S,S,S,S);;
gap> cup:=CupProduct(W);
function( p, q, vv, ww ) ... end

gap> cup(3,1,[0,0,1,0],[0,0,1,0]);
[ 1 ]

#Now test that the diagonal construction is valid.
gap> D:=DiagonalApproximation(W);;
gap> p:=D!.projection;
Map of regular CW-complexes

gap> P:=ChainMap(p);
Chain Map between complexes of length 4 .

gap> IsIsomorphismOfAbelianFpGroups(Homology(P,0));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,1));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,2));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,3));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,4));
true
```

1.6 CW maps and induced homomorphisms

A *strictly cellular* map $f{:}X \to Y$ of regular CW-complexes is a cellular map for which the image of any cell is a cell (of possibly lower dimension). Inclusions of CW-subcomplexes, and projections from a direct product to a factor, are examples of such maps. Strictly cellular maps can be represented in HAP, and their induced homomorphisms on (co)homology and on fundamental groups can be computed.

The following example begins by visualizing the trefoil knot $\kappa \in \mathbb{R}^3$. It then constructs a regular CW structure on the complement $Y = D^3 \setminus \mathrm{Nbhd}(\kappa)$ of a small tubular open neighbourhood of the knot lying inside a large closed ball D^3. The boundary of this tubular neighbourhood is a 2-dimensional CW-complex B homeomorphic to a torus $\mathbb{S}^1 \times \mathbb{S}^1$ with fundamental group $\pi_1(B) =< a,b : aba^{-1}b^{-1} = 1 >$. The inclusion map $f{:}B \hookrightarrow Y$ is constructed. Then a presentation $\pi_1(Y) =< x,y \,|\, xy^{-1}x^{-1}yx^{-1}y^{-1} >$ and the induced homomorphism \$\$\pi_1(B)\rightarrow \pi_1(Y), a\mapsto y^{-1}xy^2xy^{-1}, b\mapsto y \$\$ are computed. This induced homomorphism is an example of a *peripheral system* and is known to contain sufficient information to characterize the knot up to ambient isotopy.

Finally, it is verified that the induced homology homomorphism $H_2(B,\mathbb{Z}) \to H_2(Y,\mathbb{Z})$ is an isomorphism.

Example
```
gap> K:=PureCubicalKnot(3,1);;
gap> ViewPureCubicalKnot(K);;
```

Example
```
gap> K:=PureCubicalKnot(3,1);;
gap> f:=KnotComplementWithBoundary(ArcPresentation(K));
Map of regular CW-complexes

gap> G:=FundamentalGroup(Target(f));
<fp group of size infinity on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f1*f2^-1*f1^-1*f2*f1^-1*f2^-1 ]

gap> F:=FundamentalGroup(f);
[ f1, f2 ] -> [ f2^-1*f1*f2^2*f1*f2^-1, f1 ]


gap> phi:=ChainMap(f);
Chain Map between complexes of length 2 .

gap> H:=Homology(phi,2);
[ g1 ] -> [ g1 ]
```

Chapter 2

Cubical complexes & permutahedral complexes

2.1 Cubical complexes

A *finite simplicial complex* can be defined to be a CW-subcomplex of the canonical regular CW-structure on a simplex Δ^n of some dimension n. Analogously, a *finite cubical complex* is a CW-subcomplex of the regular CW-structure on a cube $[0,1]^n$ of some dimension n. Equivalently, but more conveniently, we can replace the unit interval $[0,1]$ by an interval $[0,k]$ with CW-structure involving $2k+1$ cells, namely one 0-cell for each integer $0 \le j \le k$ and one 1-cell for each open interval $(j, j+1)$ for $0 \le j \le k-1$. A *finite cuical complex* M is a CW-subcompex $M \subset [0,k_1] \times [0,k_2] \times \cdots [0,k_n]$ of a direct product of intervals, the direct product having the usual direct product CW-structure. The equivalence of these two definitions follows from the Gray code embedding of a mesh into a hypercube. We say that the cubical complex has *ambient dimension n*. A cubical complex M of ambient dimension n is said to be *pure* if each cell lies in the boundary of an n-cell. In other words, M is pure if it is a union of unit n-cubes in $\mathbb{R}^n$, each unit cube having vertices with integer coordinates.

HAP has a datatype for finite cubical complexes, and a slightly different datatype for pure cubical complexes.

The following example constructs the granny knot (the sum of a trefoil knot with its reflection) as a 3-dimensional pure cubical complex, and then displays it.

Example
```
gap> K:=PureCubicalKnot(3,1);
prime knot 1 with 3 crossings

gap> L:=ReflectedCubicalKnot(K);
Reflected( prime knot 1 with 3 crossings )

gap> M:=KnotSum(K,L);
prime knot 1 with 3 crossings + Reflected( prime knot 1 with 3 crossings )

gap> Display(M);
```

Next we construct the complement $Y = D^3 \setminus \mathring{M}$ of the interior of the pure cubical complex M. Here D^3 is a rectangular region with $M \subset \mathring{D}^3$. This pure cubical complex Y is a union of 5891 unit

3-cubes. We contract Y to get a homotopy equivalent pure cubical complex YY consisting of the union of just 775 unit 3-cubes. Then we convert YY to a regular CW-complex W involving 11939 cells. We contract W to obtain a homotopy equivalent regular CW-complex WW involving 5993 cells. Finally we compute the fundamental group of the complement of the granny knot, and use the presentation of this group to establish that the Alexander polynomial $P(x)$ of the granny is
$P(x) = x^4 - 2x^3 + 3x^2 - 2x + 1$.

Example

```
gap> Y:=PureComplexComplement(M);
Pure cubical complex of dimension 3.

gap> Size(Y);
5891

gap> YY:=ZigZagContractedComplex(Y);
Pure cubical complex of dimension 3.

gap> Size(YY);
775

gap> W:=RegularCWComplex(YY);
Regular CW-complex of dimension 3

gap> Size(W);
11939

gap> WW:=ContractedComplex(W);
Regular CW-complex of dimension 2

gap> Size(WW);
5993

gap> G:=FundamentalGroup(WW);
<fp group of size infinity on the generators [ f1, f2, f3 ]>

gap> AlexanderPolynomial(G);
x_1^4-2*x_1^3+3*x_1^2-2*x_1+1
```

2.2 Permutahedral complexes

A finite pure cubical complex is a union of finitely many cubes in a tessellation of $\mathbb{R}^n$ by unit cubes. One can also tessellate $\mathbb{R}^n$ by permutahedra, and we define a finite n-dimensional pure *permutahedral complex* to be a union of finitely many permutahdra from such a tessellation. There are two features of pure permutahedral complexes that are particularly useful in some situations:

- Pure permutahedral complexes are topological manifolds with boundary.
- The method used for finding a smaller pure cubical complex M' homotopy equivalent to a given pure cubical complex M retains the homeomorphism type, and not just the homotopy type, of the space M.

To illustrate these features the following example begins by reading in a protein backbone from the online Protein Database, and storing it as a pure cubical complex K. The ends of the protein have been joined, and the homology $H_i(K,\mathbb{Z}) = \mathbb{Z}$, $i = 0,1$ is seen to be that of a circle. We can thus regard the protein as a knot $K \subset \mathbb{R}^3$. The protein is visualized as a pure permutahedral complex.

Example
```
gap> file:=HapFile("data1V2X.pdb");;
gap> K:=ReadPDBfileAsPurePermutahedralComplex("file");
Pure permutahedral complex of dimension 3.

gap> Homology(K,0);
[ 0 ]
gap> Homology(K,1);
[ 0 ]

Display(K);
```

An alternative method for seeing that the pure permutahedral complex K has the homotopy type of a circle is to note that it is covered by open permutahedra (small open neighbourhoods of the closed 3-dimensional permutahedral titles) and to form the nerve $N = Nerve(\mathscr{U})$ of this open covering $\mathscr{U}$. The nerve N has the same homotopy type as K. The following commands establish that N is a 1-dimensional simplicial complex and display N as a circular graph.

Example
```
gap> N:=Nerve(K);
Simplicial complex of dimension 1.

gap> Display(GraphOfSimplicialComplex(N));
```

The boundary of the pure permutahedral complex K is a 2-dimensional CW-complex B homeomorphic to a torus. We next use the advantageous features of pure permutahedral complexes to compute the homomorphism

$\phi\colon \pi_1(B) \to \pi_1(\mathbb{R}^3 \setminus \mathring{K}), a \mapsto yx^{-3}y^2x^{-2}yxy^{-1}, b \mapsto yx^{-1}y^{-1}x^2y^{-1}$

where

$\pi_1(B) = < a,b : aba^{-1}b^{-1} = 1 >$,

$\pi_1(\mathbb{R}^3 \setminus \mathring{K}) \cong < x,y : y^2x^{-2}yxy^{-1} = 1, yx^{-2}y^{-1}x(xy^{-1})^2 = 1 >$.

Example
```
gap> Y:=PureComplexComplement(K);
Pure permutahedral complex of dimension 3.
gap> Size(Y);
418922

gap> YY:=ZigZagContractedComplex(Y);
Pure permutahedral complex of dimension 3.
gap> Size(YY);
3438

gap> W:=RegularCWComplex(YY);
Regular CW-complex of dimension 3

gap> f:=BoundaryMap(W);
```

```
Map of regular CW-complexes

gap> CriticalCells(Source(f));
[ [ 2, 1 ], [ 2, 261 ], [ 1, 1043 ], [ 1, 1626 ], [ 0, 2892 ], [ 0, 24715 ] ]

gap> F:=FundamentalGroup(f,2892);
[ f1, f2 ] -> [ f2*f1^-3*f2^2*f1^-2*f2*f1*f2^-1, f2*f1^-1*f2^-1*f1^2*f2^-1 ]

gap> G:=Target(F);
<fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f2^2*f1^-2*f2*f1*f2^-1, f2*f1^-2*f2^-1*f1*(f1*f2^-1)^2 ]
```

2.3 Constructing pure cubical and permutahedral complexes

An n-dimensional pure cubical or permutahedral complex can be created from an n-dimensional array of 0s and 1s. The following example creates and displays two 3-dimensional complexes.

Example
```
gap> A:=[[[0,0,0],[0,0,0],[0,0,0]],
>        [[1,1,1],[1,0,1],[1,1,1]],
>        [[0,0,0],[0,0,0],[0,0,0]]];;
gap> M:=PureCubicalComplex(A);
Pure cubical complex of dimension 3.

gap> P:=PurePermutahedralComplex(A);
Pure permutahedral complex of dimension 3.

gap> Display(M);
gap> Display(P);
```

2.4 Computations in dynamical systems

Pure cubical complexes can be useful for rigourous interval arithmetic calculations in numerical analysis. They can also be useful for trying to estimate approximations of certain numerical quantities. To illustrate the latter we consider the *Henon map*

$$f:\mathbb{R}^2\to\mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+1-ax^2 \\ bx \end{pmatrix}.$$

Starting with $(x_0,y_0)=(0,0)$ and iterating $(x_{n+1},y_{n+1})=f(x_n,y_n)$ with the parameter values $a=1.4$, $b=0.3$ one obtains a sequence of points which is known to be dense in the so called *strange attractor* $\mathscr{A}$ of the Henon map. The first 10 million points in this sequence are plotted in the following example, with arithmetic performed to 100 decimal places of accuracy. The sequence is stored as a 2-dimensional pure cubical complex where each 2-cell is square of side equal to $\varepsilon=1/500$.

Example
```
gap> M:=HenonOrbit([0,0],14/10,3/10,10^7,500,100);
Pure cubical complex of dimension 2.
```

```
gap> Size(M);
10287

gap> Display(M);
```

Repeating the computation but with squares of side $\varepsilon = 1/1000$

```
Example
gap> M:=HenonOrbit([0,0],14/10,3/10,10^7,1000,100);

gap> Size(M);
24949
```

we obtain the heuristic estimate
$\delta \simeq \frac{\log 24949 - \log 10287}{\log 2} = 1.277$
for the box-counting dimension of the attractor $\mathscr{A}$.

Chapter 3

Covering spaces

Let Y denote a finite regular CW-complex. Let $\widetilde{Y}$ denote its universal covering space. The covering space inherits a regular CW-structure which can be computed and stored using the datatype of a $\pi_1 Y$-equivariant CW-complex. The cellular chain complex $C_*\widetilde{Y}$ of $\widetilde{Y}$ can be computed and stored as an equivariant chain complex. Given an admissible discrete vector field on Y, we can endow Y with a smaller non-regular CW-structre whose cells correspond to the critical cells in the vector field. This smaller CW-structure leads to a more efficient chain complex $C_*\widetilde{Y}$ involving one free generator for each critical cell in the vector field.

3.1 Cellular chains on the universal cover

The following commands construct a 6-dimensional regular CW-complex $Y \simeq S^1 \times S^1 \times S^1$ homotopy equivalent to a product of three circles.

Example

```
gap> A:=[[1,1,1],[1,0,1],[1,1,1]];;
gap> S:=PureCubicalComplex(A);;
gap> T:=DirectProduct(S,S,S);;
gap> Y:=RegularCWComplex(T);;
Regular CW-complex of dimension 6

gap> Size(Y);
110592
```

The CW-somplex Y has 110592 cells. The next commands construct a free $\pi_1 Y$-equivariant chain complex $C_*\widetilde{Y}$ homotopy equivalent to the chain complex of the universal cover of Y. The chain complex $C_*\widetilde{Y}$ has just 8 free generators.

Example

```
gap> Y:=ContractedComplex(Y);;
gap> CU:=ChainComplexOfUniversalCover(Y);;
gap> List([0..Dimension(Y)],n->CU!.dimension(n));
[ 1, 3, 3, 1 ]
```

The next commands construct a subgroup $H < \pi_1 Y$ of index 50 and the chain complex $C_*\widetilde{Y} \otimes_{\mathbb{Z}H} \mathbb{Z}$ which is homotopy equivalent to the cellular chain complex $C_*\widetilde{Y}_H$ of the 50-fold cover $\widetilde{Y}_H$ of Y corresponding to H.

Example
```
gap> L:=LowIndexSubgroupsFpGroup(CU!.group,50);;
gap> H:=L[Length(L)-1];;
gap> Index(CU!.group,H);
50
gap> D:=TensorWithIntegersOverSubgroup(CU,H);
Chain complex of length 3 in characteristic 0 .

gap> List([0..3],D!.dimension);
[ 50, 150, 150, 50 ]
```

General theory implies that the 50-fold covering space $\widetilde{Y}_H$ should again be homotopy equivalent to a product of three circles. In keeping with this, the following commands verify that $\widetilde{Y}_H$ has the same integral homology as $S^1 \times S^1 \times S^1$.

Example
```
gap> Homology(D,0);
[ 0 ]
gap> Homology(D,1);
[ 0, 0, 0 ]
gap> Homology(D,2);
[ 0, 0, 0 ]
gap> Homology(D,3);
[ 0 ]
```

3.2 Spun knots and the Satoh tube map

We'll contruct two spaces Y,W with isomorphic fundamental groups and isomorphic intergal homology, and use the integral homology of finite covering spaces to establsh that the two spaces have distinct homotopy types.

By *spinning* a link $K \subset \mathbb{R}^3$ about a plane $P \subset \mathbb{R}^3$ with $P \cap K = \emptyset$, we obtain a collection $Sp(K) \subset \mathbb{R}^4$ of knotted tori. The following commands produce the two tori obtained by spinning the Hopf link K and show that the space $Y = \mathbb{R}^4 \setminus Sp(K) = Sp(\mathbb{R}^3 \setminus K)$ is connected with fundamental group $\pi_1 Y = \mathbb{Z} \times \mathbb{Z}$ and homology groups $H_0(Y) = \mathbb{Z}$, $H_1(Y) = \mathbb{Z}^2$, $H_2(Y) = \mathbb{Z}^4$, $H_3(Y,\mathbb{Z}) = \mathbb{Z}^2$. The space Y is only constructed up to homotopy, and for this reason is 3-dimensional.

Example
```
gap> Hopf:=PureCubicalLink("Hopf");
Pure cubical link.

gap> Y:=SpunAboutInitialHyperplane(PureComplexComplement(Hopf));
Regular CW-complex of dimension 3

gap> Homology(Y,0);
[ 0 ]
gap> Homology(Y,1);
[ 0, 0 ]
gap> Homology(Y,2);
[ 0, 0, 0, 0 ]
gap> Homology(Y,3);
```

```
[ 0, 0 ]
gap> Homology(Y,4);
[  ]
gap> GY:=FundamentalGroup(Y);;
gap> GeneratorsOfGroup(GY);
[ f2, f3 ]
gap> RelatorsOfFpGroup(GY);
[ f3^-1*f2^-1*f3*f2 ]
```

An alternative embedding of two tori $L \subset \mathbb{R}^4$ can be obtained by applying the 'tube map' of Shin Satoh to a welded Hopf link [Sat00]. The following commands construct the complement $W = \mathbb{R}^4 \setminus L$ of this alternative embedding and show that W has the same fundamental group and integral homology as Y above.

Example
```
gap> L:=HopfSatohSurface();
Pure cubical complex of dimension 4.

gap> W:=ContractedComplex(RegularCWComplex(PureComplexComplement(L)));
Regular CW-complex of dimension 3

gap> Homology(W,0);
[ 0 ]
gap> Homology(W,1);
[ 0, 0 ]
gap> Homology(W,2);
[ 0, 0, 0, 0 ]
gap> Homology(W,3);
[ 0, 0 ]
gap> Homology(W,4);
[  ]

gap> GW:=FundamentalGroup(W);;
gap> GeneratorsOfGroup(GW);
[ f1, f2 ]
gap> RelatorsOfFpGroup(GW);
[ f1^-1*f2^-1*f1*f2 ]
```

Despite having the same fundamental group and integral homology groups, the above two spaces Y and W were shown by Kauffman and Martins [KFM08] to be not homotopy equivalent. Their technique involves the fundamental crossed module derived from the first three dimensions of the universal cover of a space, and counts the representations of this fundamental crossed module into a given finite crossed module. This homotopy inequivalence is recovered by the following commands which involves the 5-fold covers of the spaces.

Example
```
gap> CY:=ChainComplexOfUniversalCover(Y);
Equivariant chain complex of dimension 3
gap> LY:=LowIndexSubgroups(CY!.group,5);;
gap> invY:=List(LY,g->Homology(TensorWithIntegersOverSubgroup(CY,g),2));;
```

