# SIX EXAMPLES IN COMPUTATIONAL GROUP COHOMOLOGY

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Abstract. Six examples of machine computations in group cohomology are presented using the GAP system for computational algebra. An explanation of the underlying theory for each example is given. The examples span: (1) group theory and low-dimensional cohomology; (2) linear algebra and mod-$p$ cohomology of a finite group; (3) combinatorial topology and the integral cohomology ring of a fundamental group; (4) contracting homotopies and the low dimensions of a classifying space of a finite group; (5) homological perturbation theory; (6) integral cohomology of Bianchi groups and an approach to Hecke operators. Further examples can be found at `https://gap-packages.github.io/hap/tutorial/chap0_mj.html`.

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## 1. **Lecture 1**

The cohomology of groups began with Heinz Hopf's 1941 result (published in [28] the following year) that

$$\text{cokernel}(\pi_2(X) \to H_2(X,\mathbb{Z})) \;\cong\; R \cap [F,F]/[F,R] \tag{1}$$

where $F/R \cong \pi_1 X$ is any presentation for the fundamental group $\pi_1 X$ of a path-connected topological space $X$ by a free group $F$ and normal subgroup $R \lhd F$. It follows that if the second homotopy group $\pi_2 X$ is trivial then the second homology of $X$ depends only on the fundamental group. It was already known that $H_1(X,\mathbb{Z}) \cong \pi_1 X/[\pi_1 X, \pi_1 X]$. The group construction on the right of (1) had previously arisen in Issai Schur's 1904 work [43] on projective representations – it is the *Schur multiplier* of the group $G = F/R$. Hopf, while studying in Berlin, had attended courses by Schur, but his 1942 paper [28] makes no mention of the connection with the Schur multiplier. According to Saunders Mac Lane [34], motivation seems to have come from his earlier studies of the homology of Lie groups. Hassler Whitney's review of Hopf's paper begins:

> This paper is, in the reviewer's mind, one of the most important contributions to combinatorial topology in recent years. It gives far reaching results concerning the relations between the fundamental group, the first and second homology and cohomology groups, and the products between these groups, with beautiful and simple methods. The work is based on some new constructions in groups which are undoubtedly of real significance by themselves.

Hopf's formula (1) led Eilenberg & Mac Lane [16] and others (Hopf & Eckmann, Freudenthal, Faddeev) to define

$$H_n(G,\mathbb{Z}) = H_n(BG,\mathbb{Z}) \tag{2}$$

for any group $G$ where $BG$ is any path-connected space having fundamental group $\pi_1 BG \cong G$ and $\pi_n BG = 0$ for $n \ge 2$. The space $BG$ is said to be a *classifying space* for $G$ and can be constructed from any contractible space $EG$ on which $G$ acts freely by taking $BG$ to be the quotient

$$BG = EG/G \quad .$$

In 1944 Hopf [29] essentially introduced the definition

$$H_n(G,A) = \mathrm{Tor}_*^{\mathbb{Z}[G]}(A,\mathbb{Z}) \tag{3}$$

without explicitly using the now standard language of homological algebra.

Machine algorithms for computing the homology and cohomology of groups fall very roughly into the categories of: (1) computational group/monoid theory, (2) computational topology, (3) computational homological algebra. Applications of the algorithms range from group theory (*e.g.* classification of groups of small order [4, 15]), combinatorics (*e.g.* construction of Hadamard matrices [3]), Field theory (*e.g.* Noether's problem [36]) and algebraic number theory (*e.g.* calculation of Hecke operators for automorphic forms [25]) to topics in theoretical physics (*e.g.* Bosonic string theory [30]) and physical chemistry (*e.g.* crystallography [31, 32]).

1.1. **Group theoretic approach.** One of the first implemented algorithms for group cohomology is Derek Holt's algorithm [26] for calculating the second integral homology, or Schur multiplier, $H_2(G,\mathbb{Z})$ of a finite permutation group $G$. His algorithm is based on Hopf's formula (1).

Suppose we can somehow construct a free group $F$ with normal subgroup $R \lhd F$ such that $G \cong F/R$, $F$ is generated by a finite set $\underline{x} = \{x_1, \ldots, x_n\}$ and $R$ is normally generated by a finite collection of elements $\underline{r} = \{r_1, \ldots, r_m\} \subset R$. The group $F/[R,F]$ admits the finite presentation

$$F/[R,F] \cong \langle x_1, \ldots, x_n \,|\, [r_j, x_i] = 1 \text{ for } x_i \in \underline{x}, r_j \in \underline{r} \rangle.$$

The group $R/[R,F]$ is finitely generated by the image of $\underline{r}$ and so, by Hopf's formula (1), its subgroup $H_2(G,\mathbb{Z})$ is finitely generated. The exact sequence

$$H_2(G,\mathbb{Z}) \rightarrowtail R/[R,F] \twoheadrightarrow R/R\cap[F,F]$$

is split since $R/R\cap[F,F]$ is free abelian. (It is a subgroup of the free abelian group $F/[F,F]$.) Choose a section $\sigma\colon R/R\cap[F,F] \to R/[R,F]$. There is a central extension

$$H_2(G,\mathbb{Z}) \oplus \sigma(R/R\cap[F,F]) \rightarrowtail F/[R,F] \twoheadrightarrow G.$$

Defining a *Schur cover*

$$G^* = F/[R,F]\sigma(R/R\cap[F,F])$$

we obtain a central extension

$$H_2(G,\mathbb{Z}) \rightarrowtail G^* \overset{\phi}{\twoheadrightarrow} G \tag{4}$$

with $G^*$ a finitely presented group. To see that $G^*$ is finite we need to use the transfer homomorphism

$$Tr\colon H_2(G,\mathbb{Z}) \to H_2(K,\mathbb{Z})$$

defined for any subgroup $K < G$. There are several ways to define $Tr$, one of which uses the topological definition of $H_n(G,K)$ and the $k$-fold covering space of $BG$ corresponding to a subgroup $K$ of index $k$ in $G$. It can be shown that multiplication by $k$ factors as a composite of homomorphisms

$$\times k:\ H_2(G,\mathbb{Z}) \overset{Tr}{\to} H_2(K,\mathbb{Z}) \to H_2(G,\mathbb{Z}). \tag{5}$$

Taking $K$ to be the trivial goup we see that the exponent of $H_2(G,\mathbb{Z})$ divides the order $|G|$. The abelian group $H_2(G,\mathbb{Z})$ is finite since it is finitely generated of finite exponent. Hence $G^*$ is finite. One could attempt to compute the kernel of the homomorphism $\phi$ of finite finitely presented groups using coset enumeration (the Todd-Coxeter procedure) but this is typically not practicle for groups $G$ of large order. Instead, for each prime $p$ dividing $|G|$ we could consider a Sylow subgroup $K = Syl_p(G)$. Sequence 5 implies that $H_2(Syl_p(G),\mathbb{Z})$ maps onto the $p$-part of $H_2(G,\mathbb{Z})$. Cartan & Eilenberg showed that the kernel of

$$H_2(Syl_p(G),\mathbb{Z}) \twoheadrightarrow H_2(G,\mathbb{Z})_p$$

is generated by the elements

$$h_P(a) - h_{xPx^{-1}}(a)$$

where $x$ ranges over the double coset representatives of $K$ in $G$, $P = K \cap xKx^{-1}$, the homomorphisms $h_P, h_{xPx^{-1}} : H_2(P,\mathbb{Z}) \to H_2(K,\mathbb{Z})$ are induced by the inclusion $P \to K, y \mapsto y$ and the conjugated inclusion $P \to K, y \mapsto x^{-1}yx$, and $a$ ranges over the generators of $H_2(P,\mathbb{Z})$. Thus, the computation of $H_2(G,\mathbb{Z})$ can be reduced to the computation of homomorphisms $\phi\colon H_2(P,\mathbb{Z}) \to H_2(K,\mathbb{Z})$ where $P \le K = Syl_p(G)$ and $p \mid |G|$. If $K$ and a Schur cover $K^*$ happen to be small then one could try coset enumeration to compute the kernel of $\phi$. But a more efficient approach uses the theory of polycyclic groups.

A group $G$ is *polycylic* if it admits a series of subgroups $G = G_1 > G_2 > \cdots G_{n+1} = 1$ for which $G_{i+1}$ is normal in $G_i$ with cyclic quotient $G_i/G_{i+1}$ of order $r_i$ for $0 \le i \le n-1$. Any finite group of prime-power order is polycyclic. A sequence of elements $x_1, ..., x_n$ such that $\langle x_i G_{i+1}\rangle = G_i/G_{i+1}$ for $1 \le i \le n$ is a *polycyclic generating sequence (PCGS)* for $G$. For every $g \in G$ there exists a sequence of integers $e_1, \ldots, e_n$ with $0 \le e_i \le r_i$ such that

$$g = x_1^{e_1} \cdots x_n^{e_n}. \tag{6}$$

The expression (6) is the *normal form* of $g$ with respect to the given PCGS.

