# Driven Damped Oscillator (DDO) filter

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A damped oscillator driven by a signal can be used as a filter and differentiator.

The relevant equation is

$$m\ddot{x} = -k\,(x - u) - 2\gamma\,\dot{x} \tag{1}$$

where $u$ is the driving signal.

Inserting the frequency $\omega_0 = \sqrt{k/m}$ and scaling time to $\omega_0$, i.e. $t \to \omega_0 t$, we get

$$\ddot{x} + 2\zeta\dot{x} + x = u \tag{2}$$

where $\zeta = \gamma/m\omega_0$ is a dimensionless damping parameter (large $\zeta$ = high damping). The "quality factor" $Q$, sometimes used to characterize resonances, is the inverse of $\zeta$.

## 1 ODE system

The equation can be written as a 1st order ODE system

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x} + \mathbf{b}\,u \tag{3}$$

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

The solution is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\cdot\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t dt'\,\left[e^{-\mathbf{A}t}\cdot\mathbf{b}\right]u \tag{5}$$

The matrix exp can be obtained by finding the eigensystem of $\mathbf{A}$.

The eigenvalues of $\mathbf{A}$ are

$$\lambda_{1,2} = -\zeta \pm \sqrt{\zeta^2 - 1} \tag{6}$$

For $1 > \zeta > 0$ we have complex (oscillatory) eigenvalues and the system is "under-damped" while for $\zeta > 1$ the eigenvalues are real and the system is "over-damped".

The solution for constant $u$ can be written

$$\mathbf{x}(t) = e^{\mathbf{A}t}\cdot\mathbf{x}_0 + u\,\mathbf{h}(t), \quad \mathbf{h}(t) = \int_0^t dt'\;e^{\mathbf{A}t'}\cdot\mathbf{b} \tag{7}$$

For $\zeta > 1$ we have

$$q = \sqrt{\zeta^2 - 1} \tag{8}$$

$$e^{\mathbf{A}t} = e^{-\zeta t}\cosh qt\cdot\mathbf{1} + e^{-\zeta t}\frac{\sinh qt}{q}\cdot\begin{bmatrix} \zeta & 1 \\ -1 & -\zeta \end{bmatrix} \tag{9}$$

$$\mathbf{h}(\mathbf{t}) = \left(1 - e^{-\zeta t}\cosh qt\right)\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-\zeta t}\frac{\sinh qt}{q}\cdot\begin{bmatrix} -\zeta \\ 1 \end{bmatrix} \tag{10}$$

For $0 < \zeta < 1$ we set $q = \sqrt{1 - \zeta^2}$ and change $\cosh, \sinh$ to $\cos, \sin$.

For the special case $\zeta = 1$ the equations simplify to

$$e^{\mathbf{A}t} = e^{-t} \cdot \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} \tag{11}$$

$$\mathbf{h}(\mathbf{t}) = \begin{bmatrix} 1 - e^{-t} - t\,e^{-t} \\ t\,e^{-t} \end{bmatrix} \tag{12}$$

## 2 Transfer function

$$G = \frac{x}{u} \tag{13}$$

Applying standard Laplace transforms:

$$G(s) = \frac{1}{1 + 2\zeta\, s + s^2}, \quad G(i\omega) = \frac{1}{1 + 2i\zeta\,\omega - \omega^2} \tag{14}$$

$$|G(i\omega)| = \frac{1}{\sqrt{(\omega^2 - 1)^2 + 4\zeta^2\omega^2}} \tag{15}$$

$$\arg G(i\omega) = \frac{2\zeta\omega}{\omega^2 - 1} \tag{16}$$

It can be shown from the structure of $|G(i\omega)|$ that the value $\zeta = 1/\sqrt{2}$ is the lowest damping factor that does not make any signal amplification close to the resonance. The tranfer function is then $|G(i\omega)| = (\omega^4 + 1)^{-1/2}$, which is always lower than 1.

## 3 Application to signal filtering

Assuming we have a signal $u(t)$ digitized at a regular interval $\Delta t$, $u_k = u(k \cdot \Delta t)$ then we update the system by

$$\mathbf{x}_{k+1} = e^{\mathbf{A}\omega_0\Delta t} \cdot \mathbf{x}_k + u_k\,\mathbf{h}(\omega_0\Delta t) \tag{17}$$

Then $[\mathbf{x}_{k+1}]_1$ corresponds to the filtered signal and $\omega_0\,[\mathbf{x}_{k+1}]_2$ to the 1st order time derivative $du/dt$.

Li & Ma (2014, DOI 10.1007/s00034-013-9634-z) "A New Approach for Filtering and Derivative Estimation of Noisy Signals" use the above equation to filter a signal and extract the 1st order derivative. They do not state explicitly that it is the oscillator equation but arrive at the same results from a different reasoning.

They use $\zeta = 1/\sqrt{2}$ and define a parameter $\epsilon = \sqrt{2}/\omega_0$. They write the equation

$$\ddot{x} = -\frac{2}{\epsilon^2}(x - u) - \frac{2}{\epsilon}\dot{x} \tag{18}$$