```
gap> CW:=ChainComplexOfUniversalCover(W);
Equivariant chain complex of dimension 3
gap> LW:=LowIndexSubgroups(CW!.group,5);;
gap> invW:=List(LW,g->Homology(TensorWithIntegersOverSubgroup(CW,g),2));;

gap> SSortedList(invY)=SSortedList(invW);
false
```

3.3 Cohomology with local coefficients

The $\pi_1 Y$-equivariant cellular chain complex $C_*\widetilde{Y}$ of the universal cover $\widetilde{Y}$ of a regular CW-complex Y can be used to compute the homology $H_n(Y,A)$ and cohomology $H^n(Y,A)$ of Y with local coefficients in a $\mathbb{Z}\pi_1 Y$-module A. To illustrate this we consister the space Y arising as the complement of the trefoil knot, with fundamental group $\pi_1 Y = \langle x,y : xyx = yxy\rangle$. We take $A = \mathbb{Z}$ to be the integers with non-trivial $\pi_1 Y$-action given by $x.1 = -1, y.1 = -1$. We then compute

$$\begin{aligned} H_0(Y,A) &= \mathbb{Z}_2\,, \\ H_1(Y,A) &= \mathbb{Z}_3\,, \\ H_2(Y,A) &= \mathbb{Z}\,. \end{aligned}$$

Example

```
gap> K:=PureCubicalKnot(3,1);;
gap> Y:=PureComplexComplement(K);;
gap> Y:=ContractedComplex(Y);;
gap> Y:=RegularCWComplex(Y);;
gap> Y:=SimplifiedComplex(Y);;
gap> C:=ChainComplexOfUniversalCover(Y);;
gap> G:=C!.group;;
gap> GeneratorsOfGroup(G);
[ f1, f2 ]
gap> RelatorsOfFpGroup(G);
[ f2^-1*f1^-1*f2^-1*f1*f2*f1, f1^-1*f2^-1*f1^-1*f2*f1*f2 ]
gap> hom:=GroupHomomorphismByImages(G,Group([[-1]]),[G.1,G.2],[[[-1]],[[-1]]]);;
gap> A:=function(x); return Determinant(Image(hom,x)); end;;
gap> D:=TensorWithTwistedIntegers(C,A); #Here the function A represents
gap> #the integers with twisted action of G.
Chain complex of length 3 in characteristic 0 .
gap> Homology(D,0);
[ 2 ]
gap> Homology(D,1);
[ 3 ]
gap> Homology(D,2);
[ 0 ]
```

3.4 Distinguishing between two non-homeomorphic homotopy equivalent spaces

The granny knot is the sum of the trefoil knot and its mirror image. The reef knot is the sum of two identical copies of the trefoil knot. The following commands show that the degree 1 homology homomorphisms

$H_1(p^{-1}(B),\mathbb{Z}) \rightarrow H_1(\widetilde{X}_H,\mathbb{Z})$

distinguish between the homeomorphism types of the complements $X \subset \mathbb{R}^3$ of the granny knot and the reef knot, where $B \subset X$ is the knot boundary, and where $p{:}\widetilde{X}_H \rightarrow X$ is the covering map corresponding to the finite index subgroup $H < \pi_1 X$. More precisely, $p^{-1}(B)$ is in general a union of path components

$p^{-1}(B) = B_1 \cup B_2 \cup \cdots \cup B_t$.

The function `FirstHomologyCoveringCokernels(f,c)` inputs an integer c and the inclusion $f{:}B \hookrightarrow X$ of a knot boundary B into the knot complement X. The function returns the ordered list of the lists of abelian invariants of cokernels

coker($H_1(p^{-1}(B_i),\mathbb{Z}) \rightarrow H_1(\widetilde{X}_H,\mathbb{Z})$)

arising from subgroups $H < \pi_1 X$ of index c. To distinguish between the granny and reef knots we use index $c = 6$.

Example

```
gap> K:=PureCubicalKnot(3,1);;
gap> L:=ReflectedCubicalKnot(K);;
gap> granny:=KnotSum(K,L);;
gap> reef:=KnotSum(K,K);;
gap> fg:=KnotComplementWithBoundary(ArcPresentation(granny));;
gap> fr:=KnotComplementWithBoundary(ArcPresentation(reef));;
gap> a:=FirstHomologyCoveringCokernels(fg,6);;
gap> b:=FirstHomologyCoveringCokernels(fr,6);;
gap> a=b;
false
```

3.5 Second homotopy groups of spaces with finite fundamental group

If $p:\widetilde{Y} \rightarrow Y$ is the universal covering map, then the fundamental group of $\widetilde{Y}$ is trivial and the Hurewicz homomorphism $\pi_2\widetilde{Y} \rightarrow H_2(\widetilde{Y},\mathbb{Z})$ from the second homotopy group of $\widetilde{Y}$ to the second integral homology of $\widetilde{Y}$ is an isomorphism. Furthermore, the map p induces an isomorphism $\pi_2\widetilde{Y} \rightarrow \pi_2 Y$. Thus $H_2(\widetilde{Y},\mathbb{Z})$ is isomorphic to the second homotopy group $\pi_2 Y$.

If the fundamental group of Y happens to be finite, then in principle we can calculate $H_2(\widetilde{Y},\mathbb{Z}) \cong \pi_2 Y$. We illustrate this computation for Y equal to the real projective plane. The above computation shows that Y has second homotopy group $\pi_2 Y \cong \mathbb{Z}$.

Example

```
gap> K:=[ [1,2,3], [1,3,4], [1,2,6], [1,5,6], [1,4,5],
>         [2,3,5], [2,4,5], [2,4,6], [3,4,6], [3,5,6]];;

gap> K:=MaximalSimplicesToSimplicialComplex(K);
Simplicial complex of dimension 2.

gap> Y:=RegularCWComplex(K);
```

```
Regular CW-complex of dimension 2
gap> # Y is a regular CW-complex corresponding to the projective plane.

gap> U:=UniversalCover(Y);
Equivariant CW-complex of dimension 2

gap> G:=U!.group;;
gap> # G is the fundamental group of Y, which by the next command
gap> # is finite of order 2.
gap> Order(G);
2

gap> U:=EquivariantCWComplexToRegularCWComplex(U,Group(One(G)));
Regular CW-complex of dimension 2
gap> #U is the universal cover of Y

gap> Homology(U,0);
[ 0 ]
gap> Homology(U,1);
[  ]
gap> Homology(U,2);
[ 0 ]
```

3.6 Third homotopy groups of simply connected spaces

For any path connected space Y with universal cover $\widetilde{Y}$ there is an exact sequence
$\to \pi_4\widetilde{Y} \to H_4(\widetilde{Y},\mathbb{Z}) \to H_4(K(\pi_2\widetilde{Y},2),\mathbb{Z}) \to \pi_3\widetilde{Y} \to H_3(\widetilde{Y},\mathbb{Z}) \to 0$
due to J.H.C.Whitehead. Here $K(\pi_2(\widetilde{Y}),2)$ is an Eilenberg-MacLane space with second homotopy group equal to $\pi_2\widetilde{Y}$.

3.6.1 First example

Continuing with the above example where Y is the real projective plane, we see that $H_4(\widetilde{Y},\mathbb{Z}) = H_3(\widetilde{Y},\mathbb{Z}) = 0$ since $\widetilde{Y}$ is a 2-dimensional CW-space. The exact sequence implies $\pi_3\widetilde{Y} \cong H_4(K(\pi_2\widetilde{Y},2),\mathbb{Z})$. Furthermore, $\pi_3\widetilde{Y} = \pi_3 Y$. The following commands establish that $\pi_3 Y \cong \mathbb{Z}$.

Example
```
gap> A:=AbelianPcpGroup([0]);
Pcp-group with orders [ 0 ]

gap> K:=EilenbergMacLaneSimplicialGroup(A,2,5);;
gap> C:=ChainComplexOfSimplicialGroup(K);
Chain complex of length 5 in characteristic 0 .

gap> Homology(C,4);
[ 0 ]
```

3.6.2 Second example

The following commands construct a 4-dimensional simplicial complex Y with 9 vertices and 36 4-dimensional simplices, and establish that

$\pi_1 Y = 0, \pi_2 Y = \mathbb{Z}, H_3(Y,\mathbb{Z}) = 0, H_4(Y,\mathbb{Z}) = \mathbb{Z}, H_4(K(\pi_2 Y,2),\mathbb{Z}) = \mathbb{Z}.$

Example

```
gap> Y:=[ [ 1, 2, 4, 5, 6 ], [ 1, 2, 4, 5, 9 ], [ 1, 2, 5, 6, 8 ],
>         [ 1, 2, 6, 4, 7 ], [ 2, 3, 4, 5, 8 ], [ 2, 3, 5, 6, 4 ],
>         [ 2, 3, 5, 6, 7 ], [ 2, 3, 6, 4, 9 ], [ 3, 1, 4, 5, 7 ],
>         [ 3, 1, 5, 6, 9 ], [ 3, 1, 6, 4, 5 ], [ 3, 1, 6, 4, 8 ],
>         [ 4, 5, 7, 8, 3 ], [ 4, 5, 7, 8, 9 ], [ 4, 5, 8, 9, 2 ],
>         [ 4, 5, 9, 7, 1 ], [ 5, 6, 7, 8, 2 ], [ 5, 6, 8, 9, 1 ],
>         [ 5, 6, 8, 9, 7 ], [ 5, 6, 9, 7, 3 ], [ 6, 4, 7, 8, 1 ],
>         [ 6, 4, 8, 9, 3 ], [ 6, 4, 9, 7, 2 ], [ 6, 4, 9, 7, 8 ],
>         [ 7, 8, 1, 2, 3 ], [ 7, 8, 1, 2, 6 ], [ 7, 8, 2, 3, 5 ],
>         [ 7, 8, 3, 1, 4 ], [ 8, 9, 1, 2, 5 ], [ 8, 9, 2, 3, 1 ],
>         [ 8, 9, 2, 3, 4 ], [ 8, 9, 3, 1, 6 ], [ 9, 7, 1, 2, 4 ],
>         [ 9, 7, 2, 3, 6 ], [ 9, 7, 3, 1, 2 ], [ 9, 7, 3, 1, 5 ] ];;

gap> Y:=MaximalSimplicesToSimplicialComplex(Y);
Simplicial complex of dimension 4.

gap> Y:=RegularCWComplex(Y);
Regular CW-complex of dimension 4

gap> Order(FundamentalGroup(Y));
1
gap> Homology(Y,2);
[ 0 ]
gap> Homology(Y,3);
[  ]
gap> Homology(Y,4);
[ 0 ]
```

Whitehead's sequence reduces to an exact sequence

$\mathbb{Z} \to \mathbb{Z} \to \pi_3 Y \to 0$

in which the first map is $H_4(Y,\mathbb{Z}) = \mathbb{Z} \to H_4(K(\pi_2 Y,2),\mathbb{Z}) = \mathbb{Z}$. In order to determine $\pi_3 Y$ it remains compute this first map. This computation is currently not available in HAP.

[The simplicial complex in this second example is due to W. Kiihnel and T. F. Banchoff and is of the homotopy type of the complex projective plane. So, assuming this extra knowledge, we have $\pi_3 Y = 0$.]

3.7 Computing the second homotopy group of a space with infinite fundamental group

The following commands compute the second integral homology

$H_2(\pi_1 W,\mathbb{Z}) = \mathbb{Z}$

of the fundamental group $\pi_1 W$ of the complement W of the Hopf-Satoh surface.

Example
```
gap> L:=HopfSatohSurface();
Pure cubical complex of dimension 4.

gap> W:=ContractedComplex(RegularCWComplex(PureComplexComplement(L)));
Regular CW-complex of dimension 3

gap> GW:=FundamentalGroup(W);;
gap> IsAspherical(GW);
Presentation is aspherical.
true
gap> R:=ResolutionAsphericalPresentation(GW);;
gap> Homology(TensorWithIntegers(R),2);
[ 0 ]
```

From Hopf's exact sequence
$\pi_2 W \xrightarrow{h} H_2(W,\mathbb{Z}) \twoheadrightarrow H_2(\pi_1 W,\mathbb{Z}) \to 0$
and the computation $H_2(W,\mathbb{Z}) = \mathbb{Z}^4$ we see that the image of the Hurewicz homomorphism is $\mathsf{im}(h) = \mathbb{Z}^3$. The image of h is referred to as the subgroup of *spherical homology classes* and often denoted by $\Sigma^2 W$.

The following command computes the presentation of $\pi_1 W$ corresponding to the 2-skeleton W^2 and establishes that $W^2 = S^2 \vee S^2 \vee S^2 \vee (S^1 \times S^1)$ is a wedge of three spheres and a torus.

Example
```
gap> F:=FundamentalGroupOfRegularCWComplex(W,"no simplification");
< fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(F);
[ < identity ...>, f1^-1*f2^-1*f1*f2, < identity ...>, <identity ...> ]
```

The next command shows that the 3-dimensional space W has two 3-cells each of which is attached to the base-point of W with trivial boundary (up to homotopy in W^2). Hence $W = S^3 \vee S^3 \vee S^2 \vee S^2 \vee S^2 \vee (S^1 \times S^1)$.

Example
```
gap> CriticalCells(W);
[ [ 3, 1 ], [ 3, 3148 ], [ 2, 6746 ], [ 2, 20510 ], [ 2, 33060 ],
  [ 2, 50919 ], [ 1, 29368 ], [ 1, 50822 ], [ 0, 21131 ] ]
gap> CriticalBoundaryCells(W,3,1);
[  ]
gap> CriticalBoundaryCells(W,3,3148);
[ -50919, 50919 ]
```

Therefore $\pi_1 W$ is the free abelian group on two generators, and $\pi_2 W$ is the free $\mathbb{Z}\pi_1 W$-module on three free generators.

Chapter 4

Topological data analysis

4.1 Persistent homology

Pairwise distances between 74 points from some metric space have been recorded and stored in a 74×74 matrix D. The following commands load the matrix, construct a filtration of length 100 on the first two dimensions of the assotiated clique complex (also known as the *Rips Complex*), and display the resulting degree 0 persistent homology as a barcode. A single bar with label n denotes n bars with common starting point and common end point.

Example

```
gap> file:=HapFile("data253a.txt");;
gap> Read(file);

gap> G:=SymmetricMatrixToFilteredGraph(D,100);
Filtered graph on 74 vertices.

gap> K:=FilteredRegularCWComplex(CliqueComplex(G,2));
Filtered regular CW-complex of dimension 2

gap> P:=PersistentBettiNumbers(K,0);;
gap> BarCodeCompactDisplay(P);
```

The next commands display the resulting degree 1 persistent homology as a barcode.

Example

```
gap> P:=PersistentBettiNumbers(K,1);;
gap> BarCodeCompactDisplay(P);
```

The following command displays the 1 skeleton of the simplicial complex arizing as the 65-th term in the filtration on the clique complex.

Example

```
gap> Y:=FiltrationTerm(K,65);
Regular CW-complex of dimension 1

gap> Display(HomotopyGraph(Y));
```

These computations suuggest that the dataset contains two persistent path components (or clusters), and that each path component is in some sense periodic. The final command displays one possible representation of the data as points on two circles.

4.1.1 Background to the data

Each point in the dataset was an image consisting of 732×761 pixels. This point was regarded as a vector in $\mathbb{R}^{732\times 761}$ and the matrix D was constructed using the Euclidean metric. The images were the following:

4.2 Mapper clustering

The following example reads in a set S of vectors of rational numbers. It uses the Euclidean distance $d(u,v)$ between vectors. It fixes some vector $u_0\in S$ and uses the associated function $f\colon D \to [0,b] \subset \mathbb{R}, v \mapsto d(u_0,v)$. In addition, it uses an open cover of the interval $[0,b]$ consisting of 100 uniformly distributed overlapping open subintervals of radius $r = 29$. It also uses a simple clustering algorithm implemented in the function `cluster`.

These ingredients are input into the Mapper clustering procedure to produce a simplicial complex M which is intended to be a representation of the data. The complex M is 1-dimensional and the final command uses GraphViz software to visualize the graph. The nodes of this simplicial complex are "buckets" containing data points. A data point may reside in several buckets. The number of points in the bucket determines the size of the node. Two nodes are connected by an edge when their end-point nodes contain common data points.

Example

```
gap> file:=HapFile("data134.txt");;
gap> Read(file);
gap> dx:=EuclideanApproximatedMetric;;
gap> dz:=EuclideanApproximatedMetric;;
gap> L:=List(S,x->Maximum(List(S,y->dx(x,y))));;
gap> n:=Position(L,Minimum(L));;
gap> f:=function(x); return [dx(S[n],x)]; end;;
gap> P:=30*[0..100];; P:=List(P, i->[i]);;
gap> r:=29;;
gap> epsilon:=75;;
gap>  cluster:=function(S)
>    local Y, P, C;
>    if Length(S)=0 then return S; fi;
>    Y:=VectorsToOneSkeleton(S,epsilon,dx);
>    P:=PiZero(Y);
>    C:=Classify([1..Length(S)],P[2]);
>    return List(C,x->S{x});
>   end;;
gap> M:=Mapper(S,dx,f,dz,P,r,cluster);
Simplicial complex of dimension 1.

gap> Display(GraphOfSimplicialComplex(M));
```

4.2.1 Background to the data

The datacloud S consists of the 400 points in the plane shown in the following picture.

4.3 Digital image analysis

The following example reads in a digital image as a filtered pure cubical complexex. The filtration is obtained by thresholding at a sequence of uniformly spaced values on the greyscale range. The persistent homology of this filtered complex is calculated in degrees 0 and 1 and displayed as two barcodes.

Example
```
gap> file:=HapFile("image1.3.2.png");;
gap> F:=ReadImageAsFilteredPureCubicalComplex(file,20);
Filtered pure cubical complex of dimension 2.
gap> P:=PersistentBettiNumbers(F,0);;
gap> BarCodeCompactDisplay(P);
```

Example
```
gap> P:=PersistentBettiNumbers(F,1);;
gap> BarCodeCompactDisplay(P);
```

The 20 persistent bars in the degree 0 barcode suggest that the image has 20 objects. The degree 1 barcode suggests that 14 (or possibly 17) of these objects have holes in them.

4.3.1 Background to the data

The following image was used in the example.

Chapter 5

Group theoretic computations

5.1 Third homotopy group of a supsension of an Eilenberg-MacLane space

The following example uses the nonabelian tensor square of groups to compute the third homotopy group

$\pi_3(S(K(G,1))) = \mathbb{Z}^{30}$

of the suspension of the Eigenberg-MacLane space $K(G,1)$ for G the free nilpotent group of class 2 on four generators.

Example
```
gap> F:=FreeGroup(4);;G:=NilpotentQuotient(F,2);;
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0 ]
```

5.2 Representations of knot quandles

The following example constructs the finitely presented quandles associated to the granny knot and square knot, and then computes the number of quandle homomorphisms from these two finitely prresented quandles to the 17-th quandle in HAP's library of connected quandles of order 24. The number of homomorphisms differs between the two cases. The computation therefore establishes that the complement in $\mathbb{R}^3$ of the granny knot is not homeomorphic to the complement of the square knot.

Example
```
gap> Q:=ConnectedQuandle(24,17,"import");;
gap> K:=PureCubicalKnot(3,1);;
gap> L:=ReflectedCubicalKnot(K);;
gap> square:=KnotSum(K,L);;
gap> granny:=KnotSum(K,K);;
gap> gcsquare:=GaussCodeOfPureCubicalKnot(square);;
gap> gcgranny:=GaussCodeOfPureCubicalKnot(granny);;
gap> Qsquare:=PresentationKnotQuandle(gcsquare);;
gap> Qgranny:=PresentationKnotQuandle(gcgranny);;
gap> NumberOfHomomorphisms(Qsquare,Q);
408
```