A polycyclic group $G$ admits a "consistent power-conjugate presentation" or *pc-presentation.* This pc-presentation can be used to efficiently represent the group on a computer, solve the word problem, and perform computations such as calculations of kernels of homomorphisms. Furthermore, in the case of finite $p$-groups there there is a *p-quotient algorithm* that inputs an arbitrary finite presentation for $G$ and return a pc-presentation [35]. This works by means the the $p$-lower central series $G = P_0 \ge P_1 \ge \cdots$ where $P_i = [P_{i-1}, G]P_{i-1}^p$ for $i \ge 1$. Starting with a presentation of $G$ the algorithm begins by constructing a pc-presentation for the quotient $G/[G,G]G^p$ and an epimorphism $G \twoheadrightarrow [G,G]G^p$. It recursively computes a pc-presentation for $G/P_i$ and epimorphism $G \twoheadrightarrow G/P_i$.

A refined version of the above ideas are used in Derek Holt's standalone C implementation and algorithm to compute Schur multipliers and corresponding second cohomology groups. This implementation has been made available in both the Magma [5] and GAP [21] systems for computational algebra . The following GAP example uses it to compute the Schur multiplier of the group of permutations of the Rubik's cube. This group $G$ is of order $\approx 43 \times 10^{18}$ and has $H_2(G,\mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. It provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects.

**Example 1**

```
gap> G:=Group([
>   ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19),
>   ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35),
>   (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11),
>   (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24),
>   (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27),
>   (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)
> ]);;
gap> Order(G);
43252003274489856000
gap> SSortedList(Factors(Order(G)));
[ 2, 3, 5, 7, 11 ]

gap> LoadPackage("cohomolo");
true

gap> C:=CHR(G,2);;
gap> SchurMultiplier(C);
[ 2, 2, 2 ]
```

```
gap> C:=CHR(G,3);;
gap> SchurMultiplier(C);
[  ]

gap> C:=CHR(G,5);;
gap> SchurMultiplier(C);
[  ]

gap> C:=CHR(G,7);;
gap> SchurMultiplier(C);
[  ]

gap> C:=CHR(G,11);;
gap> SchurMultiplier(C);
[  ]
```

For a finite permutation group $G = F/R$ and finitely generated abelian group $A$ (on which $G$ acts trivially) one can derive an analogue of Hopf's formula for second cohomology.

$$H^2(G,A) \cong \mathrm{Hom}_{\mathbb{Z}}(\frac{R\cap[F,F]}{[R,F]},A) \oplus \mathrm{coker}\left\{\mathrm{Hom}_{\mathbb{Z}}(\frac{F}{[F,F]},A) \to \mathrm{Hom}_{\mathbb{Z}}(\frac{R}{R\cap[F,F]},A)\right\}.$$

Using this formula, the computation of Schur multipliers can be adapted to one for second cohomology with abelian group coefficients together with one for computing free presentations of central extensions $A \rightarrowtail E \twoheadrightarrow G$ arising under the bijection

$$H^2(G,A) \cong \{\text{central extensions } A \rightarrowtail E \twoheadrightarrow G\}/\text{ Yoneda equivalence} \quad .$$

Two extensions $E, E'$ are Yoneda equivalent if there is an isomorphism $E \to E'$ inducing the identity on $A$ and $G$. An implementation for the case where $A$ is a finite-dimensional vector space over the field of $p$ elements on which $G$ acts non-trivially is described in [27] using the formula

$$H^2(G,A) \cong \mathrm{coker}\left\{H^1(\frac{F}{[R,R]R^p},A) \to \mathrm{Hom}_{\mathbb{Z}G}(\frac{R}{[R,R]R^p},A)\right\}$$

and is again available in Magma and GAP.

For more details on the role of polycyclic groups in the computation of cohomology in degrees one and two, see [14]. A pc-presentation can be regarded as a special kind of rewriting system. For an alternative approach to degrees one and two that uses more general rewriting systems, see [11]. For an example in which cocycles are used to endow extension groups with an algorithm for performing group multiplication, see `https://gap-packages.github.io/hap/tutorial/chap6_mj.html`.

1.2. **Homological algebra/ linear algebra approach.** The cup product product $\cup\colon H^p(X,\mathbb{Z}) \times H^q(X,\mathbb{Z}) \to H^{p+q}(X,\mathbb{Z})$ (introduced in the mid 1930s) makes the cohomology of a space $X$ into a ring. The multiplication is *graded comutative*: $a \cup b = (-1)^{pq}(b \cup a)$. It can be defined using the diagonal map

$$\Delta\colon X \to X \times X, x \mapsto (x,x)$$

to define

$$\cup\colon H^p(X,\mathbb{Z}) \otimes H^q(X,\mathbb{Z}) \xrightarrow{\kappa} H^{p+q}(X\times X,\mathbb{Z}) \xrightarrow{\Delta^*} H^{p+q}(X,\mathbb{Z})$$

with $\kappa$ an easily computed *cross product* and $\Delta^*$ the computationally more problematic map induced by the diagonal.

In particular, the cohomology of a group is a ring, or an algebra if one works over the field $\mathbb{F} = \mathbb{F}_p$ of $p$ elements in place of the integers. By a result of Venko [49] and Evens [18] the mod $p$ cohomology ring $H^*(G,\mathbb{F})$ is finitely generated and presented for any finite group $G$. Presentations for the rings $H^*(G,\mathbb{F})$ for groups $G$ of order 32 were calculated by Rusin [42]. His calculations were computer aided but were mainly done by hand. The rings for groups of order 64 were handled by Carlson, Valeri-Elizondo and Zhang [9] using essnetially just linear algebra in Magma and various "tests for completion". The group of order 128 have been handled by Green and King [23, 24] using Gröbner basis techniques implemented in Sage [46] and GAP. By reducing to Sylow subgroups once can attempt to handle finite groups of non prime power order. A presentation for the mod-2 cohomology ring of the third Conway group $Co_3$ of order 495 766 656 000 is computed in [17] and used to establish that the ring is Cohen-Macaulay. An approach to mod $p$ cohomology based on the theory of *basic algebras* is implemented in Magma.

Starting from the definition and isomorphism

$$H^n(G,\mathbb{F}) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},\mathbb{F}) \cong \mathrm{Ext}^n_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$$

where $\mathbb{F}G$ is the group algebra over $\mathbb{F}$, we can attempt to compute the mod $p$ cohomology of a group $G$ by constructing an exact sequence of free $\mathbb{F}G$-modules

$$R_* = \quad \cdots \to R_2 \to R_1 \to R_0$$

with $H_0(R_*) \cong \mathbb{F}$. *i.e.* a free $\mathbb{F}G$-resolution of $\mathbb{F}$. It is then a straightforward linear algebra computation to determine

$$H^n(G,\mathbb{F}) = \frac{\ker(\ \mathrm{Hom}_{\mathbb{F}G}(R_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}G}(R_{n+1},\mathbb{F})\ )}{\mathrm{image}(\ \mathrm{Hom}_{\mathbb{F}G}(R_{n-1},\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}G}(R_n,\mathbb{F})\ )}\ .$$

The resolution $R_*$ can be constructed recursively. We set $R_0 = \mathbb{F}G$ and consider the augmentation map $d_0\colon R_0 \to \mathbb{F}$. Given $d_n\colon R_n \to R_{n-1}$ we can compute $V = \ker d_n$ as a vector space by using Gaussian elimination. Now $V$ is an $\mathbb{F}G$-module and if we can find a generating set for this module involving say $d$ generators, then we simply construct $R_{n+1} = \oplus_{i=1}^d \mathbb{F}G$ and the obvious surjection $d_{n+1}\colon R_{n+1} \twoheadrightarrow V \subset R_n$.

For a $p$-group $G$ the radical of $V$ is

$$\mathrm{rad}(V) = \mathrm{span}_{\mathbb{F}}\{g \cdot v\ :\ g \in G, v \in \mathrm{Basis}(V)\}\ .$$

In fact, it suffices to let $g$ range over generators for $G$. For small $p$-groups $G$ and $V$ of small dimension as a vector space, one can use Gaussian elimination to find an $\mathbb{F}$-basis of $V$. The cosets $V/rad(V)$ then correspond to a *minimal* generating set for $V$ as an $\mathbb{F}G$-module.

