homalg - Constructive Homological Algebra

Mohamed Barakat

RWTH Aachen University

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Joint work with Markus Lange-Hegermann, Sebastian Gutsche, Sebastian Posur

The category R-fpmod

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up to equivalence.

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 - objects L, M, N, \ldots and
 - sets of morphisms $\operatorname{Hom}_{\mathcal{A}}(M,N)$.
- In fact, only the Hom sets and their compositions are relevant

$$\operatorname{Hom}_{\mathcal{A}}(L,M) \times \operatorname{Hom}_{\mathcal{A}}(M,N) \to \operatorname{Hom}_{\mathcal{A}}(L,N)$$

 $(\varphi,\psi) \mapsto \varphi\psi.$

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- The objects are only place-holders, exactly like the vertices of a graph.
- The notion "equivalence of categories" gives one even more freedom in the description of a (constructive) model of the category.

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Example

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 \leadsto from the categorical point of view, linear algebra and matrix theory are equivalent.

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$$\begin{split} R\text{-}\mathbf{fpmod} &\simeq \\ R\text{-}\mathbf{fpres} := \begin{cases} \mathsf{Obj:} & \mathtt{M} \in R^{r \times g}, \mathtt{N} \in R^{r' \times g'}, \ldots, \ r, g, r', g' \in \mathbb{N}, \\ \mathsf{Mor:} & [(\mathtt{M}, \mathtt{A}, \mathtt{N})] \ \mathsf{with} \ \mathtt{A} \in R^{g \times g'} \ \mathsf{lies} \ \mathsf{in} \ \mathsf{Hom}(\mathtt{M}, \mathtt{N}), \\ & \mathsf{if} \ \mathtt{N} \geq \mathtt{MA}, \end{cases} \\ \mathsf{and} \ (\mathtt{M}, \mathtt{A}, \mathtt{N}) &\sim (\mathtt{M}', \mathtt{A}', \mathtt{N}') :\iff \mathtt{M} = \mathtt{M}', \mathtt{N} = \mathtt{N}', \mathtt{N} \geq \mathtt{A} - \mathtt{A}'. \end{cases}$$

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Definition

A category is called **constructively** ABELian if all disjunctions (\lor) and existential quantifiers (\exists) in the axioms of an ABELian category can be realized by algorithms.

Example

Let $\varphi:M\to N$ be a morphism in \mathcal{A} .

$$M \xrightarrow{\varphi} N$$

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$$\ker \varphi$$

$$M \stackrel{\varphi}{-\!\!\!-\!\!\!-} N$$

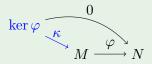
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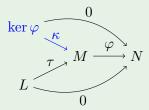
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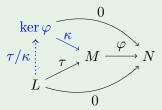
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The "hidden" existential quantifiers of "kernels"

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A is a category

A is a category:

- **1** For any object M there exists an **identity morphism** 1_M .
- 2 For any two composable morphisms φ, ψ there exists a composition $\varphi\psi$.

A is a category with zero

\mathcal{A} is a category **with zero**:

- 3 There exists a zero object 0.
- 4 For all objects M, N there exists a zero morphism 0_{MN} .

\mathcal{A} is an **additive** category

A is an **additive** category:

- **5** For all objects M, N there exists an **addition** $(\varphi, \psi) \mapsto \varphi + \psi$ in the ABELian group $\operatorname{Hom}_{\mathcal{A}}(M, N)$.
- 6 For all objects M,N there exists a subtraction $(\varphi,\psi)\mapsto \varphi-\psi$ in the ABELian group $\operatorname{Hom}_{\mathcal A}(M,N)$.
- 7 For all objects A_1, A_2 there exists a **direct sum** $A_1 \oplus A_2$ and projections $\pi_i : A_1 \oplus A_2 \to A_i$ such that
- 8 for all pairs of morphisms $\varphi_i: M \to A_i, \ i=1,2$ there exists a *unique* product morphism $\{\varphi_1, \varphi_2\}: M \to A_1 \oplus A_2$ satisfying $\{\varphi_1, \varphi_2\}\pi_i = \varphi_i$.
- 9 for all pairs of morphisms $\varphi_i: A_i \to M, i = 1, 2$ there exists a *unique* coproduct morphism^a $\langle \varphi_1, \varphi_2 \rangle : A_1 \oplus A_2 \to M$.