```
gap> NumberOfHomomorphisms(Qgranny,Q);
24
```

5.3 Aspherical 2-complexes

The following example uses Polymake's linear programming routines to establish that the 2-complex associated to the group presentation $< x,y,z : xyx = yxy, yzy = zyz, xzx = zxz >$ is aspherical (that is, has contractible universal cover). The presentation is Tietze equivalent to the presentation used in the computer code, and the associated 2-complexes are thus homotopy equivalent.

Example

```
gap> F:=FreeGroup(6);;
gap> x:=F.1;;y:=F.2;;z:=F.3;;a:=F.4;;b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x, a*x*(y*a)^-1,
>     b*y*(z*b)^-1, c*z*(x*c)^-1];;
gap> Print(IsAspherical(F,rels),"\n");
Presentation is aspherical.

true
```

5.4 Bogomolov multiplier

The Bogomolov multiplier of a group is an isoclinism invariant. Using this property, the following example shows that there are precisely three groups of order 243 with non-trivial Bogomolov multiplier. The groups in question are numbered 28, 29 and 30 in GAP's library of small groups of order 243.

Example

```
gap> L:=AllSmallGroups(3^5);;
gap> C:=IsoclinismClasses(L);;
gap> for c in C do
> if Length(BogomolovMultiplier(c[1]))>0 then
> Print(List(c,g->IdGroup(g)),"\n\n\n"); fi;
> od;
[ [ 243, 28 ], [ 243, 29 ], [ 243, 30 ] ]
```

5.5 Second group cohomology and group extensions

Any group extension $N \rightarrowtail E \twoheadrightarrow G$ gives rise to:

- an outer action $\alpha\colon G \to Out(G)$ of G on N.
- an action $G \to Aut(Z(N))$ of G on the centre of N, uniquely induced by the outer action α and the canonical action of $Out(N)$ on $Z(N)$.
- a 2-cocycle $f\colon G \times G \to Z(N)$ with values in the G-module $A = Z(N)$.

Any outer homomorphism $\alpha: G \to Out(N)$ gives rise to a cohomology class k in $H^3(G, Z(N))$. It was shown by Eilenberg and Mac Lane that the class k is trivial if and only if the outer action α arises from some group extension $N \rightarrowtail E \twoheadrightarrow G$. If k is trivial then there is a bijection between the second cohomology group $H^2(G, Z(N))$ and Yoneda equivalence classes of extensions of G by N that are compatible with α.

FIRST EXAMPLE.

Consider the group $H = SmallGroup(64,134)$. Consider the normal subgroup $N = NormalSubgroups(G)[15]$ and quotient group $G = H/N$. We have $N = C_2 \times D_4$, $A = Z(N) = C_2 \times C_2$ and $G = C_2 \times C_2$.

Suppose we wish to classify all extensions $C_2 \times D_4 \rightarrowtail E \twoheadrightarrow C_2 \times C_2$ that induce the given outer action of G on N. The following commands show that, up to Yoneda equivalence, there are two such extensions.

Example

```
gap> H:=SmallGroup(64,134);;
gap> N:=NormalSubgroups(H)[15];;
gap> A:=Centre(GOuterGroup(H,N));;
gap> G:=ActingGroup(A);;
gap> R:=ResolutionFiniteGroup(G,3);;
gap> C:=HomToGModule(R,A);;
gap> Cohomology(C,2);
[ 2 ]
```

The following additional commands return a standard 2-cocycle $f: G \times G \to A = C_2 \times C_2$ corresponding to the non-trivial element in $H^2(G,A)$. The value $f(g,h)$ of the 2-cocycle is calculated for all 16 pairs $g, h \in G$.

Example

```
gap> CH:=CohomologyModule(C,2);;
gap> Elts:=Elements(ActedGroup(CH));
[ <identity> of ..., f1 ]

gap> x:=Elts[2];;
gap> c:=CH!.representativeCocycle(x);
Standard 2-cocycle

gap> f:=Mapping(c);;
gap> for g in G do for h in G do
> Print(f(g,h),"\n");
> od;
> od;
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
f6
<identity> of ...
f6
<identity> of ...
<identity> of ...
<identity> of ...
```

```
<identity> of ...
<identity> of ...
f6
<identity> of ...
f6
```

The following commands will then construct and identify all extensions of N by G corresponding to the given outer action of G on N.

Example
```
gap> H := SmallGroup(64,134);;
gap> N := NormalSubgroups(H)[15];;
gap> ON := GOuterGroup(H,N);;
gap> A := Centre(ON);;
gap> G:=ActingGroup(A);;
gap> R:=ResolutionFiniteGroup(G,3);;
gap> C:=HomToGModule(R,A);;
gap> CH:=CohomologyModule(C,2);;
gap> Elts:=Elements(ActedGroup(CH));;

gap> lst := List(Elts{[1..Length(Elts)]},x->CH!.representativeCocycle(x));;
gap> ccgrps := List(lst, x->CcGroup(ON, x));;
gap> #So ccgrps is a list of groups, each being an extension of G by N, corresponding
gap> #to the two elements in H^2(G,A).

gap> #The following command produces the GAP identification number for each group.
gap> L:=List(ccgrps,IdGroup);
[ [ 64, 134 ], [ 64, 135 ] ]
```

SECOND EXAMPLE

The following example illustrates how to construct a cohomology class k in $H^2(G,A)$ from a cocycle $f : G\times G \to A$, where $G = SL_2(\mathbb{Z}_4)$ and $A = \mathbb{Z}_8$ with trivial action.

Example
```
gap> #We'll construct G=SL(2,Z_4) as a permutation group.
gap> G:=SL(2,ZmodnZ(4));;
gap> G:=Image(IsomorphismPermGroup(G));;

gap> #We'll construct Z_8=Z/8Z as a G-outer group
gap> z_8:=Group((1,2,3,4,5,6,7,8));;
gap> Z_8:=TrivialGModuleAsGOuterGroup(G,z_8);;

gap> #We'll compute the group h=H^2(G,Z_8)
gap> R:=ResolutionFiniteGroup(G,3);;  #R is a free resolution
gap> C:=HomToGModule(R,Z_8);; # C is a chain complex
gap> H:=CohomologyModule(C,2);; #H is the second cohomology H^2(G,Z_8)
gap> h:=ActedGroup(H);; #h is the underlying group of H

gap> #We'll compute  cocycles c2, c5 for the second and fifth cohomology classs
gap> c2:=H!.representativeCocycle(Elements(h)[2]);
Standard 2-cocycle
```

```
gap> c5:=H!.representativeCocycle(Elements(h)[5]);
Standard 2-cocycle

gap> #Now we'll construct the cohomology classes C2, C5 in the group h corresponding to the cocyc
gap> C2:=CohomologyClass(H,c2);;
gap> C5:=CohomologyClass(H,c5);;

gap> #Finally, we'll show that C2, C5 are distinct cohomology classes, both of order 4.
gap> C2=C5;
false
gap> Order(C2);
4
gap> Order(C5);
4
```

5.6 Second group cohomology and cocyclic Hadamard matrices

An *Hadamard matrix* is a square $n \times n$ matrix H whose entries are either $+1$ or -1 and whose rows are mutually orthogonal, that is $HH^t = nI_n$ where H^t denotes the transpose and I_n denotes the $n \times n$ identity matrix.

Given a group $G = \{g_1, g_2, \ldots, g_n\}$ of order n and the abelian group $A = \{1, -1\}$ of square roots of unity, any 2-cocycle $f : G \times G \to A$ corresponds to an $n \times n$ matrix $F = (f(g_i, g_j))_{1 \le i,j \le n}$ whose entries are ± 1. If F is Hadamard it is called a *cocyclic Hadamard matrix* corresponding to G.

The following commands compute all 192 of the cocyclic Hadamard matrices for the abelian group $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ of order $n = 16$.

Example

```
gap> G:=AbelianGroup([4,4]);;
gap> F:=CocyclicHadamardMatrices(G);;
gap> Length(F);
192
```

5.7 Third group cohomology and homotopy 2-types

HOMOTOPY 2-TYPES

The third cohomology $H^3(G,A)$ of a group G with coefficients in a G-module A, together with the corresponding 3-cocycles, can be used to classify homotopy 2-types. A *homotopy 2-type* is a CW-complex whose homotopy groups are trivial in dimensions $n = 0$ and $n > 2$. There is an equivalence between the two categories

1. (Homotopy category of connected CW-complexes X with trivial homotopy groups $\pi_n(X)$ for $n > 2$)

2. (Localization of the category of simplicial groups with Moore complex of length 1, where localization is with respect to homomorphisms inducing isomorphisms on homotopy groups)

which reduces the homotopy theory of 2-types to a 'computable' algebraic theory. Furthermore, a simplicial group with Moore complex of length 1 can be represented by a group H endowed with two endomorphisms $s\colon H \to H$ and $t\colon H \to H$ satisfying the axioms

- $ss = s$, $ts = s$,
- $tt = t$, $st = t$,
- $[\ker s, \ker t] = 1$.

Ths triple (H,s,t) was termed a *cat*^{1}*-group* by J.-L. Loday since it can be regarded as a group H endowed with one compatible category structure.

The *homotopy groups* of a cat^1-group H are defined as: $\pi_1(H) = \mathrm{image}(s)/t(\ker(s))$; $\pi_2(H) = \ker(s) \cap \ker(t)$; $\pi_n(H) = 0$ for $n > 2$ or $n = 0$. Note that $\pi_2(H)$ is a $\pi_1(H)$-module where the action is induced by conjugation in H.

A homotopy 2-type X can be represented by a cat^1-group H or by the homotopy groups $\pi_1 X = \pi_1 H$, $\pi_2 X = \pi_2 H$ and a cohomology class $k \in H^3(\pi_1 X, \pi_2 X)$. This class k is the *Postnikov invariant*.

RELATION TO GROUP THEORY

A number of standard group-theoretic constructions can be viewed naturally as a cat^1-group.

1. A $\mathbb{Z}G$-module A can be viewed as a cat^1-group (H,s,t) where H is the semi-direct product $A \rtimes G$ and $s(a,g) = (1,g)$, $t(a,g) = (1,g)$. Here $\pi_1(H) = G$ and $\pi_2(H) = A$.
2. A group G with normal subgroup N can be viewed as a cat^1-group (H,s,t) where H is the semi-direct product $N \rtimes G$ and $s(n,g) = (1,g)$, $t(n,g) = (1,ng)$. Here $\pi_1(H) = G/N$ and $\pi_2(H) = 0$.
3. The homomorphism $\iota\colon G \to Aut(G)$ which sends elements of a group G to the corresponding inner automorphism can be viewed as a cat^1-group (H,s,t) where H is the semi-direct product $G \rtimes Aut(G)$ and $s(g,a) = (1,a)$, $t(g,a) = (1,\iota(g)a)$. Here $\pi_1(H) = Out(G)$ is the outer automorphism group of G and $\pi_2(H) = Z(G)$ is the centre of G.

These three constructions are implemented in HAP.

EXAMPLE

The following commands begin by constructing the cat^1-group H of Construction 3 for the group $G = SmallGroup(64,134)$. They then construct the fundamental group of H and the second homotopy group of as a π_1-module. These homotopy groups have orders 8 and 2 respectively.

Example
```
gap> G:=SmallGroup(64,134);;
gap> H:=AutomorphismGroupAsCatOneGroup(G);;
gap> pi_1:=HomotopyGroup(H,1);;
gap> pi_2:=HomotopyModule(H,2);;
gap> Order(pi_1);
8
gap> Order(ActedGroup(pi_2));
2
```

The following additional commands show that there are 1024 Yoneda equivalence classes of cat^1-groups with fundamental group π_1 and π_1- module equal to π_2 in our example.

Example
```
gap> R:=ResolutionFiniteGroup(pi_1,4);;
gap> C:=HomToGModule(R,pi_2);;
gap> CH:=CohomologyModule(C,3);;
gap> AbelianInvariants(ActedGroup(CH));
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

A 3-cocycle $f\colon \pi_1 \times \pi_1 \times \pi_1 \to \pi_2$ corresponding to a random cohomology class $k \in H^3(\pi_1, \pi_2)$ can be produced using the following command.

Example
```
gap> x:=Random(Elements(ActedGroup(CH)));;
gap> f:=CH!.representativeCocycle(x);
Standard 3-cocycle
```

The 3-cocycle corresponding to the Postnikov invariant of H itself can be easily constructed directly from its definition in terms of a set-theoretic 'section' of the crossed module corresponding to H.

Chapter 6

Cohomology of groups

6.1 Finite groups

It is possible to compute the low degree (co)homology of a finite group or monoid of small order directly from the bar resolution. The following commands take this approach to computing the fifth integral homology

$H_5(Q_4,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

of the quaternion group $G = Q_4$ of order 8.

Example
```
gap> Q:=QuaternionGroup(8);;
gap> B:=BarComplexOfMonoid(Q,6);;
gap> C:=ContractedComplex(B);;
gap> Homology(C,5);
[ 2, 2 ]
```

However, this approach is of limited applicability since the bar resolution involves $|G|^k$ free generators in degree k. A range of techniques, tailored to specific classes of groups, can be used to compute the (co)homology of larger finite groups.

The following example computes the fourth integral cohomomogy of the Mathieu group M_{24}.

$H^4(M_{24},\mathbb{Z}) = \mathbb{Z}_{12}$

Example
```
gap> GroupCohomology(MathieuGroup(24),4);
[ 4, 3 ]
```

The following example computes the third integral homology of the Weyl group $W = Weyl(E_8)$, a group of order 696729600.

$H_3(Weyl(E_8),\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$

Example
```
p> L:=SimpleLieAlgebra("E",8,Rationals);;
gap> W:=WeylGroup(RootSystem(L));;
gap> Order(W);
696729600
gap> GroupHomology(W,3);
[ 2, 2, 4, 3 ]
```

The preceding calculation could be achieved more quickly by noting that $W = Weyl(E_8)$ is a Coxeter group, and by using the associated Coxeter polytope. The following example uses this approach to compute the fourth integral homology of W. It begins by displaying the Coxeter diagram of W, and then computes

$H_4(Weyl(E_8),\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus Z_2 \oplus \mathbb{Z}_2$.

Example
```
gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3],[5,3]],[5,[6,3]],[6,[7,3]],[7,[8,3]]];;
gap> CoxeterDiagramDisplay(D);
```

Example
```
gap> polytope:=CoxeterComplex_alt(D,5);;
gap> R:=FreeGResolution(polytope,5);
Resolution of length 5 in characteristic 0 for <matrix group with
8 generators> .
No contracting homotopy available.

gap> C:=TensorWithIntegers(R);
Chain complex of length 5 in characteristic 0 .

gap> Homology(C,4);
[ 2, 2, 2, 2 ]
```

The following example computes the sixth mod-2 homology of the Sylow 2-subgroup $Syl_2(M_{24})$ of the Mathieu group M_{24}.

$H_6(Syl_2(M_{24}),\mathbb{Z}_2) = \mathbb{Z}_2^{143}$

Example
```
gap> GroupHomology(SylowSubgroup(MathieuGroup(24),2),6,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

The following example constructs the Poincare polynomial

$p(x) = \frac{1}{-x^3+3*x^2-3*x+1}$

for the cohomology $H^*(Syl_2(M_{12},\mathbb{F}_2)$. The coefficient of x^n in the expansion of $p(x)$ is equal to the dimension of the vector space $H^n(Syl_2(M_{12},\mathbb{F}_2)$. The computation involves SINGULAR's Groebner basis algorithms and the Lyndon-Hochschild-Serre spectral sequence.

Example
```
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> PoincareSeriesLHS(G);
(1)/(-x_1^3+3*x_1^2-3*x_1+1)
```

The following example constructs the polynomial

$p(x) = \frac{x^4-x^3+x^2-x+1}{x^6-x^5+x^4-2*x^3+x^2-x+1}$

whose coefficient of x^n is equal to the dimension of the vector space $H^n(M_{11},\mathbb{F}_2)$ for all n in the range $0 \le n \le 14$. The coefficient is not guaranteed correct for $n \ge 15$.

Example

```
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,14);
(x_1^4-x_1^3+x_1^2-x_1+1)/(x_1^6-x_1^5+x_1^4-2*x_1^3+x_1^2-x_1+1)
```

6.2 Nilpotent groups

The following example computes

$H_4(N,\mathbb{Z}) = (Z_3)^4 \oplus \mathbb{Z}^{84}$

for the free nilpotent group N of class 2 on four generators.

Example

```
gap> F:=FreeGroup(4);; N:=NilpotentQuotient(F,2);;
gap> GroupHomology(N,4);
[ 3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

6.3 Crystallographic groups

The following example computes

$H_5(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

for the 3-dimensional crystallographic space group G with Hermann-Mauguin symbol "P62"

Example

```
gap> GroupHomology(SpaceGroupBBNWZ("P62"),5);
[ 2, 2 ]
```

6.4 Arithmetic groups

The following example computes

$H_6(SL_2(\mathcal{O},\mathbb{Z}) = \mathbb{Z}_2$

for $\mathcal{O}$ the ring of integers of the number field $\mathbb{Q}(\sqrt{-2})$.

Example

```
gap> C:=ContractibleGcomplex("SL(2,O-2)");;
gap> R:=FreeGResolution(C,7);;
gap> Homology(TensorWithIntegers(R),6);
[ 2, 12 ]
```

6.5 Artin groups

The following example computes

$H_5(G,\mathbb{Z}) = \mathbb{Z}_3$

for G the classical braid group on eight strings.

Example
```
gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3]],[4,[5,3]],[5,[6,3]],[6,[7,3]]];;
gap> CoxeterDiagramDisplay(D);;
```

Example
```
gap> R:=ResolutionArtinGroup(D,6);;
gap> C:=TensorWithIntegers(R);;
gap> Homology(C,5);
[ 3 ]
```

6.6 Graphs of groups

The following example computes

$H_5(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$

for G the graph of groups corresponding to the amalgamated product $G = S_5 *_{S_3} S_4$ of the symmetric groups S_5 and S_4 over the canonical subgroup S_3.