The cup product can constructed using *any* chain map $R_* \to R_* \otimes R_*$ over the identity on $H_0 \cong \mathbb{F}$ that is equivariant with respect to the group homomorphism $G \to G \times G$, $g \mapsto (g,g)$ where we note that $R_* \otimes R_*$ is a $\mathbb{Z}[G \times G]$-resolution. Such a chain map can be constructed from a contracting homotopy on $R_* \otimes R_*$. This contracting homotopy is readily derived from any *contracting homotopy* on $R*$ by which we mean a sequence of $\mathbb{F}$-linear homomorphisms $h_n\colon R_n \to R_{n+1}$ for $n \geq 0$ satisfying $d_{n+1}h_n + h_{n-1}d_n = 1$ $(n > 0)$, $d_1h_0 = 1 - \epsilon$ where $\epsilon : R_0 \twoheadrightarrow H_0(R_*) \cong \mathbb{F} \hookrightarrow R_0$. The contracting homotopy on $R_*$ can be found by solving systems of linear equations over $\mathbb{F}$. There is an explicit formula for a contracting homotopy on $R_* \otimes R_*$ given in terms of a contracting homotopy on $R_*$.

Contracting homotopies provide a method for solving the frequent issue in homological constructions of choosing some element $\tilde{x} \in R_{n+1}$ that maps to a given element $x \in \ker(d_n\colon R_n \to R_{n-1})$. The required element $\tilde{x}$ can be chosen by setting $\tilde{x} = h_n(x)$.

For the simple groups $PSL_2(\mathbb{F}_{125})$ of order 976500 and $PSL_2(\mathbb{F}_{343})$ of order 20176632 the following example gives presentations

$$H^*(PSL_2(\mathbb{F}_{125}),\mathbb{F}_2) \ \cong\ \mathbb{F}_2[a,b,c]/[a^3 + b^2 + bc + c^2]$$

$$H^*(PSL_2(\mathbb{F}_{343}),\mathbb{F}_2) \ \cong\ \mathbb{F}_2[a,b,c]/[bc + c^2]$$

with $\deg(a) = 2$, $\deg(b) = \deg(c) = 3$ and common Poincaré series

$$(x^2 - x + 1)/(x^4 - x^3 - x + 1)$$

that are certainly correct up to degree 10. The additional computations given in subsections 1.2.1 and 1.2.2 guarantee correctness in all degrees. Steenrod squares are also computed for the Sylow

2-subgroup of the first group. (For the mod $p$ Steenrod algebra at odd primes $p$ only the Bockstein operator is implemented in HAP.)

**Example 2A**

```
gap> G:=PSL(2,125);;
gap> p:=2;; deg:=10;;
gap> A:=ModPCohomologyRing(G,p,deg);;
gap> F:=PresentationOfGradedStructureConstantAlgebra(A);
Graded algebra GF(2)[ x_1, x_2, x_3 ] / [ x_1^3+x_2^2+x_2*x_3+x_3^2
 ] with indeterminate degrees [ 2, 3, 3 ]
gap> HilbertPoincareSeries(F);
(x_1^2-x_1+1)/(x_1^4-x_1^3-x_1+1)

gap> GG:=PSL(2,343);;
gap> AA:=ModPCohomologyRing(GG,p,deg);;
gap> ff:=PresentationOfGradedStructureConstantAlgebra(AA);
Graded algebra GF(2)[ x_1, x_2, x_3 ] / [ x_2*x_3+x_3^2
 ] with indeterminate degrees [ 2, 3, 3 ]

gap> St:=Mod2SteenrodAlgebra(SylowSubgroup(G,2),deg);;
gap> gens:=ModPRingGenerators(St);;
gap> List(gens,x->Sq(St,1,x));  #Sq^1
[ 0*v.1, v.4+v.5, v.6 ]
gap> List(gens,x->Sq(St,2,x));  #Sq^2
[ 0*v.1, 0*v.1, 0*v.1 ]
```

Alexander Rahm *et al.* use this kind of computation as one step in their study of the cohomology of certain infinite arithmetic groups (*c.f.* [8]).

1.2.1. *Proof of the above Poincaré series.* If one needs to verify that the above Poincaré series are valid in all degrees then more work is required. One readily implemented (but computationally non-optimal) approach is to use Peter Symonds result [48] that: if a non-cyclic finite group $G$ has a faithful complex representation equal to a sum of irreducibles of dimensions $n_i$ then the cohomology ring $H^*(G,\mathbb{Z}_p)$ is generated by elements of degree at most $\sum n_i^2$; a degree bound for the relations is $2\sum n_i^2$. Thus, if we use at least $\sum n_i^2$ degrees of a (minimal) resolution to construct a presentation for the cohomology ring then the presented ring maps surjectively onto the actual cohomology ring. Furthermore, if this surjection is a bijection in the first $2\sum n_i^2$ degrees then it is necessarily an isomorphism in all degrees.

Rather than apply Symonds' bounds directly to a finite group $G$ we can invoke a useful result of P.J. Webb [51] concerning the simplicial complex $\mathcal{A}_p = \mathcal{A}_p(G)$ arising as the order complex of the poset of non-trivial elementary abelian $p$-subgroups of $G$. The group $G$ acts on $\mathcal{A}_p$. Denote the orbit of a $k$-simplex $e^k$ by $[e^k]$, and the stabilizer of $e^k$ by $Stab(e^k) \le G$. For a finite abelian group $H$ let $H_p$ denote the Sylow $p$-subgroup or the "$p$-part". In Theorem 3.3 of [51] P.J. Webb proved the following.

**Theorem 1.1.** [51] *For any $G$-module $M$ there is a (non natural) isomomorphism*

$$H^n(G,M)_p \oplus \bigoplus_{[e^k]\,:\,k\ \mathrm{odd}} H^n(Stab(e^k),M)_p \cong \bigoplus_{[e^k]\,:\,k\ \mathrm{even}} H^n(Stab(e^k),M)_p$$

*for $n \ge 1$. The isomorphism can also be expressed as*

$$H^n(G,M)_p \cong \bigoplus_{[e^k]\,:\,k\ \mathrm{even}} H^n(Stab(e^k),M)_p \ - \bigoplus_{[e^k]\,:\,k\ \mathrm{odd}} H^n(Stab(e^k),M)_p$$

*where terms can often be cancelled.*

Thus the additive structure of the $p$-part of the cohomology of $G$ is determined by that of the stabilizer groups. The result also holds with cohomology replaced by homology and can be viewed as a complementary technique to the Cartan & Eilenberg formula for the kernel of the transfer.

For $G = PSL_2(\mathbb{F}_{125})$ the following GAP session recalculates the mod 2 Poincaré series $P_G(x)$ in a way that guarantees correctness in all degrees by using Symond's and Webb's results to obtain

$$P_G(x) = P_{D_{128}}(x) + P_{A_4}(x) - P_{C_2\times C_2}(x) = (x^2 - x + 1)/(x^4 - x^3 - x + 1)\ .$$

**Example 2B**

```
gap> G:=PSL(2,125);;
gap> D:=HomologicalGroupDecomposition(G,2);;
gap> [List(D[1],StructureDescription) , List(D[2],StructureDescription)];
[ [ "D124", "A4" ], [ "C2 x C2" ] ]
gap> ModPCohomologyPresentationBounds(D[1][1]);
rec( generators_degree_bound := 4, relators_degree_bound := 8 )
gap> A11:=ModPCohomologyRing(D[1][1],2,9);;
gap> F11:=PresentationOfGradedStructureConstantAlgebra(A11);;
gap> P11:=HilbertPoincareSeries(F11);
(1)/(x_1^2-2*x_1+1)
gap> ModPCohomologyPresentationBounds(D[1][2]);
rec( generators_degree_bound := 9, relators_degree_bound := 18 )
gap> A12:=ModPCohomologyRing(D[1][2],2,19);;
gap> F12:=PresentationOfGradedStructureConstantAlgebra(A12);;
gap> P12:=HilbertPoincareSeries(F12);
(x_1^2-x_1+1)/(x_1^4-x_1^3-x_1+1)
gap> ModPCohomologyPresentationBounds(D[2][1]);
rec( generators_degree_bound := 1, relators_degree_bound := 2 )
gap> A21:=ModPCohomologyRing(D[2][1],2,3);;
gap> F21:=PresentationOfGradedStructureConstantAlgebra(A21);;
gap> P21:=HilbertPoincareSeries(F21);
(1)/(x_1^2-2*x_1+1)

gap> P:=P11+P12-P21;
(x_1^2-x_1+1)/(x_1^4-x_1^3-x_1+1)
```

In principle the Poincaré series for $PSL_2(\mathbb{F}_{343})$ could be calculated in the same fashion. However, the current implementation and laptop runs out of memory when calculating the stabilizer subgroups of $\mathcal{A}_2$. A way around this is to: compute the Poincaré series for $PSL_2(\mathbb{F}_7)$ in the above fashion; observe that the canonical inclusion $PSL_2(\mathbb{F}_7) \hookrightarrow PSL_2(\mathbb{F}_{343})$ restricts to an isomorphism on a Sylow 2-subgroup and thus there is an inclusion $H^*(PSL_2(\mathbb{F}_{343}), \mathbb{F}_2) \subseteq H^*(PSL_2(\mathbb{F}_7), \mathbb{F}_2)$ of cohomology rings; compute ring generators for $H^*(PSL_2(\mathbb{F}_7), \mathbb{F}_2)$ and note that they lie in $H^*(PSL_2(\mathbb{F}_{343}), \mathbb{F}_2)$. The ring inclusion is thus an isomorphism and the computed Poincaré series for $PSL_2(\mathbb{F}_7)$ must equal that for $PSL_2(\mathbb{F}_{343})$.

