Prueba #1

Gustavo Pinto

Junio 2022

Pregunta 1:

a)

Los hogares maximizan la siguiente función:

$$\max \sum_{t=0}^{\infty} \beta^{t} U(c_t, l_t, g_t) \tag{1}$$

Sujeto a las siguientes restricciones:

$$c_t + inv_t + b_t^g = w_t(1-\tau_{w,t})l_t + r_t^k k_t + (1+r_{t-1})b_{t-1}^g \tag{2}$$

$$k_{t+1} = (1-\delta)k_t + \phi\left(\frac{inv_t}{k_t}\right)k_t \tag{3}$$

Por lo tanto, el Lagrangeano queda:

$$\begin{aligned}\mathcal{L}: \quad \sum_{t=0}^{\infty} & \beta^t U(c_t, g_t, l_t) \\ & + \beta^t \lambda_t \big[w_t(1-\tau_{w,t})l_t + r_t^k k_t + (1+r_{t-1})b_{t-1}^g \\ & - c_t - inv_t - b_t^g\big] \\ & + \beta^t \lambda_t q_t \left[(1-\delta)k_t + \phi\left(\frac{inv_t}{k_t}\right)k_t - k_{t+1}\right]\end{aligned} \tag{4}$$

Las condiciones de primer orden quedan como:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{\partial U}{\partial c_t} - \beta^t \lambda_t = 0$$

$$\frac{\partial U}{\partial c_t} = \lambda_t \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = \beta^t \frac{\partial U}{\partial l_t} + \beta^t \lambda_t w_t(1-\tau_{w,t}) = 0$$

$$\frac{\partial U}{\partial l_t} = -\lambda_t w_t(1-\tau_{w,t})$$

$$\frac{\partial U}{\partial l_t} = -\frac{\partial U}{\partial c_t} w_t(1-\tau_{w,t})$$

$$\frac{-\frac{\partial U}{\partial l_t}}{\frac{\partial U}{\partial c_t}(1-\tau_{w,t})} = w_t \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial inv_t} = -\beta^t \lambda_t + \beta^t \lambda_t q_t \frac{\partial \phi}{\partial inv_t} k_t = 0$$

$$q_t \frac{\partial \phi}{\partial inv_t} k_t = 1 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t q_t + \beta^{t+1} \lambda_{t+1} r_{t+1}^k + \beta^{t+1} \lambda_{t+1} q_{t+1} (1-\delta) + \beta^{t+1} \lambda_{t+1} q_{t+1} \left[\phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) + \frac{\partial \phi}{\partial k_{t+1}} k_{t+1} \right] = 0$$

$$\beta \lambda_{t+1} \left[r_{t+1}^k + q_{t+1}(1-\delta) + q_{t+1} \phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) + q_{t+1} \frac{\partial \phi}{\partial k_{t+1}} k_{t+1} \right] = \lambda_t q_t$$

$$\beta \frac{\partial U}{\partial c_{t+1}} \left[r_{t+1}^k + q_{t+1}(1-\delta) + q_{t+1} \phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) + q_{t+1} \frac{\partial \phi}{\partial k_{t+1}} k_{t+1} \right] = \frac{\partial U}{\partial c_t} q_t \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial b_t^g} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1 + r_t) = 0$$

$$\lambda_t + \beta \lambda_{t+1} (1 + r_t)$$

$$\frac{\partial U}{\partial c_t} = \beta \frac{\partial U}{\partial c_{t+1}} (1 + r_t) \tag{9}$$

b) Desde el lado de la firma tenemos que estas maximizan la siguiente función

$$Max\ \pi: y_t - w_t l_t^d - r_t^k k_t^d \tag{10}$$

Con:

$$y_t = A_t (k_t^d)^\alpha (l_t^d)^{1-\alpha} \tag{11}$$

Por lo tanto, la función de beneficios queda:

$$Max\ \pi: A_t (k_t^d)^\alpha (l_t^d)^{1-\alpha} - w_t l_t^d - r_t^k k_t^d$$

Las condiciones de primer orden quedan:

$$\frac{\partial \pi}{\partial k_t^d} = \alpha A_t (k_t^d)^{\alpha-1} (l_t^d)^{1-\alpha} - r_t^k = 0$$

$$\alpha A_t (k_t^d)^{\alpha-1} (l_t^d)^{1-\alpha} = r_t^k \tag{12}$$

$$\frac{\partial \pi}{\partial l_t^d} = (1-\alpha) A_t (k_t^d)^\alpha (l_t^d)^{-\alpha} - w_t = 0$$

$$(1-\alpha) A_t (k_t^d)^\alpha (l_t^d)^{-\alpha} = w_t \tag{13}$$

Donde la productividad sigue el siguiente proceso:

$$ln\, A_t = \rho_a\, ln\, A_{t-1} + \sigma_a \epsilon_{a,t} \tag{14}$$

c) Podemos resumir las ecuaciones del equilibrio de la siguiente forma:

1. La demanda de la economía:

$$y_t = c_t + inv_t + g_t$$

Con:

$$k_t^d = k_t$$

$$l_t^d = l_t$$

En el lado de los hogares, tenemos:

2. Ecuación de Euler del capital:

$$\beta \frac{\partial U}{\partial c_{t+1}} \left[r_{t+1}^k + q_{t+1}(1-\delta) + q_{t+1}\phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) + q_{t+1}\frac{\partial \phi}{\partial k_{t+1}} k_{t+1} \right] = \frac{\partial U}{\partial c_t} q_t$$

3. Oferta de trabajo:

$$\frac{-\frac{\partial U}{\partial l_t}}{\frac{\partial U}{\partial c_t}\left(1-\tau_{w,t}\right)} = w_t$$

4. Q de tobin:

$$q_t \frac{\partial \phi}{\partial inv_t} k_t = 1$$

5. Ecuación de Euler de los bonos de gobierno:

$$\frac{\partial U}{\partial c_t} = \beta \frac{\partial U}{\partial c_{t+1}}(1+r_t)$$

6. La Dinámica del Capital:

$$k_{t+1} = (1-\delta)k_t + \phi\left(\frac{inv_t}{k_t}\right) k_t$$

En el caso de las firmas tenemos:

7. Función de producción:

$$y_t = A_t (k_t^d)^\alpha (l_t^d)^{1-\alpha}$$

8. Demanda de capital:

$$\alpha A_t k_t^{\alpha-1} l_t^{1-\alpha} = r_t^k$$

9. Demanda de mano de obra:

$$(1-\alpha) A_t k_t^\alpha l_t^{-\alpha} = w_t$$

10. Proceso AR1 para la productividad:

$$\ln A_t = \rho_a \ln A_{t-1} + \sigma_a \epsilon_{a,t}$$

Por parte del gobierno, tenemos:

11. Restricción presupuestaria:

$$g_t + (1 + r_{t-1})b_{t-1}^g = \tau_{w,t} w_t l_t + b_t^g$$

12. Proceso AR1 de tasa de impuesto:

$$\tau_{w,t} = (1 - \rho_\tau)\bar{\tau}_w + \rho_\tau \tau_{w,t-1} + \sigma_\tau \epsilon_{\tau,t}$$

13. Regla Fiscal:

$$g_t = (1 - \rho_g)\bar{g} + \rho_g g_{t-1} - \theta_g b_{t-1}^g + \sigma_g \epsilon_{g,t}$$

d) Definimos la utilidad como:

$$U(c_t, g_t, l_t) = \ln\left(c_t g_t^\eta - \psi \frac{l_t^{1+\Theta}}{1+\Theta}\right)$$

Y los costos de ajuste como:

$$\phi\left(\frac{inv_t}{k_t}\right) = \frac{inv_t}{k_t} - \frac{\mu}{2}\left(\frac{inv_t}{k_t} - \delta\right)^2$$

Las derivadas respectivas quedan:

$$\frac{\partial U}{\partial c_t} = \frac{g_t^\eta}{c_t g_t^\eta - \psi \frac{l_t^{1+\Theta}}{1+\Theta}}$$

$$\frac{\partial U}{\partial c_{t+1}} = \frac{g_{t+1}^\eta}{c_{t+1} g_{t+1}^\eta - \psi \frac{l_{t+1}^{1+\Theta}}{1+\Theta}}$$

$$\frac{\partial U}{\partial l_t} = \frac{-\psi l_t^\Theta}{c_t g_t^\eta - \psi \frac{l_t^{1+\Theta}}{1+\Theta}}$$

$$\frac{\partial \phi}{\partial inv_t} = \frac{1}{k_t} - \frac{\mu}{k_t}\left(\frac{inv_t}{k_t} - \delta\right)$$

$$\frac{\partial \phi}{\partial k_t} = -\frac{inv_t}{k_t^2} + \mu\left(\frac{inv_t}{k_t} - \delta\right)\left(\frac{inv_t}{k_t^2}\right)$$

$$\frac{\partial \phi}{\partial k_{t+1}} = -\frac{inv_{t+1}}{k_{t+1}^2} + \mu\left(\frac{inv_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{inv_{t+1}}{k_{t+1}^2}\right)$$