Example
```
gap> S5:=SymmetricGroup(5);SetName(S5,"S5");
gap> S4:=SymmetricGroup(4);SetName(S4,"S4");
gap> A:=SymmetricGroup(3);SetName(A,"S3");
gap> AS5:=GroupHomomorphismByFunction(A,S5,x->x);
gap> AS4:=GroupHomomorphismByFunction(A,S4,x->x);
gap> D:=[S5,S4,[AS5,AS4]];
gap> GraphOfGroupsDisplay(D);
```

Example
```
gap> R:=ResolutionGraphOfGroups(D,6);;
gap> Homology(TensorWithIntegers(R),5);
[ 2, 2, 2, 2, 2 ]
```

Chapter 7

Cohomology operations

7.1 Steenrod operations on the classifying space of a finite 2-group

The following example determines a presentation for the cohomology ring $H^*(Syl_2(M_{12}), \mathbb{Z}_2)$. The Lyndon-Hochschild-Serre spectral sequence, and Groebner basis routines from SINGULAR, are used to determine how much of a resolution to compute for the presentation.

Example

```
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> Mod2CohomologyRingPresentation(G);
Graded algebra GF(2)[ x_1, x_2, x_3, x_4, x_5, x_6, x_7 ] /
[ x_2*x_3, x_1*x_2, x_2*x_4, x_3^3+x_3*x_5,
  x_1^2*x_4+x_1*x_3*x_4+x_3^2*x_4+x_3^2*x_5+x_1*x_6+x_4^2+x_4*x_5,
  x_1^2*x_3^2+x_1*x_3*x_5+x_3^2*x_5+x_3*x_6,
  x_1^3*x_3+x_3^2*x_4+x_3^2*x_5+x_1*x_6+x_3*x_6+x_4*x_5,
  x_1*x_3^2*x_4+x_1*x_3*x_6+x_1*x_4*x_5+x_3*x_4^2+x_3*x_4*x_5+x_3*x_5^\
2+x_4*x_6, x_1^2*x_3*x_5+x_1*x_3^2*x_5+x_3^2*x_6+x_3*x_5^2,
  x_3^2*x_4^2+x_3^2*x_5^2+x_1*x_5*x_6+x_3*x_4*x_6+x_4*x_5^2,
  x_1*x_3*x_4^2+x_1*x_3*x_4*x_5+x_1*x_3*x_5^2+x_3^2*x_5^2+x_1*x_4*x_6+\
x_2^2*x_7+x_2*x_5*x_6+x_3*x_4*x_6+x_3*x_5*x_6+x_4^2*x_5+x_4*x_5^2+x_6^\
2, x_1*x_3^2*x_6+x_3^2*x_4*x_5+x_1*x_5*x_6+x_4*x_5^2,
  x_1^2*x_3*x_6+x_1*x_5*x_6+x_2^2*x_7+x_2*x_5*x_6+x_3*x_5*x_6+x_6^2
 ] with indeterminate degrees [ 1, 1, 1, 2, 2, 3, 4 ]
```

The command `CohomologicalData(G,n)` prints complete information for the cohomology ring $H^*(G, Z_2)$ of a 2-group G provided that the integer n is at least the maximal degree of a relator in a minimal set of relators for the ring. Groebner basis routines from SINGULAR are called involved in the example.

The following example produces complete information on the Steenrod algebra of group number 8 in GAP's library of groups of order 32.

Example

```
Group number: 8
Group description: C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2)

Cohomology generators
Degree 1: a, b
Degree 2: c, d
```

```
Degree 3: e
Degree 5: f, g
Degree 6: h
Degree 8: p

Cohomology relations
1: f^2
2: c*h+e*f
3: c*f
4: b*h+c*g
5: b*e+c*d
6: a*h
7: a*g
8: a*f+b*f
9: a*e+c^2
10: a*c
11: a*b
12: a^2
13: d*e*h+e^2*g+f*h
14: d^2*h+d*e*f+d*e*g+f*g
15: c^2*d+b*f
16: b*c*g+e*f
17: b*c*d+c*e
18: b^2*g+d*f
19: b^2*c+c^2
20: b^3+a*d
21: c*d^2*e+c*d*g+d^2*f+e*h
22: c*d^3+d*e^2+d*h+e*f+e*g
23: b^2*d^2+c*d^2+b*f+e^2
24: b^3*d
25: d^3*e^2+d^2*e*f+c^2*p+h^2
26: d^4*e+b*c*p+e^2*g+g*h
27: d^5+b*d^2*g+b^2*p+f*g+g^2

Poincare series
(x^5+x^2+1)/(x^8-2*x^7+2*x^6-2*x^5+2*x^4-2*x^3+2*x^2-2*x+1)

Steenrod squares
Sq^1(c)=0
Sq^1(d)=b*b*b+d*b
Sq^1(e)=c*b*b
Sq^2(e)=e*d+f
Sq^1(f)=c*d*b*b+d*d*b*b
Sq^2(f)=g*b*b
Sq^4(f)=p*a
Sq^1(g)=d*d*d+g*b
Sq^2(g)=0
Sq^4(g)=c*d*d*d*b+g*d*b*b+g*d*d+p*a+p*b
Sq^1(h)=c*d*d*b+e*d*d
Sq^2(h)=d*d*d*b*b+c*d*d*d+g*c*b
Sq^4(h)=d*d*d*d*b*b+g*e*d+p*c
Sq^1(p)=c*d*d*d*b
```

```
Sq^2(p)=d*d*d*d*b*b+c*d*d*d*d
Sq^4(p)=d*d*d*d*d*b*b+d*d*d*d*d*d+g*d*d*d*b+g*g*d+p*d*d
```

7.2 Steenrod operations on the classifying space of a finite p-group

The following example constructs the first eight degrees of the mod-3 cohomology ring $H^*(G,\mathbb{Z}_3)$ for the group G number 4 in GAP's library of groups of order 81. It determines a minimal set of ring generators lying in degree ≤ 8 and it evaluates the Bockstein operator on these generators. Steenrod powers for $p \geq 3$ are not implemented as no efficient method of implementation is known.

Example

```
gap> G:=SmallGroup(81,4);;
gap> A:=ModPSteenrodAlgebra(G,8);;
gap> List(ModPRingGenerators(A),x->Bockstein(A,x));
[ 0*v.1, 0*v.1, v.5, 0*v.1, (Z(3))*v.7+v.8+(Z(3))*v.9 ]
```

Chapter 8

Bredon homology

8.1 Davis complex

The following example computes the Bredon homology

$\underline{H}_0(W,\mathscr{R}) = \mathbb{Z}^{21}$

for the infinite Coxeter group W associated to the Dynkin diagram shown in the computation, with coefficients in the complex representation ring.

Example
```
gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3]],[4,[5,6]]];;
gap> CoxeterDiagramDisplay(D);
```

Example
```
gap> C:=DavisComplex(D);;
gap> D:=TensorWithComplexRepresentationRing(C);;
gap> Homology(D,0);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

8.2 Arithmetic groups

The following example computes the Bredon homology

$\underline{H}_0(SL_2(\mathscr{O}_{-3}),\mathscr{R}) = \mathbb{Z}_2 \oplus \mathbb{Z}^9$

$\underline{H}_1(SL_2(\mathscr{O}_{-3}),\mathscr{R}) = \mathbb{Z}$

for $\mathscr{O}_{-3}$ the ring of integers of the number field $\mathbb{Q}(\sqrt{-3})$, and $\mathscr{R}$ the complex reflection ring.

Example
```
gap> R:=ContractibleGcomplex("SL(2,O-3)");;
gap> IsRigid(R);
false
gap> S:=BaryCentricSubdivision(R);;
gap> IsRigid(S);
true
gap> C:=TensorWithComplexRepresentationRing(S);;
gap> Homology(C,0);
[ 2, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Homology(C,1);
```

```
[ 0 ]
```

8.3 Crystallographic groups

The following example computes the Bredon homology

$\underline{H}_0(G,\mathscr{R}) = \mathbb{Z}^{17}$

for G the second crystallographic group of dimension 4 in GAP's library of crystallographic groups, and for $\mathscr{R}$ the Burnside ring.

Example

```
gap> G:=SpaceGroup(4,2);;
gap> gens:=GeneratorsOfGroup(G);;
gap> B:=CrystGFullBasis(G);;
gap> R:=CrystGcomplex(gens,B,1);;
gap> IsRigid(R);
false
gap> S:=CrystGcomplex(gens,B,0);;
gap> IsRigid(S);
true
gap> D:=TensorWithBurnsideRing(S);;
gap> Homology(D,0);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

Chapter 9

Simplicial groups

9.1 Crossed modules

A *crossed module* consists of a homomorphism of groups $\partial\colon M \to G$ together with an action $(g,m) \mapsto {}^g m$ of G on M satisfying

1. $\partial({}^g m) = gmg^{-1}$
2. ${}^{\partial m} m' = mm'm^{-1}$

for $g \in G$, $m, m' \in M$.

A crossed module $\partial\colon M \to G$ is equivalent to a cat^1-group (H,s,t) (see 5.7) where $H = M \rtimes G$, $s(m,g) = (1,g)$, $t(m,g) = (1,(\partial m)g)$. A cat^1-group is, in turn, equivalent to a simplicial group with Moore complex has length 1. The simplicial group is constructed by considering the cat^1-group as a category and taking its nerve. Alternatively, the simplicial group can be constructed by viewing the crossed module as a crossed complex and using a nonabelian version of the Dold-Kan theorem.

The following example concerns the crossed module
$\partial\colon G \to Aut(G), g \mapsto (x \mapsto gxg^{-1})$
associated to the dihedral group G of order 16. This crossed module represents, up to homotopy type, a connected space X with $\pi_i X = 0$ for $i \geq 3$, $\pi_2 X = Z(G)$, $\pi_1 X = Aut(G)/Inn(G)$. The space X can be represented, up to homotopy, by a simplicial group. That simplicial group is used in the example to compute
$H_1(X,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_2(X,\mathbb{Z}) = \mathbb{Z}_2$,
$H_3(X,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_4(X,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_5(X,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Example
```
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(16));
Cat-1-group with underlying group Group(
[ f1, f2, f3, f4, f5, f6, f7, f8, f9 ] ) .

gap> Size(C);
512
gap> Q:=QuasiIsomorph(C);
Cat-1-group with underlying group Group( [ f9, f8, f1, f2*f3, f5 ] ) .

```

```
gap> Size(Q);
32

gap> N:=NerveOfCatOneGroup(Q,6);
Simplicial group of length 6

gap> K:=ChainComplexOfSimplicialGroup(N);
Chain complex of length 6 in characteristic 0 .

gap> Homology(K,1);
[ 2, 2 ]
gap> Homology(K,2);
[ 2 ]
gap> Homology(K,3);
[ 2, 2, 2 ]
gap> Homology(K,4);
[ 2, 2, 2 ]
gap> Homology(K,5);
[ 2, 2, 2, 2, 2, 2 ]
```

9.2 Eilenberg-MacLane spaces as simplicial groups (not recommended)

The following example concerns the Eilenberg-MacLane space $X = K(\mathbb{Z}_3, 3)$ which is a path-connected space with $\pi_3 X = \mathbb{Z}_3$, $\pi_i X = 0$ for $3 \neq i \geq 1$. This space is represented by a simplicial group, and perturbation techniques are used to compute
$H_7(X,\mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Example

```
gap> A:=AbelianGroup([3]);;AbelianInvariants(A);
[ 3 ]
gap>  K:=EilenbergMacLaneSimplicialGroup(A,3,8);
Simplicial group of length 8

gap> C:=ChainComplex(K);
Chain complex of length 8 in characteristic 0 .

gap> Homology(C,7);
[ 3, 3 ]
```

9.3 Eilenberg-MacLane spaces as simplicial free abelian groups (recommended)

For integer $n > 1$ and abelian group A the Eilenberg-MacLane space $K(A,n)$ is better represented as a simplicial free abelian group. (The reason is that the functorial bar resolution of A can be replaced in computations by the smaller functorial Chevalley-Eilenberg complex of A, obviating the need for perturbation techniques.)

The following commands compute the integral homology $H_n(K(\mathbb{Z},3),\mathbb{Z})$ for $0 \le n \le 16$. (Note that one typically needs fewer than n terms of the Eilenberg-MacLance space to compute its n-th homology – an error is printed if too few terms of the space are available for a given computation.)

Example
```
gap> A:=AbelianPcpGroup([0]);; #infinite cyclic group
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,3,14);
Simplicial free abelian group of length 14

gap> for n in [0..16] do
> Print("Degree ",n," integral homology of K is ",Homology(K,n),"\n");
> od;
Degree 0 integral homology of K is [ 0 ]
Degree 1 integral homology of K is [  ]
Degree 2 integral homology of K is [  ]
Degree 3 integral homology of K is [ 0 ]
Degree 4 integral homology of K is [  ]
Degree 5 integral homology of K is [ 2 ]
Degree 6 integral homology of K is [  ]
Degree 7 integral homology of K is [ 3 ]
Degree 8 integral homology of K is [ 2 ]
Degree 9 integral homology of K is [ 2 ]
Degree 10 integral homology of K is [ 3 ]
Degree 11 integral homology of K is [ 5, 2 ]
Degree 12 integral homology of K is [ 2 ]
Degree 13 integral homology of K is [  ]
Degree 14 integral homology of K is [ 10, 2 ]
Degree 15 integral homology of K is [ 7, 6 ]
Degree 16 integral homology of K is [  ]
```

For an n-connected pointed space X the Freudenthal Suspension Theorem states that the map $X \to \Omega(\Sigma X)$ induces a map $\pi_k(X) \to \pi_k(\Omega(\Sigma X))$ which is an isomorphism for $k \le 2n$ and epimorphism for $k = 2n+1$. Thus the Eilenberg-MacLane space $K(A,n+1)$ can be constructed from the suspension $\Sigma K(A,n)$ by attaching cells in dimensions $\ge 2n+1$. In particular, there is an isomorphism $H_{k-1}(K(A,n),\mathbb{Z}) \to H_k(K(A,n+1),\mathbb{Z})$ for $k \le 2n$ and epimorphism for $k = 2n+1$.

For instance, $H_{k-1}(K(\mathbb{Z},3),\mathbb{Z}) \cong H_k(K(\mathbb{Z},4),\mathbb{Z})$ for $k \le 6$ and $H_6(K(\mathbb{Z},3),\mathbb{Z}) \twoheadrightarrow H_7(K(\mathbb{Z},4),\mathbb{Z})$. This assertion is seen in the following session.

Example
```
gap> A:=AbelianPcpGroup([0]);; #infinite cyclic group
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,4,11);
Simplicial free abelian group of length 11

gap> for n in [0..13] do
> Print("Degree ",n," integral homology of K is ",Homology(K,n),"\n");
> od;
Degree 0 integral homology of K is [ 0 ]
Degree 1 integral homology of K is [  ]
Degree 2 integral homology of K is [  ]
Degree 3 integral homology of K is [  ]
Degree 4 integral homology of K is [ 0 ]
Degree 5 integral homology of K is [  ]
Degree 6 integral homology of K is [ 2 ]
```

```
Degree 7 integral homology of K is [  ]
Degree 8 integral homology of K is [ 3, 0 ]
Degree 9 integral homology of K is [  ]
Degree 10 integral homology of K is [ 2, 2 ]
Degree 11 integral homology of K is [  ]
Degree 12 integral homology of K is [ 5, 12, 0 ]
Degree 13 integral homology of K is [ 2 ]
```

9.4 Elementary theoretical information on $H^*(K(\pi,n),\mathbb{Z})$

The cup product is not implemented for the cohomology ring $H^*(K(\pi,n),\mathbb{Z})$. Standard theoretical spectral sequence arguments have to be applied to obtain basic information relating to the ring structure. To illustrate this the following commands compute $H^n(K(\mathbb{Z},2),\mathbb{Z})$ for the first few values of n.

Example
```
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,2,10);;
gap> List([0..10],k->Cohomology(K,k));
[ [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ] ]
```

There is a fibration sequence $K(\pi,n) \hookrightarrow * \twoheadrightarrow K(\pi,n+1)$ in which $*$ denotes a contractible space. For $n=1, \pi=\mathbb{Z}$ the terms of the E_2 page of the Serre integral cohomology spectral sequence for this fibration are

- $E_2^{pq} = H^p(K(\mathbb{Z},2),H^q(K(\mathbb{Z},1),\mathbb{Z}))$.

Since $K(\mathbb{Z},1)$ can be taken to be the circle S^1 we know that it has non-trivial cohomology in degrees 0 and 1 only. The first few terms of the E_2 page are given in the following table.

1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
q/p	0	1	2	3	4	5	6	7	8	9	10

Table: E^2 cohomology page for $K(\mathbb{Z},1) \hookrightarrow * \twoheadrightarrow K(\mathbb{Z},2)$

Let x denote the generator of $H^1(K(\mathbb{Z},1),\mathbb{Z})$ and y denote the generator of $H^2(K(\mathbb{Z},2),\mathbb{Z})$. Since $*$ has zero cohomology in degrees ≥ 1 we see that the differential must restrict to an isomorphism $d_2\colon E_2^{0,1} \to E_2^{2,0}$ with $d_2(x)=y$. Then we see that the differential must restrict to an isomorphism $d_2\colon E_2^{2,1} \to E_2^{4,0}$ defined on the generator xy of $E_2^{2,1}$ by

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)}xd_2(y) = y^2 \ .$$

Hence $E_2^{4,0} \cong H^4(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^2. The argument extends to show that $H^6(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^3, $H^8(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^4, and so on.

In fact, to obtain a complete description of the ring $H^*(K(\mathbb{Z},2),\mathbb{Z})$ in this fashion there is no benefit to using computer methods at all. We only need to know the cohomology ring $H^*(K(\mathbb{Z},1),\mathbb{Z}) = H^*(S^1,\mathbb{Z})$ and the single cohomology group $H^2(K(\mathbb{Z},2),\mathbb{Z})$.

A similar approach can be attempted for $H^*(K(\mathbb{Z},3),\mathbb{Z})$ using the fibration sequence $K(\mathbb{Z},2) \hookrightarrow * \twoheadrightarrow K(\mathbb{Z},3)$ and, as explained in Chapter 5 of [Hat01], yields the computation of the group

$H^i(K(\mathbb{Z},3),\mathbb{Z})$ for $4 \le i \le 13$. The method does not directly yield $H^3(K(\mathbb{Z},3),\mathbb{Z})$ and breaks down in degree 14 yielding only that $H^{14}(K(\mathbb{Z},3),\mathbb{Z}) = 0$ or $\mathbb{Z}_3$. The following commands provide $H^3(K(\mathbb{Z},3),\mathbb{Z}) = \mathbb{Z}$ and $H^{14}(K(\mathbb{Z},3),\mathbb{Z}) = 0$.