1.2.2. *Proof of the above ring presentations.* Symond's bounds tell us that we can be guaranteed of a correct presentation for the mod 2 cohomology ring presentation if we use a minimal resolution of length 50 for $G = PSL_2(\mathbb{F}_5)$ and of length 98 for $G = PSL_2(\mathbb{F}_7)$. A more efficient approach to proving the above presentations correct in all degrees is to note that Example 2B provides a finitely presented graded ring $F$ and a ring homomorphism $\phi\colon F \to H^*(G, \mathbb{F}_2) \subset H^*(Syl_2(G), \mathbb{F}_2) = \mathbb{F}[x, y]$ where $Syl_2(G) = C_2 \times C_2$. Gröbner basis techniques available through GAP's interface to Singular [10] ) can be used to show that $\phi$ is injective. Since the presented ring $F$ has the correct Hilbert-Poincaré series we conclude that $\phi$ sends $F$ isomorphically to $H^*(G, \mathbb{F}_2)$.

1.3. **Topological approach.** For a given group $G$ we may know of a theoretical classifying space $BG$, or a contractible space $EG$ on which $G$ acts freely. By imposing a CW cellular structure on $BG$ we can use the cellular chain complex $C_*(BG)$ to compute the cohomology of $H^*(G,\mathbb{Z}) = H^*(BG,\mathbb{Z})$. By imposing a cellular structure on $EG$ that is preserved by the $G$-action, we can use the cellular chain complex $C_*(EG)$ as a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

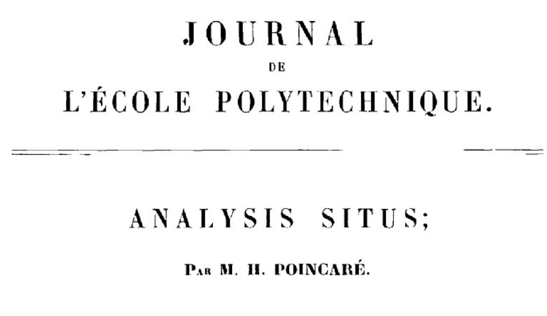
For instance, consider the orientable closed 3-manifold $M$ obtained by performing a $1/11$ Dehn surgery on the trefoil knot embedded in $S^3$. The following GAP commands show that $\pi_1 M \cong C_{11} \times SL_2(\mathbb{Z}_5)$. Hence $M$ is a spherical manifold and $G = C_{11} \times SL_2(\mathbb{Z}_5)$ acts freely on its universal cover $\widetilde{M} = S^3$. If a group $G$ acts freely on a 3-sphere it must have periodic homology of period 4 and $H_3(G,\mathbb{Z}) = \mathbb{Z}_{|G|}$. Hence

$$H_n(C_{11} \times SL_2(\mathbb{Z}_5),\mathbb{Z}) = \begin{cases} \mathbb{Z}_{11}\,, & n \equiv 1 \mod 4 \\ 0\,, & n \equiv 0,2 \mod 4 \\ \mathbb{Z}_{1330}\,, & n \equiv 3 \mod 4. \end{cases}$$

**Example 3A**

```
gap> ap:=ArcPresentation(PureCubicalKnot(3,1));;
gap> W:=ThreeManifoldViaDehnSurgery(ap,1,11);
Regular CW-complex of dimension 3

gap> G:=FundamentalGroup(W);;
gap Order(G);
1320

gap> StructureDescription(G);
C11 x SL(2,5)
```

As a second illustration let us consider Henri Poincaré's paper of 1895 on *Analysis Situs* [39].


JOURNAL

DE

L'ÉCOLE POLYTECHNIQUE.

ANALYSIS SITUS;

PAR M. H. POINCARÉ.

Poincaré constructs several spaces, his first and fourth examples being:

*Premier exemple.*

L'exemple le plus simple est celui où l'on n'a qu'un seul polyèdre $P_1$ et où ce polyèdre est un cube ABCD A'B'C'D' dont les sommets ont respectivement pour coordonnées

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | A' | 0 | 0 | 1 |
| B | 0 | 1 | 0 | B' | 0 | 1 | 1 |
| C | 1 | 0 | 0 | C' | 1 | 0 | 1 |
| D | 1 | 1 | 0 | D' | 1 | 1 | 1 |

Je suppose que les faces opposées soient conjuguées et cela de la façon suivante :

$$(1) \qquad \begin{cases} ABDC \equiv A'B'D'C', \\ ACC'A' \equiv BDD'A', \\ CDD'C' \equiv ABB'A'. \end{cases}$$

Voici ce que j'entends par cette notation : la relation

$$ABDC \equiv A'B'D'C'$$

signifie :

1° Que les faces ABDC et A'B'D'C' sont conjuguées;

2° Que les sommets se rencontrant sur la première de ces faces se rencontrent dans l'ordre circulaire ABDC;

3° Que les sommets A et A', B et B', D et D', C et C' se correspondent.

*Quatrième exemple.*

Soit maintenant

$$(5) \qquad \begin{cases} ABDC \equiv B'D'C'A', \\ ABB'A' \equiv CDD'C', \\ ACC'A' \equiv BDD'B'. \end{cases}$$

Poincaré's first example is the space $X = S^1 \times S^1 \times S^1$ with $\pi_1 X = C_\infty \times C_\infty \times C_\infty$ and universal cover $\widetilde{X} = \mathbb{R}^3$. Since $\widetilde{X}$ is contractible we can use the obvious cellular structure on $X = BG$ for $G = C_\infty \times C_\infty \times C_\infty$ to calculate

$$C_*(BG) \ : \ \ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and then calculate the homology groups $H_n(G, \mathbb{Z}) = H_n(C_*(BG))$.

For $X$ equal to Poincaré's fourth example the following GAP session computes

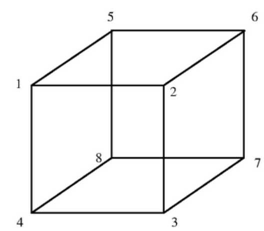
$$G = \pi_1 X \cong \langle x, y \ : \ xy^{-1}x^{-3}y^{-1} = 1, xy^{-1}xy^3 = 1 \rangle$$

and

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[a, b, c]/[ab + b^2, a^2 + b^2, bc, a^3 + ac] \ .$$

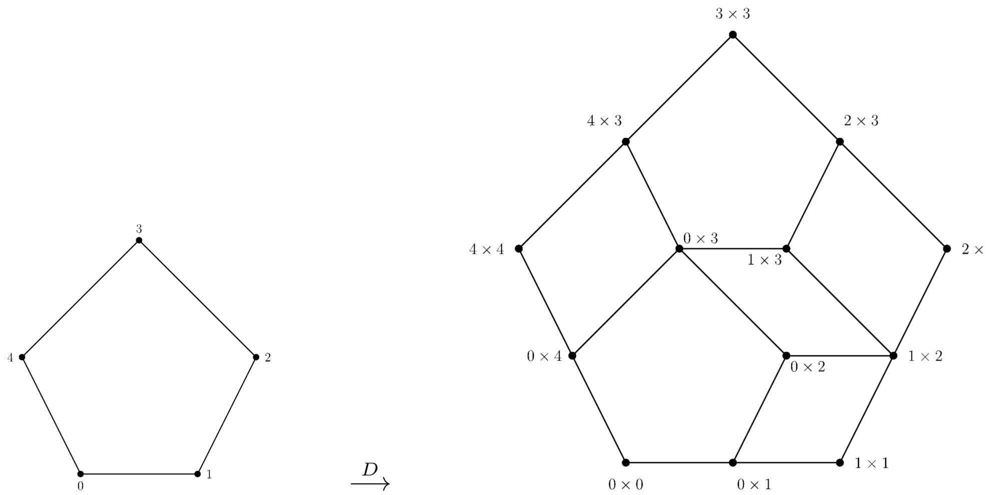
**Example 3B**

```
gap> L:=[[1,4,3,2],[8,7,6,5]];;
gap> M:=[[1,4,8,5],[2,3,7,6]];;
gap> N:=[[1,2,6,5],[4,3,7,8]];;
gap> Ex4:=PoincareCubeCWComplex(L,M,N);
Regular CW-complex of dimension 3

gap> IsClosedManifold(Ex4);
true

gap> ManifoldType(Ex4);
"euclidean"

gap> G:=FundamentalGroup(Ex4);
<fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f1*f2^-1*f1^-3*f2^-1, f1*f2^-1*f1*f2^3 ]

gap> p:=2;;A:=CohomologyRing(Ex4,p);
<algebra of dimension 6 over GF(2)>
gap> PresentationOfGradedStructureConstantAlgebra(A);
Graded algebra GF(2)[ x_1, x_2, x_3 ] /
                    [ x_1*x_2+x_2^2, x_1^2+x_2^2, x_2*x_3, x_2^3+x_1*x_3 ]
with indeterminate degrees [ 1, 1, 2 ]
```