^afollows from the above axioms [HS97, Prop. II.9.1].

\mathcal{A} is a **pre-Abelian** category

A is a **pre-Abelian** category:

- **10** For any morphism $\varphi: M \to N$ there exists a **kernel** $\ker \varphi \overset{\kappa}{\hookrightarrow} M$, such that
- for any morphism $\tau:L\to M$ with $\tau\varphi=0$ there exists a unique lift $\tau_0:L\to\ker\varphi$ of τ along κ , i.e., $\tau_0\kappa=\tau$.
- $\text{ For any morphism } \varphi: M \to N \text{ there exists a cokernel } N \overset{\varepsilon}{\to} \operatorname{coker} \varphi, \text{ such that }$
- **(8)** for any morphism $\eta: N \to L$ with $\varphi \eta = 0$ there exists a unique colift $\eta_0: \operatorname{coker} \varphi \to L$ of η along ε , i.e., $\varepsilon \eta_0 = \eta$.

A is an **ABELian** category

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- 14 Each mono is a kernel mono.
- 15 Each epi is a cokernel epi.

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Determining a syzygy matrix S of A:

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Theorem ([BLH11])

If R is left computable then the category R-fpres $\simeq R$ -fpmod is constructively ABELian.

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 Deciding the solvability of the inhomogeneous linear system XA = B for a single row matrix B is thus nothing but the **submodule membership problem** for the submodule generated by the rows of the matrix A.

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- Deciding the solvability of the inhomogeneous linear system XA = B for a single row matrix B is thus nothing but the submodule membership problem for the submodule generated by the rows of the matrix A.
- Finding a particular solution X (in case one exists) solves the submodule membership problem **effectively**.

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In particular, the equation $\mathtt{XA} = \mathtt{B}$ is solvable iff

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 DecideZeroRowsEffectively(B, A) computes a matrix T satisfying B + TA = B', where B' = DecideZeroRows(B, A).

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DecideZeroRowsEffectively(B, A) computes a matrix T satisfying B + TA = B', where B' = DecideZeroRows(B, A).
 In particular, if the equation XA = B is solvable then we recover

$$X := -T =: RightDivide(B, A).$$

Example

gap> ?SyzygiesOfRows

Example

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gap> ?SyzygiesGeneratorsOfRows

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Example (computable rings)	
ring	algorithm
a	'
b	

Example (computable rings)	
ring	algorithm
a constructive field k	
а b	

Example (computable rings)	
ring	algorithm
a constructive field k ring of rational integers $\mathbb Z$	
b	

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	l
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a constructive field k	
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a polynomial ring ^a $R[x_1, \ldots, x_n]$	
AD and of the other days	
^a R any of the above rings	

Example (computable rings)	
ring a constructive field k ring of rational integers \mathbb{Z} a univariate polynomial ring $k[x]$ a polynomial ring ^a $R[x_1,\ldots,x_n]$ many noncommutative rings	algorithm
${}^{a}R$ any of the above rings	

Example	(computable rings)
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ring	algorithm
a constructive field k	
ring of rational integers $\mathbb Z$	
a univariate polynomial ring $k[x]$	
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many noncommutative rings	
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In this context any algorithm to compute a GRÖBNER basis is a substitute for the GAUSS resp. HERMITE normal form algorithm.

BasisOfRows

Exercise

Use BasisOfRows to program

- DecideZeroRows,
- DecideZeroRowsEffectively,
- and SyzygiesOfRows.