Example
```
gap> A:=AbelianPcpGroup([0]);;
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,3,15);;
gap> Cohomology(K,3);
[ 0 ]
gap> Cohomology(K,14);
[  ]
```

However, the implementation of these commands is currently a bit naive, and computationally inefficient, since they do not currently employ any homological perturbation techniques.

9.5 The first three non-trivial homotopy groups of spheres

The Hurewicz Theorem immediately gives

$$\pi_n(S^n) \cong \mathbb{Z} \quad (n \ge 1)$$

and

$$\pi_k(S^n) = 0 \quad (k \le n-1).$$

As a CW-complex the Eilenberg-MacLane space $K = K(\mathbb{Z},n)$ can be obtained from an n-sphere $S^n = e^0 \cup e^n$ by attaching cells in dimensions $\ge n+2$ so as to kill the higher homotopy groups of S^n. From the inclusion $\iota: S^n \hookrightarrow K(\mathbb{Z},n)$ we can form the mapping cone $X = C(\iota)$. The long exact homotopy sequence

$\cdots \to \pi_{k+1}K \to \pi_{k+1}(K,S^n) \to \pi_k S^n \to \pi_k K \to \pi_k(K,S^n) \to \cdots$

implies that $\pi_k(K,S^n) = 0$ for $0 \le k \le n+1$ and $\pi_{n+2}(K,S^n) \cong \pi_{n+1}(S^n)$. The relative Hurewicz Theorem gives an isomorphism $\pi_{n+2}(K,S^n) \cong H_{n+2}(K,S^n,\mathbb{Z})$. The long exact homology sequence

$\cdots H_{n+2}(S^n,\mathbb{Z}) \to H_{n+2}(K,\mathbb{Z}) \to H_{n+2}(K,S^n,\mathbb{Z}) \to H_{n+1}(S^n,\mathbb{Z}) \to \cdots$

arising from the cofibration $S^n \hookrightarrow K \twoheadrightarrow X$ implies that $\pi_{n+1}(S^n) \cong \pi_{n+2}(K,S^n) \cong H_{n+2}(K,S^n,\mathbb{Z}) \cong H_{n+2}(K,\mathbb{Z})$. From the GAP computations in 9.3 and the Freudenthal Suspension Theorem we find:

$$\pi_3 S^2 \cong \mathbb{Z}, \qquad \pi_{n+1}(S^n) \cong \mathbb{Z}_2 \quad (n \ge 3).$$

The Hopf fibration $S^3 \to S^2$ has fibre $S^1 = K(\mathbb{Z},1)$. It can be constructed by viewing S^3 as all pairs $(z_1,z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ and viewing S^2 as $\mathbb{C} \cup \infty$; the map sends $(z_1,z_2) \mapsto z_1/z_2$. The homotopy exact sequence of the Hopf fibration yields $\pi_k(S^3) \cong \pi_k(S^2)$ for $k \ge 3$, and in particular

$$\pi_4(S^2) \cong \pi_4(S^3) \cong \mathbb{Z}_2 .$$

It will require further techniques (such as the Postnikov tower argument in Section 9.8 below) to establish that $\pi_5(S^3) \cong \mathbb{Z}_2$. Once we have this isomorphism for $\pi_5(S^3)$, the generalized Hopf fibration $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ comes into play. This fibration is contructed as for the classical fibration, but using pairs (z_1,z_2) of quaternions rather than pairs of complex numbers. The Hurewicz Theorem gives $\pi_3(S^7) = 0$; the fibre S^3 is thus homotopic to a point in S^7 and the inclusion of the fibre induces the zero homomorphism $\pi_k(S^3) \xrightarrow{0} \pi_k(S^7) \quad (k \ge 1)$. The exact homotopy sequence of the generalized Hopf fibration then gives $\pi_k(S^4) \cong \pi_k(S^7) \oplus \pi_{k-1}(S^3)$. On taking $k=6$ we obtain $\pi_6(S^4) \cong \pi_5(S^3) \cong \mathbb{Z}_2$. Freudenthal suspension then gives

$$\pi_{n+2}(S^n) \cong \mathbb{Z}_2, \quad (n \geq 2).$$

9.6 The first two non-trivial homotopy groups of the suspension and double suspension of a $K(G,1)$

For any group G we consider the homotopy groups $\pi_n(\Sigma K(G,1))$ of the suspension $\Sigma K(G,1)$ of the Eilenberg-MacLance space $K(G,1)$. On taking $G=\mathbb{Z}$, and observing that $S^2=\Sigma K(\mathbb{Z},1)$, we specialize to the homotopy groups of the 2-sphere S^2.

By construction,

$$\pi_1(\Sigma K(G,1)) = 0\ .$$

The Hurewicz Theorem gives

$$\pi_2(\Sigma K(G,1)) \cong G_{ab}$$

via the isomorphisms $\pi_2(\Sigma K(G,1)) \cong H_2(\Sigma K(G,1),\mathbb{Z}) \cong H_1(K(G,1),\mathbb{Z}) \cong G_{ab}$. R. Brown and J.-L. Loday [BL87] obtained the formulae

$$\pi_3(\Sigma K(G,1)) \cong \ker(G\otimes G \to G, x\otimes y \mapsto [x,y])\ ,$$

$$\pi_4(\Sigma^2 K(G,1)) \cong \ker(G\widetilde{\otimes} G \to G, x\widetilde{\otimes} y \mapsto [x,y])$$

involving the nonabelian tensor square and nonabelian symmetric square of the group G. The following commands use the nonabelian tensor and symmetric product to compute the third and fourth homotopy groups for $G=Syl_2(M_{12})$ the Sylow 2-subgroup of the Mathieu group M_{12}.

Example
```
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
gap>
gap> FourthHomotopyGroupOfDoubleSuspensionB(G);
[ 2, 2, 2, 2, 2, 2 ]
```

9.7 Postnikov towers and $\pi_5(S^3)$

A Postnikov system for the sphere S^3 consists of a sequence of fibrations $\cdots X_3 \stackrel{p_3}{\to} X_2 \stackrel{p_2}{\to} X_1 \stackrel{p_1}{\to} *$ and a sequence of maps $\phi_n\colon S^3 \to X_n$ such that

- $p_n \circ \phi_n = \phi_{n-1}$
- The map $\phi_n\colon S^3 \to X_n$ induces an isomorphism $\pi_k(S^3) \to \pi_k(X_n)$ for all $k \leq n$
- $\pi_k(X_n) = 0$ for $k > n$

- and consequently each fibration p_n has fibre an Eilenberg-MacLane space $K(\pi_n(S^3),n)$.

The space X_n is obtained from S^3 by adding cells in dimensions $\geq n+2$ and thus

- $H_k(X_n,\mathbb{Z}) = H_k(S^3,\mathbb{Z})$ for $k \leq n+1$.

So in particular $X_1 = X_2 = *, X_3 = K(\mathbb{Z},3)$ and we have a fibration sequence $K(\pi_4(S^3),4) \hookrightarrow X_4 \twoheadrightarrow K(\mathbb{Z},3)$. The terms in the E_2 page of the Serre integral cohomology spectral sequence of this fibration are

- $E_2^{p,q} = H^p(\,K(\mathbb{Z},3),\, H_q(K(\mathbb{Z}_2,4),\mathbb{Z})\,)$.

The first few terms in the E_2 page can be computed using the commands of Sections 9.2 and 9.3 and recorded as follows.

q/p	0	1	2	3	4	5	6	7	8	9
8	$\mathbb{Z}_2$	0	0							
7	$\mathbb{Z}_2$	0	0							
6	0	0	0							
5	$\pi_4(S^3)$	0	0	$\pi_4(S^3)$	0	0	0			
4	0	0	0	0	0	0				
3	0	0	0	0	0	0				
2	0	0	0	0	0	0	0	0		
1	0	0	0	0	0	0	0	0		
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$

Table: E_2 cohomology page for $K(\pi_4(S^3),4) \hookrightarrow X_4 \twoheadrightarrow X_3$

Since we know that $H^5(X_4,\mathbb{Z}) = 0$, the differentials in the spectral sequence must restrict to an isomorphism $E_2^{0,5} = \pi_4(S^3) \xrightarrow{\cong} E_2^{6,0} = \mathbb{Z}_2$. This provides an alternative derivation of $\pi_4(S^3) \cong \mathbb{Z}_2$. We can also immediately deduce that $H^6(X_4,\mathbb{Z}) = 0$. Let x be the generator of $E_2^{0,5}$ and y the generator of $E_2^{3,0}$. Then the generator xy of $E_2^{3,5}$ gets mapped to a non-zero element $d_7(xy) = d_7(x)y - xd_7(y)$. Hence the term $E_2^{0,7} = \mathbb{Z}_2$ must get mapped to zero in $E_2^{3,5}$. It follows that $H^7(X_4,\mathbb{Z}) = \mathbb{Z}_2$.

The integral cohomology of Eilenberg-MacLane spaces yields the following information on the E_2 page $E_2^{p,q} = H_p(X_4, H^q(K(\pi_5 S^3,5),\mathbb{Z})\,)$ for the fibration $K(\pi_5(S^3),5) \hookrightarrow X_5 \twoheadrightarrow X_4$.

q/p	0	1	2	3	4	5	6	7
6	$\pi_5(S^3)$	0	0	$\pi_5(S^3)$	0	0		
5	0	0	0	0	0	0	0	
4	0	0	0	0	0	0	0	
3	0	0	0	0	0	0	0	
2	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	$H^7(X_4,\mathbb{Z})$

Table: E_2 cohomology page for $K(\pi_5(S^3),5) \hookrightarrow X_5 \twoheadrightarrow X_4$

Since we know that $H^6(X_5,\mathbb{Z}) = 0$, the differentials in the spectral sequence must restrict to an isomorphism $E_2^{0,6} = \pi_5(S^3) \xrightarrow{\cong} E_2^{7,0} = H^7(X_4,\mathbb{Z})$. We can conclude the desired result:

$$\pi_5(S^3) = \mathbb{Z}_2\ .$$

Note that the fibration $X_4 \twoheadrightarrow K(\mathbb{Z},3)$ is determined by a cohomology class $\kappa \in H^5(K(\mathbb{Z},3),\mathbb{Z}_2) = \mathbb{Z}_2$. If $\kappa = 0$ then we'd have $X_4 = K(\mathbb{Z}_2,4) \times K(\mathbb{Z},3)$ and, as the following commands show, we'd then have $H_4(X_4,\mathbb{Z}) = \mathbb{Z}_2$.

Example
```
gap> K:=EilenbergMacLaneSimplicialGroup(AbelianPcpGroup([0]),3,7);;
gap> L:=EilenbergMacLaneSimplicialGroup(CyclicGroup(2),4,7);;
gap> CK:=ChainComplex(K);;
gap> CL:=ChainComplex(L);;
gap> T:=TensorProduct(CK,CL);;
gap> Homology(T,4);
[ 2 ]
```

Since we know that $H_4(X_4,\mathbb{Z}) = 0$ we can conclude that the Postnikov invariant κ is the non-zero class in $H^5(K(\mathbb{Z},3),\mathbb{Z}_2) = \mathbb{Z}_2$.

9.8 Towards $\pi_4(\Sigma K(G,1))$

Consider the suspension $X = \Sigma K(G,1)$ of a classifying space of a group G once again. This space has a Postnikov system in which $X_1 = *$, $X_2 = K(G_{ab},2)$. We have a fibration sequence $K(\pi_3 X,3) \hookrightarrow X_3 \twoheadrightarrow K(G_{ab},2)$. The corresponding integral cohomology Serre spectral sequence has E_2 page with terms

- $E_2^{p,q} = H^p(K(G_{ab},2),H^q(K(\pi_3 X,3)),\mathbb{Z})\,)$.

As an example, for the Alternating group $G = A_4$ of order 12 the following commands of Section 9.6 compute $G_{ab} = \mathbb{Z}_3$ and $\pi_3 X = \mathbb{Z}_6$.

Example
```
gap> AbelianInvariants(G);
[ 3 ]
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 2, 3 ]
```

The first terms of the E_2 page can be calculated using the commands of Sections 9.2 and 9.3.

q/p	0	1	2	3	4	5	6	7
7	$\mathbb{Z}_2$	0						
6	$\mathbb{Z}_2$	0	0	0				
5	0	0	0	0				
4	$\mathbb{Z}_6$	0	0	$\mathbb{Z}_3$				
3	0	0	0	0	0	0		
2	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_9$

Table: E^2 cohomology page for $K(\pi_3 X,3) \hookrightarrow X_3 \twoheadrightarrow X_2$

We know that $H^1(X_3,\mathbb{Z})=0$, $H^2(X_3,\mathbb{Z})=H^1(G,\mathbb{Z})=0$, $H^3(X_3,\mathbb{Z})=H^2(G,\mathbb{Z})=\mathbb{Z}_3$, and that $H^4(X_3,\mathbb{Z})$ is a subgroup of $H^3(G,\mathbb{Z})=\mathbb{Z}_2$. It follows that the differential induces a surjection $E_2^{0,4}=\mathbb{Z}_6 \twoheadrightarrow E_2^{5,0}=\mathbb{Z}_3$. Consequently $H^4(X_3,\mathbb{Z})=\mathbb{Z}_2$ and $H^5(X_3,\mathbb{Z})=0$ and $H^6(X_3,\mathbb{Z})=\mathbb{Z}_2$.

The E_2 page for the fibration $K(\pi_4X,4)\hookrightarrow X_4 \twoheadrightarrow X_3$ contains the following terms.

5	π_4X	0	0				
4	0	0	0	0			
3	0	0	0	0	0	0	
2	0	0	0	0	0	0	
1	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
q/p	0	1	2	3	4	5	6

Table: E^2 cohomology page for $K(\pi_4X,4)\hookrightarrow X_4 \twoheadrightarrow X_3$

We know that $H^5(X_4,\mathbb{Z})$ is a subgroup of $H^4(G,\mathbb{Z})=\mathbb{Z}_6$, and hence that there is a homomorphisms $\pi_4X\rightarrow\mathbb{Z}_2$ whose kernel is a subgroup of $\mathbb{Z}_6$. If follows that $|\pi_4X|\le 12$.

9.9 Enumerating homotopy 2-types

A *2-type* is a CW-complex X whose homotopy groups are trivial in dimensions $n=0$ and $n>2$. As explained in 5.7 the homotopy type of such a space can be captured algebraically by a cat^1-group G. Let X, Y be 2-tytpes represented by cat^1-groups G, H. If X and Y are homotopy equivalent then there exists a sequence of morphisms of cat^1-groups

$$G\rightarrow K_1\rightarrow K_2\leftarrow K_3\rightarrow\cdots\rightarrow K_n\leftarrow H$$

in which each morphism induces isomorphisms of homotopy groups. When such a sequence exists we say that G is *quasi-isomorphic* to H. We have the following result.

THEOREM. The 2-types X and Y are homotopy equivalent if and only if the associated cat^1-groups G and H are quasi-isomorphic.

The following commands produce a list L of all of the 62 non-isomorphic cat^1-groups whose underlying group has order 16.

Example
```
gap> L:=[];;
gap> for G in AllSmallGroups(16) do
> Append(L,CatOneGroupsByGroup(G));
> od;
gap> Length(L);
62
```

The next commands use the first and second homotopy groups to prove that the list L contains at least 37 distinct quasi-isomorphism types.

Example
```
gap> Invariants:=function(G)
> local inv;
> inv:=[];
> inv[1]:=IdGroup(HomotopyGroup(G,1));
```

```
> inv[2]:=IdGroup(HomotopyGroup(G,2));
> return inv;
> end;;

gap> C:=Classify(L,Invariants);;
gap> Length(C);
```

The following additional commands use second and third integral homology in conjunction with the first two homotopy groups to prove that the list L contains AT LEAST 49 distinct quasi-isomorphism types.

Example
```
gap> Invariants2:=function(G)
> local inv;
> inv:=[];
> inv[1]:=Homology(G,2);
> inv[2]:=Homology(G,3);
> return inv;
> end;;
gap> C:=RefineClassification(C,Invariants2);;

gap> Length(C);
49
```

The following commands show that the above list L contains AT MOST 51 distinct quasi-isomorphism types.

Example
```
gap> Q:=List(L,QuasiIsomorph);;
gap> M:=[];;

gap> for q in Q do
> bool:=true;;
> for m in M do
> if not IsomorphismCatOneGroups(m,q)=fail then bool:=false; break; fi;
> od;
> if bool then Add(M,q); fi;
> od;

gap> Length(M);
51
```

9.10 Identifying cat^1-groups of low order

Let us define the *order* of a cat^1-group to be the order of its underlying group. The function `IdQuasiCatOneGroup(C)` inputs a cat^1-group C of "low order" and returns an integer pair $[n,k]$ that uniquely idenifies the quasi-isomorphism type of C. The integer n is the order of a smallest cat^1-group quasi-isomorphic to C. The integer k identifies a particular cat^1-group of order n.

The following commands use this function to show that there are precisely 49 distinct quasi-isomorphism types of cat^1-groups of order 16.

Example
```
gap> L:=[];;
gap> for G in AllSmallGroups(16) do
> Append(L,CatOneGroupsByGroup(G));
> od;
gap> M:=List(L,IdQuasiCatOneGroup);
[ [ 16, 1 ], [ 16, 2 ], [ 16, 3 ], [ 16, 4 ], [ 16, 5 ], [ 4, 4 ], [ 1, 1 ],
  [ 16, 6 ], [ 16, 7 ], [ 16, 8 ], [ 16, 9 ], [ 16, 10 ], [ 16, 11 ],
  [ 16, 9 ], [ 16, 12 ], [ 16, 13 ], [ 16, 14 ], [ 16, 15 ], [ 4, 1 ],
  [ 4, 2 ], [ 16, 16 ], [ 16, 17 ], [ 16, 18 ], [ 16, 19 ], [ 16, 20 ],
  [ 16, 21 ], [ 16, 22 ], [ 16, 23 ], [ 16, 24 ], [ 16, 25 ], [ 16, 26 ],
  [ 16, 27 ], [ 16, 28 ], [ 4, 3 ], [ 4, 1 ], [ 4, 4 ], [ 4, 4 ], [ 4, 2 ],
  [ 4, 5 ], [ 16, 29 ], [ 16, 30 ], [ 16, 31 ], [ 16, 32 ], [ 16, 33 ],
  [ 16, 34 ], [ 4, 3 ], [ 4, 4 ], [ 4, 4 ], [ 16, 35 ], [ 16, 36 ], [ 4, 3 ],
  [ 16, 37 ], [ 16, 38 ], [ 16, 39 ], [ 16, 40 ], [ 16, 41 ], [ 16, 42 ],
  [ 16, 43 ], [ 4, 3 ], [ 4, 4 ], [ 1, 1 ], [ 4, 5 ] ]
gap> Length(SSortedList(M));
49
```

The next example first identifies the order and the identity number of the cat^1-group C corresponding to the crossed module (see 9.1)

$$\iota : G \longrightarrow Aut(G), g \mapsto (x \mapsto gxg^{-1})$$

for the dihedral group G of order 10. It then realizes a smallest possible cat^1-group D of this quasi-isomorphism type.