The computation of the cup product in this example merits discussion. The given regular CW-complex has very few cells and so one could easily use barycentric subdivision to produce a homeomorphic simplicial complex for which cup products could be computed using the Alexander-Whitney formula. For other CW complexes $X$ involving large numbers of cells barycentric subdivision is not a practical option. In such cases a more practical approach is to find a *diagonal approximation* $D\colon X \to X \times X$ that is homotopic to the actual diagonal $\Delta\colon X \to X \times X$, that is a cellular map, and that satisfies $D(x) = (x, x)$ on vertices. This is the technique used in the GAP example. In fact, it suffices to construct (in a 'consistent' fashion) a diagonal approximation $D\colon \overline{e^k} \to \overline{e^k} \times \overline{e^k}$ on the closure of each cell $e^k$ in $X$.

To illustrate the nature of such a diagonal approximation let us consider the case of a space $X$ involving a closed 2-cell $\overline{e^2}$ equal to a pentagonal disk involving five vertices, five edges and one face. The vertices of $\overline{e^2}$ are labelled $0, \ldots, 4$ in the following diagram and the edge incident with vertices $i, j$ is labelled $ij$. The vertices of $\overline{e^2} \times \overline{e^2}$ are labelled $i \times j$ for $0 \le i, j \le 4$. The diagram illustrates a diagonal approximation $D\colon \overline{e^2} \to \overline{e^2} \times \overline{e^2}$ which maps $\overline{e^2}$ homeomorphically onto the closure of the union $0 \times e^2 \ \cup\ e^2 \times 3 \ \cup\ 01 \times 43 \ \cup\ 01 \times 12 \ \cup\ 01 \times 23 \ \cup\ 12 \times 23$.




Details of this approach to computing cup products can be found in [1]. The main feature of the algorithm is that it uses an explicit contracting homotopy on the cellular chain complex $C_*(\overline{e^k})$ of the closure of each cell $e^k$ in $X$.

1.4. **Prelude to the next lecture.** From the point of view of computational group cohomology Examples 3A, 3B are perhaps a bit of a cheat because they start with a space rather than with a group. A more realistic example would have been to start with, say, a space group $G$ acting on Euclidean space and to compute its cohomology. Consider for instance the crystallographic group $G$ acting on $\mathbb{R}^4$ described by the BBNWZ parameters $[1,2,1,1]$ used in the catalogue [7]. In the next lecture we'll consider computations such as

$$H^n(G,\mathbb{Z}) = \begin{cases} 0, & \text{odd } n \geq 1 \\ \mathbb{Z}_2^5 \oplus \mathbb{Z}^6, & n = 2 \\ \mathbb{Z}_2^{15} \oplus \mathbb{Z}, & n = 4 \\ \mathbb{Z}_2^{16}, & \text{even } n \geq 6 \end{cases}$$

peformed in the following GAP session. This GAP session also computes that there are 11 ring generators in degree 2 and no further generators, and it computes the Poincaré series

$$(x^4 + 4x^3 + 6x^2 + 4x + 1)/(-x + 1)$$

for mod 2 cohomology. The group $G$ acts on the contractible space $\mathbb{R}^4$ but not freely − it contains non-trivial finite point stabilizer groups. So in addition to having to construct a suitable equivariant cell structure on $\mathbb{R}^4$ we also need to take account somehow of the stabilizer groups.

The cell structure on $\mathbb{R}^n$ corresponding to a Dirichlet-Voronoi fundamental domain for a crystallographic group can be constructued using Polymake [22] or, in some cases such as this example, by using a brute force search to try to subdivide a fundamental domain for the translation subgroup $T \leq G$ into parallelepiped subregions. Note that the *point group* $G/T$ is finite. Geometric considerations lead to a contracting homotopy on the resulting cellular chain complex $C_*(\mathbb{R}^n)$ of (not necessarily free) $\mathbb{Z}G$-modules.

**GAP Session**

```
gap> G:=SpaceGroupBBNWZ(4,1,2,1,1);;
gap> R:=ResolutionCubicalCrystGroup(G,12);;
gap> R!.dimension(5);
16
gap> R!.dimension(7);
16
gap> List([1..16],k->R!.boundary(5,k)=R!.boundary(7,k));
[ true, true, true, true, true, true, true, true, true, true, true, true,
```

```
  true, true, true, true ]
gap> C:=HomToIntegers(R);;
gap> List([0..8],i->Cohomology(C,i));
[ [ 0 ], [  ], [ 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0 ], [  ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0 ], [  ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ], [  ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ] ]
gap> List([0..8],i->Length(IntegralRingGenerators(R,i)));
[ 1, 0, 11, 0, 0, 0, 0, 0, 0 ]

gap> D:=List([0..11],n->Cohomology(HomToIntegersModP(R,2),n));;
gap> PoincareSeries(D,11);
(x_1^4+4*x_1^3+6*x_1^2+4*x_1+1)/(-x_1+1)
```

## 2. **Lecture 2**

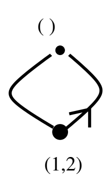
2.1. **Classifying space for a finite group.** Let $G$ be a finite group. To construct a cellular classifying space $BG$ we can attempt to construct a cellular contractible space $X = EG$ on which $G$ acts by permuting cells and then take $BG = EG\,/\,G$. We can construct the skeleta of $X^n$ inductively as follows, assuming that $G$ is not too large. As a running example we consider the group $G = S_3$.

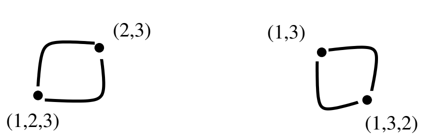
First, let $X^0$ consist of $|G|$ vertices, one vertex for each $g$ in $G$. Deem the vertex corresponding to 1 to be the base vertex and deem this base vertex to be *contracted*. Deem all other cells to be *uncontracted*.

Set $Y_1 = X^0$ and for $n = 1$ do

> **Step 1.** Choose any uncontracted $n-1$-cell $e^{n-1}$ in $Y_n$. This cell represents a homotopy class $\alpha\pi_{n-1}Y_n$. Set
> $$Y_n = Y_n \cup e^n$$
> where $n$-cell $e^n$ is attached in a way that kills $\alpha$. Deem both $e^{n-1}$ and $e^n$ to be contracted and represent this by an arrow $e^{n-1} \to e^n$.
>
> **Step 2.** For the cell $e^n$ in Step 1 and for all $1 \neq g \in G$ set
> $$Y_n = Y_n \cup g \cdot e^n$$
> where the $n$-cell $g \cdot e^n$ is attached such that its boundary cells are those obtained from the boundary cells of $e^n$ under the action of $g$ and such that $G$ acts freely on $Y^n$. Deem the cells $g \cdot e^n$ to be uncontracted.

Suppose that we have chosen $e^0 = (1,2)$ in Step 1. Then at this stage our space $Y_1$ will look as follows.



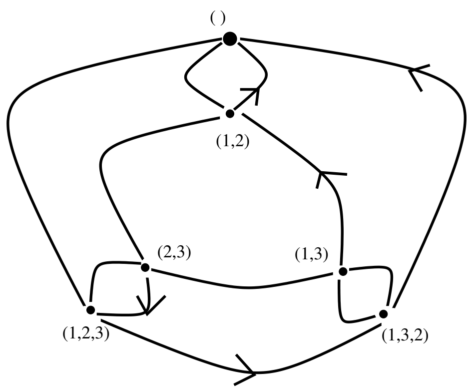



Now, for $n = 1$ do

> **Step 3.** While there is some uncontracted $n$-cell $e^n$ with precisely one uncontracted cell $e^{n-1}$ in its boundary, deem both $e^n$ and $e^{n-1}$ to be contracted and record this by placing an arrow $e^{n-1} \to e^n$.