Por lo tanto, reemplazamos estas expresiones en nuestras ecuaciones de equilibrio y obtenemos:

$$q_t\left(1 - \mu\left(\frac{inv_t}{k_t} - \delta\right)\right) = 1$$

$$q_{t+1}\left(1 - \mu\left(\frac{inv_{t+1}}{k_{t+1}} - \delta\right)\right) = 1$$

$$\beta \frac{g_{t+1}^{\eta}}{c_{t+1}g_{t+1}^{\eta} - \psi \frac{l_{t+1}^{1+\Theta}}{1+\Theta}} \left[r_{t+1}^{k} + q_{t+1}(1-\delta) + q_{t+1}\phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) + q_{t+1}\left[-\frac{inv_{t+1}}{k_{t+1}} + \mu\left(\frac{inv_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{inv_{t+1}}{k_{t+1}}\right)\right]\right] = \frac{g_t^{\eta}}{c_t g_t^{\eta} - \psi \frac{l_t^{1+\Theta}}{1+\Theta}} q_t$$

$$\beta \frac{g_{t+1}^{\eta}}{c_{t+1}g_{t+1}^{\eta} - \psi \frac{l_{t+1}^{1+\Theta}}{1+\Theta}} \left[r_{t+1}^{k} + q_{t+1}(1-\delta) + q_{t+1}\phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) - q_{t+1}\frac{inv_{t+1}}{k_{t+1}}\left[1 - \mu\left(\frac{inv_{t+1}}{k_{t+1}} - \delta\right)\right]\right] = \frac{g_t^{\eta}}{c_t g_t^{\eta} - \psi \frac{l_t^{1+\Theta}}{1+\Theta}} q_t$$

$$\beta \frac{g_{t+1}^{\eta}}{c_{t+1}g_{t+1}^{\eta} - \psi \frac{l_{t+1}^{1+\Theta}}{1+\Theta}} \left[r_{t+1}^{k} + q_{t+1}(1-\delta) + q_{t+1}\phi\left(\frac{inv_{t+1}}{k_{t+1}}\right) - \frac{inv_{t+1}}{k_{t+1}}\right] = \frac{g_t^{\eta}}{c_t g_t^{\eta} - \psi \frac{l_t^{1+\Theta}}{1+\Theta}} q_t$$

$$\frac{\psi l_t^{\Theta}}{g_t^{\eta}(1-\tau_{w,t})} = w_t$$

$$\frac{g_t^{\eta}}{c_t g_t^{\eta} - \psi \frac{l_t^{1+\Theta}}{1+\Theta}} = \beta \frac{g_{t+1}^{\eta}}{c_{t+1}g_{t+1}^{\eta} - \psi \frac{l_{t+1}^{1+\Theta}}{1+\Theta}}(1+r_t)$$

En estado estacionario tenemos las siguientes condiciones:

$$A_t = A_{t+1} = \bar{A} = 1$$

$$c_t = c_{t+1} = \bar{c}$$

$$k_t = k_{t+1} = \bar{k}$$

$$g_t = g_{t+1} = \bar{g}$$

$$l_t = l_{t+1} = \bar{l}$$

$$r_t^k = r_{t+1}^k = \overline{r^k}$$

$$w_t = w_{t+1} = \overline{w}$$

$$b_t^g = b_{t+1}^g = \bar{b}^g = 0$$

$$\frac{inv_t}{k_t} = \frac{inv_{t+1}}{k_{t+1}} = \delta$$

$$\frac{inv_t}{k_t} = \frac{inv_{t+1}}{k_{t+1}} = \delta$$

Si sustituimos esta última condición en la q de Tobin, llegamos a:

$$\bar{q} = 1$$

$$\phi\left(\frac{inv_t}{k_t}\right) = \phi(\delta) = \delta$$

Introducimos estas condiciones en nuestra ecuación de Euler:

$$\beta \frac{\bar{g}^{\eta}}{\bar{c}\bar{g}^{\eta} - \psi \frac{\bar{l}^{1+\Theta}}{1+\Theta}} \left[\overline{r^k} + \bar{q}(1-\delta) + \bar{q}\phi(\delta) - \delta\right] = \frac{\bar{g}^{\eta}}{\bar{c}\bar{g}^{\eta} - \psi \frac{\bar{l}^{1+\Theta}}{1+\Theta}} \bar{q}$$

$$\beta\left[\overline{r^k} + (1-\delta) + \delta - \delta\right] = 1$$

$$\beta\left[\overline{r^k} + 1 - \delta\right] = 1$$

$$\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} = \overline{r^k}$$

$$\overline{r^k} = \frac{1}{\beta} + \delta - 1$$

$$\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} = \frac{1}{\beta} + \delta - 1$$

$$\bar{k} = \left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha\bar{A}\bar{l}^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$$

$$(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha} = \bar{w}$$

$$\bar{g} = \bar{w}\bar{\tau}_w\bar{l} - \frac{\bar{b}^g}{\beta} + \bar{b}^g$$

$$\psi = \frac{\bar{w}\bar{g}^{\eta}\left(1 - \tau_{w,t}\right)}{\bar{l}^{\Theta}}$$