Example
```
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(10));
Cat-1-group with underlying group Group( [ f1, f2, f3, f4, f5 ] ) .

gap> Order(C);
200
gap> IdCatOneGroup(C);
[ 200, 42, 4 ]
gap>
gap> IdQuasiCatOneGroup(C);
[ 2, 1 ]
gap> D:=SmallCatOneGroup(2,1);
Cat-1-group with underlying group Group( [ f1 ] ) .
```

9.11 Identifying crossed modules of low order

The following commands construct the crossed module $\partial : G \otimes G \to G$ involving the nonabelian tensor square of the dihedral group \$G\$ of order 10, identify it as being number 71 in the list of crossed modules of order 100, create a quasi-isomorphic crossed module of order 4, and finally construct the corresponding cat^1-group of order 100.

Example

```
gap> G:=DihedralGroup(10);;
gap> T:=NonabelianTensorSquareAsCrossedModule(G);
Crossed module with group homomorphism GroupHomomorphismByImages( Group(
[ f3*f1*f3^-1*f1^-1, f3*f2*f3^-1*f2^-1 ] ), Group( [ f1, f2 ] ),
[ f3*f1*f3^-1*f1^-1, f3*f2*f3^-1*f2^-1 ], [ <identity> of ..., f2^3 ] )

gap> IdCrossedModule(T);
[ 100, 71 ]
gap> Q:=QuasiIsomorph(T);
Crossed module with group homomorphism Pcgs([ f2 ]) -> [ <identity> of ... ]

gap> Order(Q);
4
gap> C:=CatOneGroupByCrossedModule(T);
Cat-1-group with underlying group Group( [ F1, F2, F1 ] ) .
```

Chapter 10

Congruence Subgroups, Cuspidal Cohomology and Hecke Operators

In this chapter we explain how HAP can be used to make computions about modular forms associated to congruence subgroups Γ of $SL_2(\mathbb{Z})$. Also, in Subsection 10.8 onwards, we demonstrate cohomology computations for the *Picard group* $SL_2(\mathbb{Z}[i])$, some *Bianchi groups* $PSL_2(\mathcal{O}_{-d})$ where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ for square free positive integer d, and some other groups of the form $SL_m(\mathcal{O})$, $GL_m(\mathcal{O})$, $PSL_m(\mathcal{O})$, $PGL_m(\mathcal{O})$, for $m = 2,3,4$ and certain $\mathcal{O} = \mathbb{Z}, \mathcal{O}_{-d}$.

10.1 Eichler-Shimura isomorphism

We begin by recalling the Eichler-Shimura isomorphism [Eic57][Shi59]

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \oplus E_k(\Gamma) \cong_{\mathsf{Hecke}} H^1(\Gamma, P_{\mathbb{C}}(k-2))$$

which relates the cohomology of groups to the theory of modular forms associated to a finite index subgroup Γ of $SL_2(\mathbb{Z})$. In subsequent sections we explain how to compute with the right-hand side of the isomorphism. But first, for completeness, let us define the terms on the left-hand side.

Let N be a positive integer. A subgroup Γ of $SL_2(\mathbb{Z})$ is said to be a *congruence subgroup* of level N if it contains the kernel of the canonical homomorphism $\pi_N : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$. So any congruence subgroup is of finite index in $SL_2(\mathbb{Z})$, but the converse is not true.

One congruence subgroup of particular interest is the group $\Gamma_1(N) = \ker(\pi_N)$, known as the *principal congruence subgroup* of level N. Another congruence subgroup of particular interest is the group $\Gamma_0(N)$ of those matrices that project to upper triangular matrices in $SL_2(\mathbb{Z}/N\mathbb{Z})$.

A *modular form* of weight k for a congruence subgroup Γ is a complex valued function on the upper-half plane, $f : \mathfrak{h} = \{z \in \mathbb{C} : Re(z) > 0\} \to \mathbb{C}$, satisfying:

- $f(\dfrac{az+b}{cz+d}) = (cz+d)^k f(z)$ for $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma$,
- f is 'holomorphic' on the *extended upper-half plane* $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ obtained from the upper-half plane by 'adjoining a point at each cusp'.

The collection of all weight k modular forms for Γ form a vector space $M_k(\Gamma)$ over $\mathbb{C}$.

A modular form f is said to be a *cusp form* if $f(\infty) = 0$. The collection of all weight k cusp forms for Γ form a vector subspace $S_k(\Gamma)$. There is a decomposition

$$M_k(\Gamma) \cong S_k(\Gamma) \oplus E_k(\Gamma)$$

involving a summand $E_k(\Gamma)$ known as the *Eisenstein space*. See [Ste07] for further introductory details on modular forms.

The Eichler-Shimura isomorphism is more than an isomorphism of vector spaces. It is an isomorphism of Hecke modules: both sides admit notions of *Hecke operators*, and the isomorphism preserves these operators. The bar on the left-hand side of the isomorphism denotes complex conjugation, or *anti-holomorphic* forms. See [Wie78] for a full account of the isomorphism.

On the right-hand side of the isomorphism, the $\mathbb{Z}\Gamma$-module $P_{\mathbb{C}}(k-2) \subset \mathbb{C}[x,y]$ denotes the space of homogeneous degree $k-2$ polynomials with action of Γ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(dx-by, -cx+ay) .$$

In particular $P_{\mathbb{C}}(0) = \mathbb{C}$ is the trivial module. Below we shall compute with the integral analogue $P_{\mathbb{Z}}(k-2) \subset \mathbb{Z}[x,y]$.

In the following sections we explain how to use the right-hand side of the Eichler-Shimura isomorphism to compute eigenvalues of the Hecke operators restricted to the subspace $S_k(\Gamma)$ of cusp forms.

10.2 Generators for $SL_2(\mathbb{Z})$ and the cubic tree

The matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$ and it is not difficult to devise an algorithm for expressing an arbitrary integer matrix A of determinant 1 as a word in S, T and their inverses. The following illustrates such an algorithm.

Example

```
gap> A:=[[4,9],[7,16]];;
gap> word:=AsWordInSL2Z(A);
[ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, -1 ], [ 0, 1 ] ],
  [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, 1 ], [ 0, 1 ] ], [ [ 0, 1 ], [ -1, 0 ] ],
  [ [ 1, -1 ], [ 0, 1 ] ], [ [ 1, -1 ], [ 0, 1 ] ], [ [ 1, -1 ], [ 0, 1 ] ],
  [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, 1 ], [ 0, 1 ] ], [ [ 1, 1 ], [ 0, 1 ] ] ]
gap> Product(word);
[ [ 4, 9 ], [ 7, 16 ] ]
```

It is convenient to introduce the matrix $U = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. The matrices S and U also generate $SL_2(\mathbb{Z})$. In fact we have a free presentation $SL_2(\mathbb{Z}) = \langle S, U \mid S^4 = U^6 = 1, S^2 = U^3 \rangle$.

The *cubic tree* $\mathscr{T}$ is a tree (*i.e.* a 1-dimensional contractible regular CW-complex) with countably infinitely many edges in which each vertex has degree 3. We can realize the cubic tree $\mathscr{T}$ by taking the left cosets of $\mathscr{U} = \langle U \rangle$ in $SL_2(\mathbb{Z})$ as vertices, and joining cosets $x\mathscr{U}$ and $y\mathscr{U}$ by an edge if, and only if, $x^{-1}y \in \mathscr{U}\,S\,\mathscr{U}$. Thus the vertex $\mathscr{U}$ is joined to $S\mathscr{U}$, $US\mathscr{U}$ and $U^2S\mathscr{U}$. The vertices of this tree are in one-to-one correspondence with all reduced words in S, U and U^2 that, apart from the identity, end in S.

From our realization of the cubic tree $\mathscr{T}$ we see that $SL_2(\mathbb{Z})$ acts on $\mathscr{T}$ in such a way that each vertex is stabilized by a cyclic subgroup conjugate to $\mathscr{U} = \langle U \rangle$ and each edge is stabilized by a cyclic subgroup conjugate to $\mathscr{S} = \langle S \rangle$.

In order to store this action of $SL_2(\mathbb{Z})$ on the cubic tree $\mathscr{T}$ we just need to record the following finite amount of information.

10.3 One-dimensional fundamental domains and generators for congruence subgroups

The modular group $\mathscr{M} = PSL_2(\mathbb{Z})$ is isomorphic, as an abstract group, to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. By the Kurosh subgroup theorem, any finite index subgroup $M \subset \mathscr{M}$ is isomorphic to the free product of finitely many copies of $\mathbb{Z}_2$s, $\mathbb{Z}_3$s and $\mathbb{Z}$s. A subset $\underline{x} \subset M$ is an *independent* set of subgroup generators if M is the free product of the cyclic subgroups $< x >$ as x runs over $\underline{x}$. Let us say that a set of elements in $SL_2(\mathbb{Z})$ is *projectively independent* if it maps injectively onto an independent set of subgroup generators $\underline{x} \subset \mathscr{M}$. The generating set $\{S, U\}$ for $SL_2(\mathbb{Z})$ given in the preceding section is projectively independent.

We are interested in constructing a set of generators for a given congruence subgroup Γ. If a small generating set for Γ is required then we should aim to construct one which is close to being projectively independent.

It is useful to invoke the following general result which follows from a perturbation result about free $\mathbb{Z}G$-resolutons in [EHS06, Theorem 2] and an old observation of John Milnor that a free $\mathbb{Z}G$-resolution can be realized as the cellular chain complex of a CW-complex if it can be so realized in low dimensions.

THEOREM. Let X be a contractible CW-complex on which a group G acts by permuting cells. The cellular chain complex C_*X is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which typically is not free. Let $[e^n]$ denote the orbit of the n-cell e^n under the action. Let $G^{e^n} \le G$ denote the stabilizer subgroup of e^n, in which group elements are not required to stabilize e^n point-wise. Let Y_{e^n} denote a contractible CW-complex on which G^{e^n} acts cellularly and freely. Then there exists a contractible CW-complex W on which G acts cellularly and freely, and in which the orbits of n-cells are labelled by $[e^p] \otimes [f^q]$ where $p+q=n$ and $[e^p]$ ranges over the G-orbits of p-cells in X, $[f^q]$ ranges over the G^{e^p}-orbits of q-cells in Y_{e^p}.

Let W be as in the theorem. Then the quotient CW-complex $B_G = W/G$ is a classifying space for G. Let T denote a maximal tree in the 1-skeleton B^1_G. Basic geometric group theory tells us that the 1-cells in $B^1_G \setminus T$ correspond to a generating set for G.

Suppose we wish to compute a set of generators for a principal congruence subgroup $\Gamma = \Gamma_1(N)$. In the above theorem take $X = \mathscr{T}$ to be the cubic tree, and note that Γ acts freely on $\mathscr{T}$ and thus that $W = \mathscr{T}$. To determine the 1-cells of $B_\Gamma \setminus T$ we need to determine a cellular subspace $D_\Gamma \subset \mathscr{T}$ whose images under the action of Γ cover $\mathscr{T}$ and are pairwise either disjoint or identical. The subspace D_Γ will not be a CW-complex as it won't be closed, but it can be chosen to be connected, and hence contractible. We call D_Γ a *fundamental region* for Γ. We denote by $\mathring{D}_\Gamma$ the largest CW-subcomplex of D_Γ. The vertices of $\mathring{D}_\Gamma$ are the same as the vertices of D_Γ. Thus $\mathring{D}_\Gamma$ is a subtree of the cubic tree with $|\Gamma|/6$ vertices. For each vertex v in the tree $\mathring{D}_\Gamma$ define $\eta(v) = 3 - \mathrm{degree}(v)$. Then the number of generators for Γ will be $(1/2)\sum_{v\in \mathring{D}_\Gamma} \eta(v)$.

The following commands determine projectively independent generators for $\Gamma_1(6)$ and display $\mathring{D}_{\Gamma_1(6)}$. The subgroup $\Gamma_1(6)$ is free on 13 generators.

Example

```
gap> G:=HAP_PrincipalCongruenceSubgroup(6);;
```

```
gap> gens:=GeneratorsOfGroup(G);
[ [ [ -83, -18 ], [ 60, 13 ] ], [ [ -77, -18 ], [ 30, 7 ] ],
  [ [ -65, -12 ], [ 168, 31 ] ], [ [ -53, -12 ], [ 84, 19 ] ],
  [ [ -47, -18 ], [ 222, 85 ] ], [ [ -41, -12 ], [ 24, 7 ] ],
  [ [ -35, -6 ], [ 6, 1 ] ], [ [ -11, -18 ], [ 30, 49 ] ],
  [ [ -11, -6 ], [ 24, 13 ] ], [ [ -5, -18 ], [ 12, 43 ] ],
  [ [ -5, -12 ], [ 18, 43 ] ], [ [ -5, -6 ], [ 6, 7 ] ],
  [ [ 1, 0 ], [ -6, 1 ] ] ]
```

An alternative but very related approach to computing generators of congruence subgroups of $SL_2(\mathbb{Z})$ is described in [Kul91].

The congruence subgroup $\Gamma_0(N)$ does not act freely on the vertices of $\mathscr{T}$, and so one needs to incorporate a generator for the cyclic stabilizer group according to the above theorem. Alternatively, we can replace the cubic tree by a six-fold cover $\mathscr{T}'$ on whose vertex set $\Gamma_0(N)$ acts freely. This alternative approach will produce a redundant set of generators. The following commands display $\mathring{D}_{\Gamma_0(39)}$ for a fundamental region in $\mathscr{T}'$. They also use the corresponding generating set for $\Gamma_0(39)$, involving 18 generators, to compute the abelianization $\Gamma_0(39)^{ab} = \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}^9$. The abelianization shows that any generating set has at least 11 generators.

Example
```
gap> G:=HAP_CongruenceSubgroupGamma0(39);;
gap> HAP_SL2TreeDisplay(G);
gap> Length(GeneratorsOfGroup(G));
18
gap> AbelianInvariants(G);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 3, 3 ]
```

Note that to compute D_Γ one only needs to be able to test whether a given matrix lies in Γ or not. Given an inclusion $\Gamma' \subset \Gamma$ of congruence subgroups, it is straightforward to use the trees $\mathring{D}_{\Gamma'}$ and $\mathring{D}_\Gamma$ to compute a system of coset representative for $\Gamma' \setminus \Gamma$.

10.4 Cohomology of congruence subgroups

To compute the cohomology $H^n(\Gamma, A)$ of a congruence subgroup Γ with coefficients in a $\mathbb{Z}\Gamma$-module A we need to construct $n+1$ terms of a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. We can do this by first using perturbation techniques (as described in [BE14]) to combine the cubic tree with resolutions for the cyclic groups of order 4 and 6 in order to produce a free $\mathbb{Z}G$-resolution R_* for $G = SL_2(\mathbb{Z})$. This resolution is also a free $\mathbb{Z}\Gamma$-resolution with each term of rank

$$\mathrm{rank}_{\mathbb{Z}\Gamma} R_k = |G : \Gamma| \times \mathrm{rank}_{\mathbb{Z}G} R_k \ .$$

For congruence subgroups of lowish index in G this resolution suffices to make computations. The following commands compute

$$H^1(\Gamma_0(39), \mathbb{Z}) = \mathbb{Z}^9 \ .$$

Example
```
gap> R:=ResolutionSL2Z_alt(2);
Resolution of length 2 in characteristic 0 for SL(2,Integers) .
```

```
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> S:=ResolutionFiniteSubgroup(R,gamma);
Resolution of length 2 in characteristic 0 for
CongruenceSubgroupGamma0( 39)  .

gap> Cohomology(HomToIntegers(S),1);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

This computation establishes that the space $M_2(\Gamma_0(39))$ of weight 2 modular forms is of dimension 9.

The following commands show that $\mathrm{rank}_{\mathbb{Z}\Gamma_0(39)} R_1 = 112$ but that it is possible to apply 'Tietze like' simplifications to R_* to obtain a free $\mathbb{Z}\Gamma_0(39)$-resolution T_* with $\mathrm{rank}_{\mathbb{Z}\Gamma_0(39)} T_1 = 11$. It is more efficient to work with T_* when making cohomology computations with coefficients in a module A of large rank.

Example

```
gap> S!.dimension(1);
112
gap> T:=TietzeReducedResolution(S);
Resolution of length 2 in characteristic 0 for CongruenceSubgroupGamma0(
39)  .

gap> T!.dimension(1);
11
```

The following commands compute

$$H^1(\Gamma_0(39), P_{\mathbb{Z}}(8)) = \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{168} \oplus \mathbb{Z}^{84} ,$$

$$H^1(\Gamma_0(39), P_{\mathbb{Z}}(9)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 .$$

Example