In our running example Step 3 has no effect at this stage. Repeat Steps 1,2,3 for $n = 1$ until all $n-1$-cells $e^{n-1}$ are contracted.

For our running example, if we choose $e^0 = (1, 3, 2)$ in the second application of Step 1 then we end up with $Y_1$ looking as follows and with all vertices contracted. The arrows on $Y_1$ involve some arbitrary choice in Step 3.




For $n = 1$ do

> **Step 4.** Set $X^n = Y_n$ .

Now repeat Steps 1–4 for $n = 2, 3, \cdots, n + 1$ where $n$ is the degree of the homology/cohomology computation for which $X = EG$ is to be used. By construction, $\pi_i(X^{n+1}) = 0$ for $0 \le i \le n$. The arrows introduced in the construction of $X$ determine an explicit contracting homotopy on the cellular chain complex $C_*X$. This chain complex is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

The quotient space $X^2 / G$ corresponds to a free presentation for the group $G$, with 1-cells corresponding to generators and 2-cells corresponding to relators. For the running example, and for appropriate choices in Steps 1–3, the presentation is

$$S_3 = \langle x, y \,|\, x^2 = 1, xy^2x^{-1}y^{-1} \rangle$$

with $x$ corresponding to the permutation $(1, 2)$ and $y$ corresponding to the permutation $(1, 3, 2)$.

The attaching maps for the 3-cells in this construction of $X = EG$ correspond to *identities between the relators*. In general, the attaching maps of $n$-cells can be viewed as what J.-L. Loday refers to as *homotopical syzygies* [33].

This construction of $EG$ is not practical when applied directly to large finite groups $G$. But when applied in conjunction with other techniques it can lead to useful calculations. For instance, the following example applies the construction to Sylow subgroups to compute the first few integral homology groups of the Mathieu simple group $M_{23}$ of order 10200960 .

$$H_n(M_{23}, \mathbb{Z}) = \begin{cases} 0\,, & n = 1, 2, 3, 4 \\ \mathbb{Z}_7\,, & n = 5 \\ \mathbb{Z}_2\,, & n = 6 \\ \mathbb{Z}_{240}\,, & n = 7. \end{cases}$$

These calculations for $n \le 6$ were first obtained by James Milgram [37] using theoretical arguments, and provided a negative answer to a long-standing question attributed in [37] to J.-L. Loday: is the trivial group $G$ the only finite group having $H_n(G, \mathbb{Z}) = 0$ for $n = 1, 2, 3$? The computation for $n = 7$ takes a couple of hours to terminate. An analogous computation

$$H^3(M_{24}, U(1)) \cong H_3(M_{24}, \mathbb{Z}) \cong \mathbb{Z}_{12}$$

for the Mathieu simple group of order 244823040 has been used in the study of generalized Mathieu Moonshine [19, 20].

**Example 4**

```
gap> GroupHomology(MathieuGroup(23),1);
[  ]
gap> GroupHomology(MathieuGroup(23),2);
[  ]
gap> GroupHomology(MathieuGroup(23),3);
[  ]
gap> GroupHomology(MathieuGroup(23),4);
[  ]
gap> GroupHomology(MathieuGroup(23),5);
[ 7 ]
gap> GroupHomology(MathieuGroup(23),6);
[ 2 ]
gap> GroupHomology(MathieuGroup(23),6);
[ 16, 3, 5 ]
```

The above construction of a contractible cellular space $EG$ with free action of $G$ can be applied to the more general situation of a finite group $G$ with subgroup $H < G$ and where we take $X^0$ to be the set of cosets of $H$ in $G$. In this case the output will be a contractible space $X$ on which $G$ acts with finite stabilizer groups. This generalization is not yet implemented in HAP but could prove extremely fruitful when used in conjunction with the perturbation technique described in the next section.

2.2. **Homological perturbation theory.** Suppose that $X$ is a contractible CW-complex on which a group $G$ acts, not necessarily freely, by permuting cells. Some examples to have in mind are:

(1) $G = SL_m(\mathbb{Z})$, $X =$ polyhedral complex in the space of positive definite real symmetric $m \times m$ matrices.
(2) $G = PSL_2(\mathcal{O}_d)$ for $\mathcal{O}_d$ the ring of integers of $\mathbb{Q}(\sqrt{-d})$, $d$ a square free positive integer, $X =$ polyhedral complex in the closure of hyperbolic space $\mathcal{H}^3$.
(3) $G =$ crystallographic group acting on $X = \mathbb{R}^n$.
(4) $\theta\colon G \to GL_n(\mathbb{R})$ is a representation of a finite group $G$ and $X$ is the polytope in $\mathbb{R}^n$ arising as the convex hull of the set $\{\theta(g) \ : \ g \in G\}$.
(5) $G$ is a graph of groups and $X$ is the corresponding tree.
(6) $G \twoheadrightarrow Q$ is a surjective group homomorphism and $X = EQ$.
(7) $H$ is a subgroup of a finite group $G$ and $X$ is the contractible space constructed in Section 2.1 with $X^0$ equal to the set of cosets of $H$ in $G$.

The cellular chain complex $C_*X$ is then an exact sequence of $\mathbb{Z}G$-modules with $H_0(C_*X) \cong \mathbb{Z}$. Each chain group has a $\mathbb{Z}G$-module structure

$$C_n = \bigoplus_{e^n \subset X/G} \mathbb{Z}G \otimes_{\mathbb{Z}[Stab(e^n)]} \mathbb{Z}$$

where $e^n$ ranges over representatives of orbits of $n$-cells in $X$ and $Stab(e^n)$ is the cell stabilizer. For each cell stabilizer group $Stab(e^n)$ let $R_*^{e^n}$ be a free $\mathbb{Z}[Stab(e^n)]$-resolution of $\mathbb{Z}$, and set

$$R_{n,*} = \bigoplus_{e^n \subset X/G} R_*^{e^n} \otimes_{\mathbb{Z}[Stab(e^n)]} \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z}^e$$

where $\mathbb{Z}^e$ denotes the integers with a 'suitable' action of $G$. The boundary homomorphisms in $C_*X$ induce a sequence of $\mathbb{Z}G$-equivariant chain maps

$$R_{*,*}: \qquad \cdots \to R_{n,*} \to R_{n-1,*} \to \cdots \to R_{0,*}$$

If $R_{*,*}$ happened to be a bi-complex then its total complex $\mathrm{Tot}(R_{*,*})$ would be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. In general, $R_{*,*}$ is not a bi-complex because the horizontal homomorphisms don't necessarily square to zero. However, using an idea of C.T.C. Wall [50], one can add equivariant homomorphism 'correction terms' $d^k_{p,q}\colon R_{p,q} \to R_{p-k,q+k-1}$ that make $\mathrm{Tot}(R_{*,*})$ into a filtered chain complex and hence a free $\mathbb{Z}G$-resolution. Choices needed in the construction of the correction terms $d^k_{p,q}$ are made using contracting homotopies on the stabilizer resolutions $R^{e^n}_*$. A contracting homotopy on $\mathrm{Tot}(R_{*,*})$ can be constructed by additionally using an explicit contracting homotopy on $C_*X$.

The above perturbation technique is illustrated in the following GAP Example 5A which computes

$$H_5(PSL_3(\mathcal{O}(\mathbb{Z}[\omega]), \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$$

for $\omega$ a primitive cube root of unity. The 8-dimensional contractible space $X$ used in the example was computed by Mathieu Dutour-Sikiric using reduction theory and the theory of perfect forms [12] and its data set is available in HAP. The GAP commands show that the space $X$ has 3 orbits of vertices, 10 orbits of edges, and so on. All stabilizer groups are finite and the method of Section 2.1 is used to compute resolutions for them. Note that no contracting homotopy has (yet) been implemented for the space $X$ and so the resulting group resolution is not endowed with an explicit contracting homotopy.