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Hint:

$$\left(\begin{smallmatrix}1&-X\\0&Y\\0&S\end{smallmatrix}\right)\left(\begin{smallmatrix}1&B&0\\0&A&1\end{smallmatrix}\right)\xrightarrow{\mathsf{BasisOfRows}}\left(\begin{smallmatrix}1&B'&-X\\0&A'&Y\\0&0&S\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&-X\\0&Y\\0&S\end{smallmatrix}\right)\left(\begin{smallmatrix}1&B&0\\0&A&1\end{smallmatrix}\right)$$

```
Example (homalg rings)
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gap> ?Ring Constructions
gap> Q := HomalgFieldOfRationalsInSingular();
Q
```

```
gap> LoadPackage( "RingsForHomalg" );;
gap> ?Ring Constructors
gap> 0 := HomalgFieldOfRationals();
gap> F2 := HomalgRingOfIntegers( 2 );
GF (2)
gap> F4 := HomalgRingOfIntegers(2, 2);
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gap> ZZ := HomalgRingOfIntegersInSingular();
7.
qap> R := F4 * "x, y, z";
GF(2^2)[x,y,z]
```

```
gap> ?HomalgMatrix
```

```
gap> ?HomalgMatrix
gap> ZZ := HomalgRingOfIntegers();
Z
```

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gap> ?HomalgMatrix
gap> ZZ := HomalgRingOfIntegers();
Z
gap> m := HomalgMatrix( "[ 1, 2, 3, 4, 5, 6 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
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```

A is a **pre-ABELian** category

A is a **pre-Abelian** category:

- For any morphism $\varphi:M\to N$ there exists a **kernel** $\ker \varphi \overset{\kappa}{\hookrightarrow} M$, such that
- for any morphism $\tau:L\to M$ with $\tau\varphi=0$ there exists a unique lift $\tau_0:L\to\ker\varphi$ of τ along κ , i.e., $\tau_0\kappa=\tau$.
- $\hbox{ For any morphism } \varphi: M \to N \hbox{ there exists a cokernel } N \stackrel{\varepsilon}{\twoheadrightarrow} \operatorname{coker} \varphi, \hbox{ such that }$
- **(8)** for any morphism $\eta: N \to L$ with $\varphi \eta = 0$ there exists a unique colift $\eta_0: \operatorname{coker} \varphi \to L$ of η along ε , i.e., $\varepsilon \eta_0 = \eta$.

${ t S} = { t Syzygie} { t sOfRows}({ t A},{ t N})$

For the stacked matrix $({A\atop N})$ we write

$${\tt SyzygiesOfRows}((\begin{smallmatrix} A\\ N \end{smallmatrix})) = (\verb"K"L")$$

with KA + LN = 0 and define^a

$$SyzygiesOfRows(A, N) := K$$
,

for which we need a matrix algorithm CertainColumns to extract K.

^aAgain, one can derive more efficient algorithms to compute the relative version of SyzygiesOfRows.

How to compute $\ker \varphi \stackrel{\kappa}{\hookrightarrow} M$ of $\varphi: M \to N$?

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First compute

$$K = SyzygiesOfRows(A, N),$$

the matrix representing κ .

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2 Then $\ker \varphi$ is presented by the matrix

SyzygiesOfRows(K, M).

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- $\hbox{ For any morphism } \varphi: M \to N \hbox{ there exists a cokernel } N \stackrel{\varepsilon}{\to} \operatorname{coker} \varphi, \hbox{ such that }$
- **(3)** for any morphism $\eta:N\to L$ with $\varphi\eta=0$ there exists a unique colift $\eta_0:\operatorname{coker}\varphi\to L$ of η along ε , i.e., $\varepsilon\eta_0=\eta$.

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Let $\kappa: \mathtt{K} \overset{\mathtt{K}}{\hookrightarrow} \mathtt{M}$ be the kernel monomorphism and $\tau: \mathtt{L} \xrightarrow{\mathtt{T}} \mathtt{M}$ a morphism with $\tau \varphi = 0$ for $\varphi = \operatorname{coker} \kappa$. Then the matrix

$$X := RightDivide(T, K)$$

represents $\tau_0: L \to K$, the lift of τ along κ .

¹Cf. [BR08, 3.1.1, case (2)]).

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It is an easy exercise¹ to check that X represents a *morphism*.

¹Cf. [BR08, 3.1.1, case (2)]).

Thank you

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