```
gap> P:=HomogeneousPolynomials(gamma,8);;
gap> c:=Cohomology(HomToIntegralModule(T,P),1);
[ 3, 6, 168, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Length(c);
87

gap> P:=HomogeneousPolynomials(gamma,9);;
gap> c:=Cohomology(HomToIntegralModule(T,P),1);
[ 2, 2 ]
```

This computation establishes that the space $M_{10}(\Gamma_0(39))$ of weight 10 modular forms is of dimension 84, and $M_{11}(\Gamma_0(39))$ is of dimension 0. (There are never any modular forms of odd weight, and so $M_k(\Gamma) = 0$ for all odd k and any congruence subgroup Γ.)

10.5 Cuspidal cohomology

To define and compute cuspidal cohomology we consider the action of $SL_2(\mathbb{Z})$ on the upper-half plane $\mathfrak{h}$ given by

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) z = \frac{az+b}{cz+d}\ .$$

A standard 'fundamental domain' for this action is the region

$$\begin{array}{rl} D = & \{z\in\mathfrak{h}\ :\ |z|>1, |\mathrm{Re}(z)|<\frac{1}{2}\} \\ & \cup\,\{z\in\mathfrak{h}\ :\ |z|\ge 1, \mathrm{Re}(z) = -\frac{1}{2}\} \\ & \cup\,\{z\in\mathfrak{h}\ :\ |z| = 1, -\frac{1}{2}\le \mathrm{Re}(z)\le 0\} \end{array}$$

illustrated below.

The action factors through an action of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle\left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right)\rangle$. The images of D under the action of $PSL_2(\mathbb{Z})$ cover the upper-half plane, and any two images have at most a single point in common. The possible common points are the bottom left-hand corner point which is stabilized by $\langle U\rangle$, and the bottom middle point which is stabilized by $\langle S\rangle$.

A congruence subgroup Γ has a 'fundamental domain' D_Γ equal to a union of finitely many copies of D, one copy for each coset in $\Gamma\setminus SL_2(\mathbb{Z})$. The quotient space $X=\Gamma\setminus\mathfrak{h}$ is not compact, and can be compactified in several ways. We are interested in the Borel-Serre compactification. This is a space X^{BS} for which there is an inclusion $X\hookrightarrow X^{BS}$ and this inclusion is a homotopy equivalence. One defines the *boundary* $\partial X^{BS} = X^{BS} - X$ and uses the inclusion $\partial X^{BS}\hookrightarrow X^{BS}\simeq X$ to define the cuspidal cohomology group, over the ground ring $\mathbb{C}$, as

$$H^n_{cusp}(\Gamma, P_{\mathbb{C}}(k-2)) = \ker(\ H^n(X, P_{\mathbb{C}}(k-2)) \to H^n(\partial X^{BS}, P_{\mathbb{C}}(k-2))\ ).$$

Strictly speaking, this is the definition of *interior cohomology* $H^n_!(\Gamma, P_{\mathbb{C}}(k-2))$ which in general contains the cuspidal cohomology as a subgroup. However, for congruence subgroups of $SL_2(\mathbb{Z})$ there is equality $H^n_!(\Gamma, P_{\mathbb{C}}(k-2)) = H^n_{cusp}(\Gamma, P_{\mathbb{C}}(k-2))$.

Working over $\mathbb{C}$ has the advantage of avoiding the technical issue that Γ does not necessarily act freely on $\mathfrak{h}$ since there are points with finite cyclic stabilizer groups in $SL_2(\mathbb{Z})$. But it has the disadvantage of losing information about torsion in cohomology. So HAP confronts the issue by working with a contractible CW-complex $\tilde{X}^{BS}$ on which Γ acts freely, and Γ-equivariant inclusion $\partial\tilde{X}^{BS}\hookrightarrow\tilde{X}^{BS}$. The definition of cuspidal cohomology that we use, which coincides with the above definition when working over $\mathbb{C}$, is

$$H^n_{cusp}(\Gamma, A) = \ker(\ H^n(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(\tilde{X}^{BS}), A)\,) \to H^n(\ \mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(\tilde{\partial} X^{BS}), A)\ )\ ).$$

The following data is recorded and, using perturbation theory, is combined with free resolutions for C_4 and C_6 to constuct $\tilde{X}^{BS}$.

The following commands calculate

$$H^1_{cusp}(\Gamma_0(39), \mathbb{Z}) = \mathbb{Z}^6\ .$$

Example
```
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> k:=2;; deg:=1;; c:=CuspidalCohomologyHomomorphism(gamma,deg,k);
[ g1, g2, g3, g4, g5, g6, g7, g8, g9 ] -> [ g1^-1*g3, g1^-1*g3, g1^-1*g3,
  g1^-1*g3, g1^-1*g2, g1^-1*g3, g1^-1*g4, g1^-1*g4, g1^-1*g4 ]
```

```
gap> AbelianInvariants(Kernel(c));
[ 0, 0, 0, 0, 0, 0 ]
```

From the Eichler-Shimura isomorphism and the already calculated dimension of $M_2(\Gamma_0(39)) \cong \mathbb{C}^9$, we deduce from this cuspidal cohomology that the space $S_2(\Gamma_0(39))$ of cuspidal weight 2 forms is of dimension 3, and the Eisenstein space $E_2(\Gamma_0(39)) \cong \mathbb{C}^3$ is of dimension 3.

The following commands show that the space $S_4(\Gamma_0(39))$ of cuspidal weight 4 forms is of dimension 12.

Example
```
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> k:=4;; deg:=1;; c:=CuspidalCohomologyHomomorphism(gamma,deg,k);;
gap> AbelianInvariants(Kernel(c));
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

10.6 Hecke operators

A congruence subgroup $\Gamma \le SL_m(\mathbb{Z})$ and element $g \in SL_m(\mathbb{Q})$ determine the subgroup $\Gamma' = \Gamma \cap g\Gamma g^{-1}$ and homomorphisms

$$\Gamma \hookleftarrow \Gamma' \xrightarrow{\gamma \mapsto g^{-1}\gamma g} g^{-1}\Gamma' g \hookrightarrow \Gamma\,.$$

These homomorphisms give rise to homomorphisms of cohomology groups

$$H^n(\Gamma,\mathbb{Z}) \xleftarrow{tr} H^n(\Gamma',\mathbb{Z}) \xleftarrow{\alpha} H^n(g^{-1}\Gamma' g,\mathbb{Z}) \xleftarrow{\beta} H^n(\Gamma,\mathbb{Z})$$

with α, β functorial maps, and tr the transfer map. We define the composite $T_g = tr \circ \alpha \circ \beta \colon H^n(\Gamma,\mathbb{Z}) \to H^n(\Gamma,\mathbb{Z})$ to be the *Hecke operator* determined by g. Further details on this description of Hecke operators can be found in [Ste07, Appendix by P. Gunnells].

For each integer $s \ge 1$ we set $T_s = T_s$ with for $g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$.

The following commands compute T_2 and T_5 for $n = 1$ and $\Gamma = \Gamma_0(39)$. The commands also compute the eigenvalues of these two Hecke operators. The final command confirms that T_2 and T_5 commute. (It is a fact that $T_pT_q = T_qT_p$ for all integers p,q.)

Example
```
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> p:=2;;N:=1;;h:=HeckeOperator(gamma,p,N);;
gap> AbelianInvariants(Source(h));
[ 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> T2:=HomomorphismAsMatrix(h);;
gap> Display(T2);
[ [  -2,  -2,   2,   2,   1,   2,   0,   0,   0 ],
  [  -2,   0,   1,   2,  -2,   2,   2,   2,  -2 ],
  [  -2,  -1,   2,   2,  -1,   2,   1,   1,  -1 ],
  [  -2,  -1,   2,   2,   1,   1,   0,   0,   0 ],
  [  -1,   0,   0,   2,  -3,   2,   3,   3,  -3 ],
  [   0,   1,   1,   1,  -1,   0,   1,   1,  -1 ],
  [  -1,   1,   1,  -1,   0,   1,   2,  -1,   1 ],
  [  -1,  -1,   0,   2,  -3,   2,   1,   4,  -1 ],
```

```
  [   0,   1,   0,  -1,  -2,   1,   1,   1,   2 ] ]
gap> Eigenvalues(Rationals,T2);
[ 3, 1 ]

gap> p:=5;;N:=1;;h:=HeckeOperator(gamma,p,N);;
gap> T5:=HomomorphismAsMatrix(h);;
gap> Display(T5);
[ [  -1,  -1,   3,   4,   0,   0,   1,   1,  -1 ],
  [  -5,  -1,   5,   4,   0,   0,   3,   3,  -3 ],
  [  -2,   0,   4,   4,   1,   0,  -1,  -1,   1 ],
  [  -2,   0,   3,   2,  -3,   2,   4,   4,  -4 ],
  [  -4,  -2,   4,   4,   3,   0,   1,   1,  -1 ],
  [  -6,  -4,   5,   6,   1,   2,   2,   2,  -2 ],
  [   1,   5,   0,  -4,  -3,   2,   5,  -1,   1 ],
  [  -2,  -2,   2,   4,   0,   0,  -2,   4,   2 ],
  [   1,   3,   0,  -4,  -4,   2,   2,   2,   4 ] ]
gap> Eigenvalues(Rationals,T5);
[ 6, 2 ]

gap>T2*T5=T5*T2;
true
```

10.7 Reconstructing modular forms from cohomology computations

Given a modular form $f\colon\mathfrak{h}\to\mathbb{C}$ associated to a congruence subgroup Γ, and given a compact edge e in the tessellation of $\mathfrak{h}$ (*i.e.* an edge in the cubic tree $\mathscr{T}$) arising from the above fundamental domain for $SL_2(\mathbb{Z})$, we can evaluate

$$\int_e f(z)\,dz\,.$$

In this way we obtain a cochain $f_1\colon C_1(\mathscr{T})\to\mathbb{C}$ in $Hom_{\mathbb{Z}\Gamma}(C_1(\mathscr{T}),\mathbb{C})$ representing a cohomology class $c(f)\in H^1(Hom_{\mathbb{Z}\Gamma}(C_*(\mathscr{T}),\mathbb{C}))=H^1(\Gamma,\mathbb{C})$. The correspondence $f\mapsto c(f)$ underlies the Eichler-Shimura isomorphism. Hecke operators can be used to recover modular forms from cohomology classes.

Hecke operators restrict to operators on cuspidal cohomology. On the left-hand side of the Eichler-Shimura isomorphism Hecke operators restrict to operators $T_s\colon S_2(\Gamma)\to S_2(\Gamma)$ for $s\geq 1$.

Let us now introduce the function $q=q(z)=e^{2\pi iz}$ which is holomorphic on $\mathbb{C}$. For any modular form $f(z)$ there are numbers a_n such that

$$f(z)=\sum_{s=0}^{\infty}a_sq^s$$

for all $z\in\mathfrak{h}$. The form f is a cusp form if $a_0=0$.

A non-zero cusp form $f\in S_2(\Gamma)$ is an *eigenform* if it is simultaneously an eigenvector for the Hecke operators T_s for all $s=1,2,3,\cdots$. An eigenform is said to be *normalized* if its coefficient $a_1=1$. It turns out that if f is a normalized eigenform then the coefficient a_s is an eigenvalue for T_s (see for instance [Ste07] for details). It can be shown [AL70] that $f\in S_2(\Gamma_0(N))$ admits a basis of eigenforms.

This all implies that, in principle, we can construct an approximation to an explicit basis for the space $S_2(\Gamma)$ of cusp forms by computing eigenvalues for Hecke operators.

Suppose that we would like a basis for $S_2(\Gamma_0(11))$. The following commands first show that $H^1_{cusp}(\Gamma_0(11),\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$ from which we deduce that $S_2(\Gamma_0(11)) = \mathbb{C}$ is 1-dimensional. Then eigenvalues of Hecke operators are calculated to establish that the modular form

$$f = q - 2q^2 - q^3 + q^4 + q^5 + 2q^6 - 2q^7 + 2q^8 - 3q^9 - 2q^{10} + \cdots$$

constitutes a basis for $S_2(\Gamma_0(11))$.

Example

```
gap> gamma:=HAP_CongruenceSubgroupGamma0(11);;
gap> AbelianInvariants(Kernel(CuspidalCohomologyHomomorphism(gamma,1,2)));
[ 0, 0 ]

gap> T1:=HomomorphismAsMatrix(HeckeOperator(gamma,1,1));; Display(T1);
[ [  1,  0,  0 ],
  [  0,  1,  0 ],
  [  0,  0,  1 ] ]
gap> T2:=HomomorphismAsMatrix(HeckeOperator(gamma,2,1));; Display(T2);
[ [   3,  -4,   4 ],
  [   0,  -2,   0 ],
  [   0,   0,  -2 ] ]
gap> T3:=HomomorphismAsMatrix(HeckeOperator(gamma,3,1));; Display(T3);
[ [   4,  -4,   4 ],
  [   0,  -1,   0 ],
  [   0,   0,  -1 ] ]
gap> T4:=HomomorphismAsMatrix(HeckeOperator(gamma,4,1));; Display(T4);
[ [   6,  -4,   4 ],
  [   0,   1,   0 ],
  [   0,   0,   1 ] ]
gap> T5:=HomomorphismAsMatrix(HeckeOperator(gamma,5,1));; Display(T5);
[ [   6,  -4,   4 ],
  [   0,   1,   0 ],
  [   0,   0,   1 ] ]
gap> T6:=HomomorphismAsMatrix(HeckeOperator(gamma,6,1));; Display(T6);
[ [  12,  -8,   8 ],
  [   0,   2,   0 ],
  [   0,   0,   2 ] ]
gap> T7:=HomomorphismAsMatrix(HeckeOperator(gamma,7,1));; Display(T7);
[ [   8,  -8,   8 ],
  [   0,  -2,   0 ],
  [   0,   0,  -2 ] ]
gap> T8:=HomomorphismAsMatrix(HeckeOperator(gamma,8,1));; Display(T8);
[ [  12,  -8,   8 ],
  [   0,   2,   0 ],
  [   0,   0,   2 ] ]
gap> T9:=HomomorphismAsMatrix(HeckeOperator(gamma,9,1));; Display(T9);
[ [   12,  -12,   12 ],
  [    0,   -3,    0 ],
  [    0,    0,   -3 ] ]
gap> T10:=HomomorphismAsMatrix(HeckeOperator(gamma,10,1));; Display(T10);
[ [   18,  -16,   16 ],
```

```
  [    0,   -2,    0 ],
  [    0,    0,   -2 ] ]
```

For a normalized eigenform $f = 1 + \sum_{s=2}^{\infty} a_s q^s$ the coefficients a_s with s a composite integer can be expressed in terms of the coefficients a_p for prime p. If r,s are coprime then $T_{rs} = T_r T_s$. If p is a prime that is not a divisor of the level N of Γ then $a_{p^m} = a_{p^{m-1}} a_p - p a_{p^{m-2}}$. If the prime p divides N then $a_{p^m} = (a_p)^m$. It thus suffices to compute the coefficients a_p for prime integers p only.

10.8 The Picard group

Let us now consider the *Picard group* $G = SL_2(\mathbb{Z}[i])$ and its action on *upper-half space*

$$\mathfrak{h}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\} .$$

To describe the action we introduce the symbol j satisfying $j^2 = -1$, $ij = -ji$ and write $z + tj$ instead of (z,t). The action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z+tj) \;=\; (a(z+tj)+b)\,(c(z+tj)+d)^{-1} \;.$$

Alternatively, and more explicitly, the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z+tj) \;=\; \frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2}{|cz+d|^2 + |c|^2y^2} \;+\; \frac{y}{|cz+d|^2+|c|^2y^2}\, j \;.$$

A standard 'fundamental domain' D for this action is the following region (with some of the boundary points removed).

$$\{z+tj \in \mathfrak{h}^3 \mid 0 \le |\mathrm{Re}(z)| \le \frac{1}{2}, 0 \le \mathrm{Im}(z) \le \frac{1}{2}, z\bar{z}+t^2 \ge 1\}$$

The four bottom vertices of D are $a = -\frac{1}{2} + \frac{1}{2}i + \frac{\sqrt{2}}{2}j$, $b = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$, $c = \frac{1}{2} + \frac{\sqrt{3}}{2}j$, $d = \frac{1}{2} + \frac{1}{2}i + \frac{\sqrt{2}}{2}j$.

The upper-half space $\mathfrak{h}^3$ can be retracted onto a 2-dimensional subspace $\mathscr{T} \subset \mathfrak{h}^3$. The space $\mathscr{T}$ is a contractible 2-dimensional regular CW-complex, and the action of the Picard group G restricts to a cellular action of G on $\mathscr{T}$. Under this action there is one orbit of 2-cells, represented by the curvilinear square with vertices a, b, c and d in the picture. This 2-cell has stabilizer group isomorphic to the quaternion group Q_4 of order 8. There are two orbits of 1-cells, both with stabilizer group isomorphic to a semi-direct product $C_3 : C_4$. There is one orbit of 0-cells, with stabilizer group isomorphic to $SL(2,3)$.

Using perturbation techniques, the 2-complex $\mathscr{T}$ can be combined with free resolutions for the cell stabilizer groups to contruct a regular CW-complex X on which the Picard group G acts freely. The following commands compute the first few terms of the free $\mathbb{Z}G$-resolution $R_* = C_*X$. Then R_* is used to compute

$$H^1(G,\mathbb{Z}) = 0 \;,$$

$$H^2(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \;,$$

$$H^3(G,\mathbb{Z}) = \mathbb{Z}_6\ ,$$

$$H^4(G,\mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_{24}\ ,$$

and compute a free presentation for G involving four generators and seven relators.

Example

```
gap> K:=ContractibleGcomplex("SL(2,O-1)");;
gap> R:=FreeGResolution(K,5);;
gap> Cohomology(HomToIntegers(R),1);
[  ]
gap> Cohomology(HomToIntegers(R),2);
[ 2, 2 ]
gap> Cohomology(HomToIntegers(R),3);
[ 6 ]
gap> Cohomology(HomToIntegers(R),4);
[ 4, 24 ]
gap> P:=PresentationOfResolution(R);
rec( freeGroup := <free group on the generators [ f1, f2, f3, f4 ]>,
  gens := [ 184, 185, 186, 187 ],
  relators := [ f1^2*f2^-1*f1^-1*f2^-1, f1*f2*f1*f2^-2,
      f3*f2^2*f1*(f2*f1^-1)^2*f3^-1*f1^2*f2^-2,
      f1*(f2*f1^-1)^2*f3^-1*f1^2*f2^-1*f3^-1,
      f4*f2*f1*(f2*f1^-1)^2*f4^-1*f1*f2^-1, f1*f4^-1*f1^-2*f4^-1,
      f3*f2*f1*(f2*f1^-1)^2*f4^-1*f1*f2^-1*f3^-1*f4*f2 ] )
```

We can also compute the cohomology of $G = SL_2(\mathbb{Z}[i])$ with coefficients in a module such as the module $P_{\mathbb{Z}[i]}(k)$ of degree k homogeneous polynomials with coefficients in $\mathbb{Z}[i]$ and with the action described above. For instance, the following commands compute

$$H^1(G,P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{40} \oplus \mathbb{Z}_{80}\,,$$

$$H^2(G,P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^{24} \oplus \mathbb{Z}_{520030} \oplus \mathbb{Z}_{1040060} \oplus \mathbb{Z}^2\,,$$

$$H^3(G,P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^{22} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_{12})^2\,.$$

Example

```
gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 4, 8, 40, 80 ]
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  520030, 1040060, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 12, 12
 ]
```

10.9 Bianchi groups

The *Bianchi groups* are the groups $G = PSL_2(\mathcal{O}_{-d})$ where d is a square free positive integer and $\mathcal{O}_{-d}$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. More explicitly,

$$\mathcal{O}_{-d} = \mathbb{Z}\left[\sqrt{-d}\right] \qquad \text{if } d \equiv 1 \bmod 4\,,$$

$$\mathcal{O}_{-d} = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right] \qquad \text{if } d \equiv 2,3 \bmod 4\,.$$

These groups act on upper-half space $\mathfrak{h}^3$ in the same way as the Picard group. Upper-half space can be tessellated by a 'fundamental domain' for this action. Moreover, as with the Picard group, this tessellation contains a 2-dimensional cellular subspace $\mathcal{T} \subset \mathfrak{h}^3$ where $\mathcal{T}$ is a contractible CW-complex on which G acts cellularly. It should be mentioned that the fundamental domain and the contractible 2-complex $\mathcal{T}$ are not uniquely determined by G. Various algorithms exist for computing $\mathcal{T}$ and its cell stabilizers. One algorithm due to Swan [Swa71] has been implemented by Alexander Rahm [Rah10] and the output for various values of d are stored in HAP. Another approach is to use Voronoi's theory of perfect forms. This approach has been implemented by Sebastian Schoennenbeck [BCNS15] and, again, its output for various values of d are stored in HAP. The following commands combine data from Schoennenbeck's algorithm with free resolutions for cell stabiliers to compute

$$H^1(PSL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{9240} \oplus \mathbb{Z}_{55440} \oplus \mathbb{Z}^4\,,$$

$$H^2(PSL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = \begin{array}{l} (\mathbb{Z}_2)^{26} \oplus (Z_6)^8 \oplus (Z_{12})^9 \oplus \mathbb{Z}_{24} \oplus (\mathbb{Z}_{120})^2 \oplus (\mathbb{Z}_{840})^3 \\ \oplus\mathbb{Z}_{2520} \oplus (\mathbb{Z}_{27720})^2 \oplus (\mathbb{Z}_{24227280})^2 \oplus (\mathbb{Z}_{411863760})^2 \\ \oplus\mathbb{Z}_{245443824374892865187742514283666449812984 0} \\ \oplus\mathbb{Z}_{1472662946249357191126455085701998698877904 0} \\ \oplus\mathbb{Z}^4 \end{array}\,,$$

$$H^3(PSL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = (\mathbb{Z}_2)^{23} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_{12})^2\,.$$

Note that the action of $SL_2(\mathcal{O}_{-d})$ on $P_{\mathcal{O}_{-d}}(k)$ induces an action of $PSL_2(\mathcal{O}_{-d})$ provided k is even.