**Example 5A**

```
gap> _X:=ContractibleGcomplex("PGL3Eisenstein_Integers");
Non-free resolution of length 1000 in characteristic 0 for matrix group .
No contracting homotopy available.

gap> List([0..10],_X!.dimension);
[ 3, 10, 18, 24, 21, 14, 8, 5, 3, 0, 0 ]

gap> R:=FreeGResolution(_X,6);
Resolution of length 6 in characteristic 0 for matrix group .
No contracting homotopy available.

gap> Homology(TensorWithIntegers(R),5);
[ 3, 6, 6 ]
```

C.T.C. Wall [50] was interested in constructing a free resolution $R^G_*$ for a group $G$ arising in a group extension $N \rightarrowtail G \twoheadrightarrow Q$. He used the perturbation method to construct the resolution as a 'twisted tensor product' $R^G_* = R^B_* \tilde{\otimes} R^Q_*$ of resolutions for $N$ and $Q$. In the case of a direct product $G = N \times Q$ the twisted tensor product becomes the standard tensor product (over $\mathbb{Z}$); . Working recursively, one can construct a resolution for $G$ using a sequence of subnormal subgroups $G_0 \lhd G_1 \lhd \cdots \lhd G$ and the resulting group extensions $G_{i-1} \rightarrowtail G_i \twoheadrightarrow G_i/G_{i-1}$. This is analogous to the polycyclic group approach used in Section 1 for the Schur multiplier. See these examples for more details on this analogy/generalisation. If the subgroups are all normal in $G$ then one can alternatively work recursively with the extensions $G_i/G_{i-1} \rightarrowtail G/G_{i-1} \twoheadrightarrow G/G_i$. Both approaches are implemented in HAP. The following example uses the second approach, with $G_i$ the terms of the upper central series, to compute

$$H_2(Syl_2(G), \mathbb{Z}) = \oplus_{i=1}^{35} \mathbb{Z}_2$$

for $G$ the Rubik's cube group. It also computes the 2-primary part

$$H_2(G, \mathbb{Z})_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

of the Schur multiplier of $G$.

**Example 5B**

```
gap> G:=Group([
>   ( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19),
>   ( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35),
>   (17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11),
>   (25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24),
>   (33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27),
>   (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)
> ]);;
gap> S:=SylowSubgroup(G,2);
<permutation group of size 134217728 with 24 generators>

gap> L:=UpperCentralSeries(S);;
gap> R:=ResolutionNormalSeries(L,3);
Resolution of length 3 in characteristic 0 for <permutation group with
134217728 generators> .

gap> Homology(TensorWithIntegers(R),2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]

gap> PrimePartDerivedFunctorViaSubgroupChain(G,R,TensorWithIntegers,2);
[ 2, 2, 2 ]
```

The *Bianchi groups* are the groups $G = PSL_2(\mathcal{O}_d)$ where $d$ is a square free positive integer and $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Consider the $\mathbb{Z}G$-module $P_{\mathcal{O}_d}(k) \subset \mathcal{O}_d[x,y]$ consisting of the homogeneous degree $k$ polynomials in two variables $x, y$ with action of $G$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(dx - by, -cx + ay) \ .$$

Thus the module $P_{\mathcal{O}_d}(0) = \mathcal{O}_d \cong \mathbb{Z} \oplus \mathbb{Z}$ is the free abelian group of rank 2 with trivial action of $G$. The perturbation method is used in Example 5C to compute:

$$H^2(PSL_2(\mathcal{O}_6), P_{\mathcal{O}_6}(24)) = \begin{array}{l} (\mathbb{Z}_2)^{26} \oplus (Z_6)^8 \oplus (Z_{12})^9 \oplus \mathbb{Z}_{24} \oplus (\mathbb{Z}_{120})^2 \oplus (\mathbb{Z}_{840})^3 \\ \oplus \mathbb{Z}_{2520} \oplus (\mathbb{Z}_{27720})^2 \oplus (\mathbb{Z}_{24227280})^2 \oplus (\mathbb{Z}_{411863760})^2 \\ \oplus \mathbb{Z}_{24544382437489286518774251428366644498129840} \\ \oplus \mathbb{Z}_{14726629462493571911264550857019986988779040} \\ \oplus \mathbb{Z}^4 \end{array} ,$$

The contractible space $X$ in this example is a 2-complex living in $\mathfrak{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$. Several methods exist for computing $X$, one of which is discussed in Section 2.3.

**Example 5C**

```
gap> R:=ResolutionPSL2QuadraticIntegers(-6,3);
Resolution of length 3 in characteristic 0 for PSL(2,O-6) .
No contracting homotopy available.

gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 120,
```

```
120, 840, 840, 840, 2520, 27720, 27720, 24227280, 24227280, 411863760,
411863760, 2454438243748928651877425142836664498129840,
1472662946249357191126455085701998698877904O, 0, 0, 0 ]
```

2.3. **Modular forms.** The Eichler-Shimura isomorphism [13, 44]

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \oplus E_k(\Gamma) \cong_{\mathsf{Hecke}} H^1(\Gamma, P_{\mathbb{C}}(k-2))$$

relates the cohomology of groups to the theory of modular forms associated to a finite index subgroup $\Gamma$ of $SL_2(\mathbb{Z})$. Here we assume that $\Gamma$ contains the kernel of the homomorphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/\mathbb{N}Z)$ for some integer $N > 0$ or, in other words, that $\Gamma$ is a *congruence subgroup* of *level* $N$. The $\mathbb{Z}\Gamma$-module $P_{\mathbb{C}}(k-2) \subset \mathbb{C}[x,y]$ consists of the homogeneous degree $k-2$ polynomials with action described above. In particular $P_{\mathbb{C}}(0) = \mathbb{C}$ is the trivial module. The left-hand side of the isomorphism involves the vector space $S_k(\Gamma)$ of *cuspidal modular forms of weight* $k$ and the *Eisenstein space* $E_k(\Gamma)$ *of weight* $k$. Details of these terms on the left-hand side are not needed for our present purposes, other than to note that a modular form is a complex valued function $f\colon \mathfrak{H} = \{z \in \mathbb{C} \ :\ \mathrm{Re}(z) > 0\} \to \mathbb{C}$ on the upper-half plane satisfying certain properties. The Eichler-Shimura isomorphism preserves Hecke operators which, on the right-hand side, can be defined as follows.

A congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$ and element $g \in SL_2(\mathbb{Q})$ determine the subgroup $\Gamma' = \Gamma \cap g\Gamma g^{-1}$ and homomorphisms

$$\Gamma \hookleftarrow \Gamma' \stackrel{\gamma\mapsto g^{-1}\gamma g}{\longrightarrow} g^{-1}\Gamma' g \hookrightarrow \Gamma\ .$$

These homomorphisms give rise to homomorphisms of cohomology groups

$$H^1(\Gamma, P_{\mathbb{C}}(k-2)) \stackrel{tr}{\leftarrow} H^1(\Gamma', P_{\mathbb{C}}(k-2)) \stackrel{\alpha}{\leftarrow} H^1(g^{-1}\Gamma' g, P_{\mathbb{C}}(k-2)) \stackrel{\beta}{\leftarrow} H^1(\Gamma, P_{\mathbb{C}}(k-2))$$

with $\alpha$, $\beta$ functorial maps, and $tr$ the transfer map. For each integer $s \geq 1$ we define the composite $T_s = tr \circ \alpha \circ \beta\colon H^1(\Gamma, P_{\mathbb{C}}(k-2)) \to H^1(\Gamma, P_{\mathbb{C}}(k-2))$ to be the *Hecke operator* determined by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}\ .$$

Let us introduce the function $q = q(z) = e^{2\pi i z}$. For any modular form $f(z)$ there are numbers $a_s$ such that

$$f(z) = \sum_{s=0}^{\infty} a_s q^s$$

for all $z \in \mathfrak{H}$. The form $f$ is a cusp form if $a_0 = 0$. A non-zero cusp form $f \in S_k(\Gamma)$ is an *eigenform* if it is simultaneously an eigenvector for the Hecke operators $T_s$ for all $s = 1, 2, 3, \cdots$ (see for instance [45] for details). An eigenform is said to be *normalized* if its coefficient $a_1 = 1$. Moreover, the coefficients $a_s$ with $s$ a composite integer can be expressed in terms of the coefficients $a_p$ for prime $p$. It can be shown [2] that $f \in S_k(\Gamma)$ admits a basis of eigenforms. This all implies that, in principle, we can construct an approximation to an explicit basis for the space $S_k(\Gamma)$ of cusp forms by computing eigenvalues for Hecke operators.

The following computer Example 6 establishes that $S_{12}(SL_2(\mathbb{Z}))$ has a basis consisting of one cusp eigenform

$f = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + 401856q^{14} + 1217160q^{15} + 987136q^{16} - 6905934q^{17} + 2727432q^{18} + 10661420q^{19} + \ldots$ .

Note that the coeficient of $q^n$ is the value of Ramanujan's tau function $\tau(n)$ and that the number 691 appears in the eigenvectors!