Example

```
gap> R:=ResolutionPSL2QuadraticIntegers(-6,4);
Resolution of length 4 in characteristic 0 for PSL(2,O-6) .
No contracting homotopy available.

gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 12, 24, 9240, 55440, 0, 0, 0, 0 ]
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 120, 120,
  840, 840, 840, 2520, 27720, 27720, 24227280, 24227280, 411863760, 411863760,
  2454438243748928651877425142836664498129840,
```

```
  14726629462493571911264550857019986988779040, 0, 0, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 12,
  12 ]
```

We can also consider the coefficient module

$$P_{\mathcal{O}_{-d}}(k,\ell) = P_{\mathcal{O}_{-d}}(k) \otimes_{\mathcal{O}_{-d}} \overline{P_{\mathcal{O}_{-d}}(\ell)}$$

where the bar denotes a twist in the action obtained from complex conjugation. For an action of the projective linear group we must insist that $k+\ell$ is even. The following commands compute

$$H^2(PSL_2(\mathcal{O}_{-11}), P_{\mathcal{O}_{-11}}(5,5)) = (\mathbb{Z}_2)^8 \oplus \mathbb{Z}_{60} \oplus (\mathbb{Z}_{660})^3 \oplus \mathbb{Z}^6,$$

a computation which was first made, along with many other cohomology computationsfor Bianchi groups, by Mehmet Haluk Sengun [Sen11].

Example
```
gap> R:=ResolutionPSL2QuadraticIntegers(-11,3);;
gap> M:=HomogeneousPolynomials(R!.group,5,5);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 60, 660, 660, 660, 0, 0, 0, 0, 0, 0 ]
```

The function ResolutionPSL2QuadraticIntegers(-d,n) relies on a limited data base produced by the algorithms implemented by Schoennenbeck and Rahm. The function also covers some cases covered by entering a sring "-d+I" as first variable. These cases correspond to projective special groups of module automorphisms of lattices of rank 2 over the integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ with non-trivial Steinitz-class. In the case of a larger class group there are cases labelled "-d+I2",...,"-d+Ik" and the Ij together with O-d form a system of representatives of elements of the class group modulo squares and Galois action. For instance, the following commands compute

$$H_2(PSL(\mathcal{O}_{-21+I2}), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^6.$$

Example
```
gap> R:=ResolutionPSL2QuadraticIntegers("-21+I2",3);
Resolution of length 3 in characteristic 0 for PSL(2,O-21+I2)) .
No contracting homotopy available.

gap> Homology(TensorWithIntegers(R),2);
[ 2, 0, 0, 0, 0, 0, 0 ]
```

10.10 Some other infinite matrix groups

Analogous to the functions for Bianchi groups, HAP has functions

- ResolutionSL2QuadraticIntegers(-d,n)
- ResolutionSL2ZInvertedInteger(m,n)

- `ResolutionGL2QuadraticIntegers(-d,n)`
- `ResolutionPGL2QuadraticIntegers(-d,n)`
- `ResolutionGL3QuadraticIntegers(-d,n)`
- `ResolutionPGL3QuadraticIntegers(-d,n)`

for computing free resolutions for certain values of $SL_2(\mathcal{O}_{-d})$, $SL_2(\mathbb{Z}[\frac{1}{m}])$, $GL_2(\mathcal{O}_{-d})$ and $PGL_2(\mathcal{O}_{-d})$. Additionally, the function

- `ResolutionArithmeticGroup("string",n)`

can be used to compute resolutions for groups whose data (provided by Sebastian Schoennenbeck, Alexander Rahm and Mathieu Dutour) is stored in the directory `gap/pkg/Hap/lib/Perturbations/Gcomplexes` .

For instance, the following commands compute

$$H^1(SL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{9240} \oplus \mathbb{Z}_{55440} \oplus \mathbb{Z}^4 ,$$

$$H^2(SL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = \begin{array}{l} (\mathbb{Z}_2)^{26} \oplus (Z_6)^7 \oplus (Z_{12})^{10} \oplus \mathbb{Z}_{24} \oplus (\mathbb{Z}_{120})^2 \oplus (\mathbb{Z}_{840})^3 \\ \oplus \mathbb{Z}_{2520} \oplus (\mathbb{Z}_{27720})^2 \oplus (\mathbb{Z}_{24227280})^2 \oplus (\mathbb{Z}_{411863760})^2 \\ \oplus \mathbb{Z}_{245443824374892865187742514283666449812984 0} \\ \oplus \mathbb{Z}_{147266294624935719112645508570199869887790 40} \\ \oplus \mathbb{Z}^4 \end{array} ,$$

$$H^3(SL_2(\mathcal{O}_{-6}), P_{\mathcal{O}_{-6}}(24)) = (\mathbb{Z}_2)^{58} \oplus (\mathbb{Z}_4)^4 \oplus (\mathbb{Z}_{12}) .$$

Example

```
gap> R:=ResolutionSL2QuadraticIntegers(-6,4);
Resolution of length 4 in characteristic 0 for PSL(2,O-6) .
No contracting homotopy available.

gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 12, 24, 9240, 55440, 0, 0, 0, 0 ]
gap> Cohomology(C,2);
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 120,
  120, 840, 840, 840, 2520, 27720, 27720, 24227280, 24227280, 411863760,
  411863760, 2454438243748928651877425142836664498129840,
  1472662946249357191126455085701998698877904 0, 0, 0, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 12, 12 ]
```

The following commands construct free resolutions up to degree 5 for the groups $SL_2(\mathbb{Z}[\frac{1}{2}])$, $GL_2(\mathcal{O}_{-2})$, $GL_2(\mathcal{O}_2)$, $PGL_2(\mathcal{O}_2)$, $GL_3(\mathcal{O}_{-2})$, $PGL_3(\mathcal{O}_{-2})$. The final command constructs a free resolution up to degree 3 for $PSL_4(\mathbb{Z})$.

Example
```
gap> R1:=ResolutionSL2ZInvertedInteger(2,5);
Resolution of length 5 in characteristic 0 for SL(2,Z[1/2]) .

gap> R2:=ResolutionGL2QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for GL(2,O-2) .
No contracting homotopy available.

gap> R3:=ResolutionGL2QuadraticIntegers(2,5);
Resolution of length 5 in characteristic 0 for GL(2,O2) .
No contracting homotopy available.

gap> R4:=ResolutionPGL2QuadraticIntegers(2,5);
Resolution of length 5 in characteristic 0 for PGL(2,O2) .
No contracting homotopy available.

gap> R5:=ResolutionGL3QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for GL(3,O-2) .
No contracting homotopy available.

gap> R6:=ResolutionPGL3QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for PGL(3,O-2) .
No contracting homotopy available.

gap> R7:=ResolutionArithmeticGroup("PSL(4,Z)",3);
Resolution of length 3 in characteristic 0 for <matrix group with 655 generators> .
No contracting homotopy available.
```

10.11 Ideals and finite quotient groups

The following commands first construct the number field $\mathbb{Q}(\sqrt{-7})$, its ring of integers $\mathcal{O}_{-7} = \mathcal{O}(\mathbb{Q}(\sqrt{-7}))$, and the principal ideal $I = \langle 5+2\sqrt{-7}\rangle \triangleleft \mathcal{O}(\mathbb{Q}(\sqrt{-7}))$ of norm $\mathcal{N}(I) = 53$. The ring I is prime since its norm is a prime number. The primality of I is also demonstrated by observing that the quotient ring $R = \mathcal{O}_{-7}/I$ is an integral domain and hence isomorphic to the unique finite field of order 53, $R \cong \mathbb{Z}/53\mathbb{Z}$. (In a ring of quadratic integers *prime ideal* is the same as *maximal ideal*).

The finite group $G = SL_2(\mathcal{O}_{-7}/I)$ is then constructed and confirmed to be isomorphic to $SL_2(\mathbb{Z}/53\mathbb{Z})$. The group G is shown to admit a periodic $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of period dividing 52.

Finally the integral homology

$$H_n(G,\mathbb{Z}) = \begin{cases} 0 & n \neq 3,7, \text{ for } 0 \leq n \leq 8, \\ \mathbb{Z}_{2808} & n = 3,7, \end{cases}$$

is computed.

Example
```
gap> Q:=QuadraticNumberField(-7);
Q(Sqrt(-7))
```

```
gap> OQ:=RingOfIntegers(Q);
O(Q(Sqrt(-7)))

gap> I:=QuadraticIdeal(OQ,5+2*Sqrt(-7));
ideal of norm 53 in O(Q(Sqrt(-7)))

gap> R:=OQ mod I;
ring mod ideal of norm 53

gap> IsIntegralRing(R);
true

gap> gens:=GeneratorsOfGroup( SL2QuadraticIntegers(-7) );;
gap> G:=Group(gens*One(R));;G:=Image(IsomorphismPermGroup(G));;
gap> StructureDescription(G);
"SL(2,53)"

gap> IsPeriodic(G);
true
gap> CohomologicalPeriod(G);
52

gap> GroupHomology(G,1);
[  ]
gap> GroupHomology(G,2);
[  ]
gap> GroupHomology(G,3);
[ 8, 27, 13 ]
gap> GroupHomology(G,4);
[  ]
gap> GroupHomology(G,5);
[  ]
gap> GroupHomology(G,6);
[  ]
gap> GroupHomology(G,7);
[ 8, 27, 13 ]
gap> GroupHomology(G,8);
[  ]
```

The following commands show that the rational prime 7 is not prime in $\mathcal{O}_{-5} = \mathcal{O}(\mathbb{Q}(\sqrt{-5}))$. Moreover, 7 totally splits in $\mathcal{O}_{-5}$ since the final command shows that only the rational primes 2 and 5 ramify in $\mathcal{O}_{-5}$.

Example

```
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,7);;
gap> IsPrime(I);
false

gap> Factors(Discriminant(OQ));
```

```
[ -2, 2, 5 ]
```

For $d < 0$ the rings $\mathcal{O}_d = \mathcal{O}(\mathbb{Q}(\sqrt{d}))$ are unique factorization domains for precisely

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

This result was conjectured by Gauss, and essentially proved by Kurt Heegner, and then later proved by Harold Stark.

The following commands construct the classic example of a prime ideal I that is not principal. They then illustrate reduction modulo I.

Example
```
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,[2,1+Sqrt(-5)]);
ideal of norm 2 in O(Q(Sqrt(-5)))

gap> 6 mod I;
0
```

10.12 Congruence subgroups for ideals

Given a ring of integers $\mathcal{O}$ and ideal $I \triangleleft \mathcal{O}$ there is a canonical homomorphism $\pi_I : SL_2(\mathcal{O}) \to SL_2(\mathcal{O}/I)$. A subgroup $\Gamma \le SL_2(\mathcal{O})$ is said to be a *congruence subgroup* if it contains $\ker \pi_I$. Thus congruence subgroups are of finite index. Generalizing the definition in 10.1 above, we define the *principal congruence subgroup* $\Gamma_1(I) = \ker \pi_I$, and the congruence subgroup $\Gamma_0(I)$ consisting of preimages of the upper triangular matrices in $SL_2(\mathcal{O}/I)$.

The following commands construct $\Gamma = \Gamma_0(I)$ for the ideal $I \triangleleft \mathcal{O}\mathbb{Q}(\sqrt{-5})$ generated by 12 and $36\sqrt{-5}$. The group Γ has index 385 in $SL_2(\mathcal{O}\mathbb{Q}(\sqrt{-5}))$. The final command displays a tree in a Cayley graph for $SL_2(\mathcal{O}\mathbb{Q}(\sqrt{-5}))$ whose nodes represent a transversal for Γ.

Example
```
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,[36*Sqrt(-5), 12]);;
gap> G:=HAP_CongruenceSubgroupGamma0(I);
CongruenceSubgroupGamma0(ideal of norm 144 in O(Q(Sqrt(-5))))

gap> IndexInSL2O(G);
385

gap> HAP_SL2TreeDisplay(G);
```

The next commands first construct the congruence subgroup $\Gamma_0(I)$ of index 144 in $SL_2(\mathcal{O}\mathbb{Q}(\sqrt{-2}))$ for the ideal I in $\mathcal{O}\mathbb{Q}(\sqrt{-2})$ generated by $4 + 5\sqrt{-2}$. The commands then compute

$$H_1(\Gamma_0(I), \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}^8,$$

$$H_2(\Gamma_0(I),\mathbb{Z}) = (\mathbb{Z}_2)^9 \oplus \mathbb{Z}^7\,,$$

$$H_3(\Gamma_0(I),\mathbb{Z}) = (\mathbb{Z}_2)^9\,.$$

Example

```
gap> Q:=QuadraticNumberField(-2);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,4+5*Sqrt(-2));;
gap> G:=HAP_CongruenceSubgroupGamma0(I);
CongruenceSubgroupGamma0(ideal of norm 66 in O(Q(Sqrt(-2))))

gap> IndexInSL2O(G);
144

gap> R:=ResolutionSL2QuadraticIntegers(-2,4,true);;
gap> S:=ResolutionFiniteSubgroup(R,G);;

gap> Homology(TensorWithIntegers(S),1);
[ 3, 6, 30, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Homology(TensorWithIntegers(S),2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0 ]
gap> Homology(TensorWithIntegers(S),3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

10.13 First homology

The isomorphism $H_1(G,\mathbb{Z}) \cong G_{ab}$ allows for the computation of first integral homology using computational methods for finitely presented groups. Such methods underly the following computation of

$$H_1(\Gamma_0(I),\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_{4078793513671}$$

where I is the prime ideal in the Gaussian integers generated by $41+56\sqrt{-1}$.

Example

```
gap> Q:=QuadraticNumberField(-1);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,41+56*Sqrt(-1));
ideal of norm 4817 in O(GaussianRationals)
gap> G:=HAP_CongruenceSubgroupGamma0(I);;
gap> AbelianInvariants(G);
[ 2, 2, 4, 5, 7, 16, 29, 43, 157, 179, 1877, 7741, 22037, 292306033,
  4078793513671 ]
```

We write G^{ab}_{tors} to denote the maximal finite summand of the first homology group of G and refer to this as the *torsion subgroup*. Nicholas Bergeron and Akshay Venkatesh [Ber16] have conjectured relationships between the torsion in congruence subgroups Γ and the volume of their quotient manifold $\mathfrak{h}^3/\Gamma$. For instance, for the Gaussian integers they conjecture

$$\frac{\log|\Gamma_0(I)^{ab}_{tors}|}{\mathrm{Norm}(I)} \to \frac{\lambda}{18\pi},\ \lambda = L(2,\chi_{\mathbb{Q}(\sqrt{-1})}) = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots$$

as the norm of the prime ideal I tends to ∞. The following approximates $\lambda/18\pi = 0.0161957$ and $\frac{\log|\Gamma_0(I)^{ab}_{tors}|}{\mathrm{Norm}(I)} = 0.00913432$ for the above example.

Example

```
gap> Q:=QuadraticNumberField(-1);;
gap> Lfunction(Q,2)/(18*3.142);
0.0161957

gap> 1.0*Log(Product(AbelianInvariants(F)),10)/Norm(I);
0.00913432
```

The link with volume is given by the Humbert volume formula

$$\mathrm{Vol}(\mathfrak{h}^3/PSL_2(\mathcal{O}_d)) = \frac{|D|^{3/2}}{24}\zeta_{\mathbb{Q}(\sqrt{d})}(2)/\zeta_{\mathbb{Q}}(2)$$

valid for square-free $d < 0$, where D is the discriminant of $\mathbb{Q}(\sqrt{d})$. The volume of a finite index subgroup Γ is obtained by multiplying the right-hand side by the index $|PSL_2(\mathcal{O}_d) : \Gamma|$.

Chapter 11

Parallel computation

11.1 An embarassingly parallel computation

The following example creates five child processes and uses them simultaneously to compute the second integral homology of each of the 267 groups of order 64. The final command shows that

$H_2(G,\mathbb{Z}) = \mathbb{Z}_2^{15}$

for the 267-th group G in GAP's library of small groups.

Example
```
gap> Processes:=List([1..5],i->ChildProcess());;
gap> fn:=function(i);return GroupHomology(SmallGroup(64,i),2);end;;
gap> for p in Processes do
> ChildPut(fn,"fn",p);
> od;

gap> NrSmallGroups(64);
267

gap> L:=ParallelList([1..267],"fn",Processes);;

gap> L[267];
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

The function `ParallelList()` is built from HAP's six core functions for parallel computation.

11.2 An non-embarassingly parallel computation

The following commands use core functions to compute the product $A = M \times N$ of two random matrices by distributing the work over two processors.

Example
```
gap> M:=RandomMat(2000,2000);;
gap> N:=RandomMat(2000,2000);;

gap> s:=ChildProcess();;

gap> Mtop:=M{[1..1000]};;
gap> Mbottom:=M{[1001..2000]};;
```

```
gap> ChildPut(Mtop,"Mtop",s);
gap> ChildPut(N,"N",s);
gap> NextAvailableChild([s]);;

gap> ChildCommand("Atop:=Mtop*N;;",s);;
gap> Abottom:=Mbottom*N;;
gap> A:=ChildGet("Atop",s);;
gap> Append(A,Abottom);;
```

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