**Example 6**

```
gap> G:=SL(2,Integers);;
gap> for p in [2,3,5,7,11,13,17,19] do
T:=HeckeOperator(H,p,12);;
Print("eigenvalues= ",Eigenvalues(Rationals,T), " and eigenvectors = ",
Eigenvectors(Rationals,T)," for p= ",p,"\n");
od;

eigenvalues= [ 2049, -24 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 2
eigenvalues= [ 177148, 252 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 3
eigenvalues= [ 48828126, 4830 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 5
eigenvalues= [ 1977326744, -16744 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 7
eigenvalues= [ 285311670612, 534612 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 11
eigenvalues= [ 1792160394038, -577738 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 13
eigenvalues= [ 34271896307634, -6905934 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 17
eigenvalues= [ 116490258898220, 10661420 ] and eigenvectors =
[ [ 1, 6075/691, 23580/691 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] for p= 19
```

The contractible space $X \subset \mathfrak{H}$ used in the computations of Example 6 is the *cubic tree* – the tree in which each vertex has degree 3. It has one $SL_2(\mathbb{Z})$-orbit of vertices with cyclic stabilizer group of order 3. The workhorse of the computations is a contracting homotopy on this space $X$ implemented using the Euclidean algorithm in $\mathbb{Z}$. Via perturbation theory this leads to a free $SL_2(\mathbb{Z})$-resolution with contracting homotopy.

In principle, Hecke operators for congruence subgroups of *Bianchi groups* $G = PSL_2(\mathcal{O}_d)$ should be computable using the same method. The one step not yet implemented is the contracting homotopy on the 2-complex $X \subset \mathfrak{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$ underlying the free resolution used (*c.f.* see Example 5C). For the Euclidean cases $d = 1, 2, 3, 7, 11$ it should be possible to use the Euclidean algorithm in $\mathcal{O}_d$ in the contracting homotopy on $X$. Computer experiments suggest that one brute force approach that appears to work generally is simply to construct a ball $B_r \subset X$ consisting of the closures of all finitely many 2-cells connected to the root vertex of $X$ by paths of at most some given length $r > 0$ in the Cayley graph and then apply the naive collapsing procedure described in [1] to try to produce a contracting discrete vector field on the ball $B_r$. Experiments have not yielded an example where the procedure fails to produce a contracting vector field on the ball. Any given cohomology computation involves at most a finite number of cells of $X$, so if $r$ is chosen large enough these cells will lie in $B_r$ and the partial contracting homotopy on $X$ should suffice.

The space $X \subset \mathfrak{H}^3$ is constructed from a fundamental domain for the action of $G$. Explicit fundamental domains for certain values of $d$ were calculated by Bianchi in the 1890s and further calculations were made by Swan in 1971 [47]. In the 1970s, building on Swan's work, Robert Riley [41] developed a computer program for computing fundamental domains of certain Kleinian groups (including Bianchi groups). In their 2010 PhD theses Alexander Rahm [40] and M.T. Aranes independently developed Pari/GP and Sage software based on Swan's ideas. In 2011 Dan Yasaki [52] used a different approach based on Voronoi's theory of perfect forms in his Magma software for fundamental domains of Bianchi groups. Aurel Page [38] developed software for fundamental domains of Kleinian groups in his 2010 masters thesis. In 2018 Sebastian Schoennenbeck [6] used a more general approach based on perfect forms in his Magma software for computing fundamental domains of Bianchi and other groups. Output from the code of Alexander Rahm and Sebastian

Schoennenbeck for certain Bianchi groups has been stored in HAP for use in constructing free resolutions.

More recently an implementation of Swan's algorithm has been included in HAP. The implementation uses exact computations in $\mathbb{Q}(\sqrt{-d})$ and in $\mathbb{Q}(\sqrt{d})$. A bespoke implementation of these two fields is part of the implementation so as to avoid making apparently slower computations with cyclotomic numbers. The account of Swan's algorithm in the thesis of Alexander Rahm was the main reference during the implementation. The space $X$ is constructed in HAP using the following description/method of Swan [47].

We take the boundary $\partial\mathfrak{H}^3$ to be the Riemann sphere $\mathbb{C}\cup\infty$ and let $\overline{\mathfrak{H}}^3$ denote the union of $\mathfrak{H}^3$ and its boundary. The element $\infty$ and each element of the number field $\mathbb{Q}(\sqrt{-d})$ are thought of as lying in the boundary $\partial\mathfrak{h}^3$ and are referred to as *cusps*. Let $Y$ denote the union of $\mathfrak{H}^3$ with the set of cusps, $Y=\mathfrak{H}^3\cup\{\infty\}\cup\mathbb{Q}(\sqrt{-d})$.

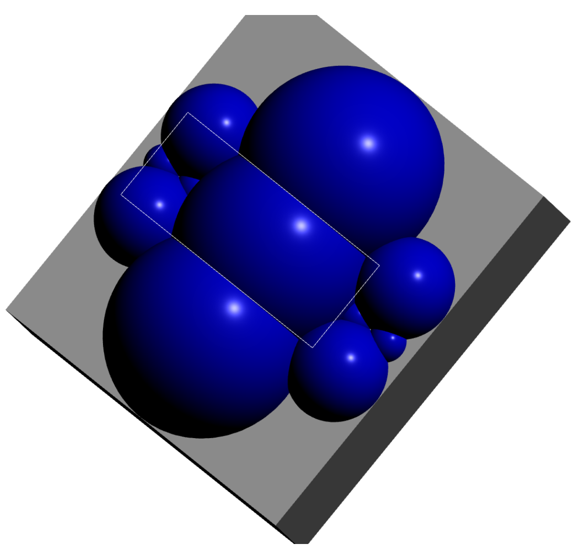
A pair $(a,b)$ of elements in $\mathcal{O}_d$ is said to be *unimodular* if the ideal generated by $a,b$ is the whole ring $\mathcal{O}_d$ and $a\neq 0$. A unimodular pair can be represented by a hemisphere in $\overline{\mathfrak{H}}^3$ with base centred at the point $b/a\in\mathbb{C}$ and of radius $|a|$. The radius is $\leq 1$. Think of the points in $\mathfrak{H}^3$ as lying strictly above $\mathbb{C}$. Let $B$ denote the space obtained by removing all such hemispheres from $\mathfrak{H}^3$.

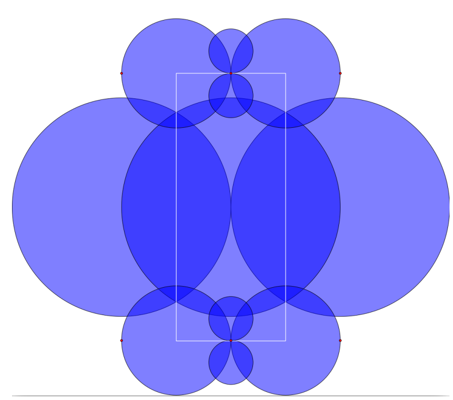
When $d\equiv 3 \mod 4$ let $F$ be the subspace of $\overline{\mathfrak{H}}^3$ consisting of the points $x+iy+jt$ with $-1/2\leq x\leq 1/2$, $-1/4\leq y\leq 1/4$, $t\geq 0$. Otherwise, let $F$ be the subspace of $\overline{\mathfrak{H}}^3$ consisting of the points $x+iy+jt$ with $-1/2\leq x\leq 1/2$, $-1/2\leq y\leq 1/2$, $t\geq 0$.

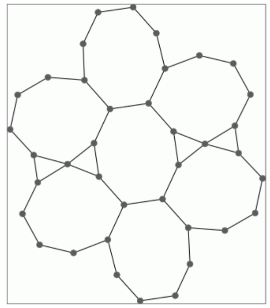
It is explained in [47] that $F\cap B$ is a 3-cell in a $PSL_2(\mathcal{O}_d)$-equivariant CW structure on $Y$. There is just one orbit of 3-cells, and any 3-cell has trivial stabilizer. Following [40] we refer to $F\cap B$ as the *Bianchi fundamental polyhedron*. The 2-dimensional retract $X$ is obtained from $Y$ by removing every cell whose boundary intersects with the orbit of the 0-cell $\infty$.

In the following visualizations for $d=6$, the blue surface below the white rectangular region depicts those cells of $X$ in the boundary of $F\cap B$. Red points indicate cusps in this blue surface. These red cusps form one orbit, indicating that the class group of $\mathbb{Q}(\sqrt{-6})$ is of order 2. The CW structure is computed using exact arithmetic over $\mathbb{Q}(\sqrt{-6})$ and over $\mathbb{Q}(\sqrt{6})$. The 1-skeleton of this CW-structure is also visualized.

```
gap> D:=BianchiPolyhedron(-6);;
gap> Display3D(D);;
gap> Display2D(D);;
gap> Display(GraphOfRegularCWComplex(RegularCWComplex(D)));
```







By considering neighbours of the Bianchi fundamental polytope in its orbit under the group action we can construct a finite set of generators for the group $PSL_2(\mathcal{O}_d)$. As mentioned above, the construction of an explicit contracting homotopy on the 2-complex $X$ has yet to be implemented.

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