4. Odds and Ends

This section is devoted to an overview of a number of interesting questions. For fuller coverage see one of the classics: Morse and Feshbach *Methods of Theoretical Physics,* 1953, or a bit more rigorously, Carrier, Krook, and Pearson, *Functions of a Complex Variable,* 1966.

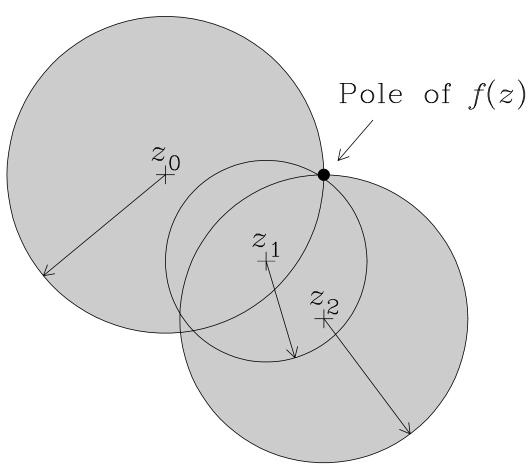
Recall that one way of defining analyticity is to say there is a convergent power series for the function about a point z_0. What is the size of the region of convergence, its shape, and what determines them? The answer is surprisingly easy to state and to prove. The region of convergence is always a circular disk centered on z_0; the radius of the disk, called the **radius of convergence**, is the distance from z_0 to the nearest singularity (excluding, the branch cuts, which can always be moved out of the way). The Taylor series always converges uniformly within the disk; this means it can be differentiated term by term, for example, and the resultant series remains valid. Whether the series converges on the circle is a delicate matter that depends in detail on the singularities.

In this way we see at once that the Taylor series about the point $z = 0$ for a complicated function like

$$h(z) = \frac{(5 + 33z^4)\,e^{-2\sin z}}{\sqrt{4 + z^2}} \tag{27}$$

must converge inside the disk radius 2, without evaluating even the first term! A general result is that entire functions have power series that converge everywhere, the radius of convergence is infinite.

These ideas lead to the process of **analytic continuation**. Suppose $f(z)$ has a power series with radius of convergence r_0 about the point z_0. We can concentrate on another point z_1 within the disk, evaluate f and all of its derivatives there, and obtain a new Taylor series for the function now centered on z_1. By choosing the location of z_1 carefully with respect to the singularity that set the radius r_0 in the first series, we can arrange that the new series converges in parts of the complex plane where the old series didn't work, and so we can extend the definition of f to a bigger portion of the plane. This process can usually be repeated indefinitely, until the entire plane is covered, except of course for the singularities. One tricky point concerns the branch cuts, which some authors slip past. During analytic continuation it may happen that the series evaluations in a region continued by two different routes will not agree — a decision is needed to make a unique function, and a branch cut will result where the transition is made from one definition to the other. A possible failure of analytic continuation into the whole plane arises because there are some weird



functions, analytic in a region bounded by barrier of infinitely dense singularities, a wall that cannot be crossed. An example from the theory of elliptic functions is:

$$\sigma(z) = \frac{1}{\prod_{n=1}^{\infty} (1 - z^n)}, \quad |z| < 1 . \tag{28}$$

A few second's thought will convince you that there is a singularity at every point on the unit circle in the form $z = \exp(2im\pi/n)$ with m, n integers. Returning to less pathological territory, you will see that analytic continuation is a way to give a definite meaning to apparently divergent series or integrals. This is another way of extending the meaning of the Γ function, whose integral definition

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt \tag{29}$$

only works for $\mathrm{Re}\, z > 0$, into the rest of the complex plane.

The Cauchy-Riemann relations imply something stated earlier in the notes: if f is analytic in a region, its real and imaginary parts satisfy the two-dimensional **Laplace's equation** $\nabla^2 u = \nabla^2 v = 0$. To see this differentiate the first part of (16) with respect to x, and second against y:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \, \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x} . \tag{30}$$

Since v is continuously differentiable to all orders, $\partial^2 v/\partial x \, \partial y = \partial^2 v/\partial y \, \partial x$, and therefore it follows from (30) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{31}$$

which proves the result for the real part of f; the imaginary part follows by taking the derivative of (16) in the other order and applying the same argument. Then analytic functions immediately become solutions of electrostatic or flow problems, with logarithmic singularities playing the role of line charges or line vortices, and simple poles as dipoles, or source-sink pairs.

Considerable power is added to this notion when we see that a solution of Laplace's equation under boundary conditions with some very simple geometry, can be transformed by **conformal mapping** into another solution with remarkably complicated boundaries. We sketch the idea. So far we have been thinking of the function $f(z)$ as a kind of contour map, with real and imaginary parts contoured, or perhaps magnitude and phase plotted; Matlab makes this very easy to do in practice; recall Exercise 3.1. In this picture, poles appear as pointy spikes and branch cuts as cliffs of discontinuity. But there is another way of visualizing the results of the action of $f(z)$: for every point z in the complex plane, $w = f(z)$ is a corresponding point in an image plane. Points and lines in the z-plane are mapped into points and lines (usually) in the w-plane. As the simplest illustration we look at the function $w = z^2$. If we write $w = u + iv$, $u, v \in \mathbb{R}$, then $w = z^2$ becomes

$$u + iv = x^2 - y^2 + 2ixy, \text{ or } u = x^2 - y^2, \text{ and } v = 2xy . \tag{32}$$

Now consider the horizontal lines in the z-plane given by $\mathrm{Im}\, z = y =$ constant.

Eliminating x between the equations for u and v gives us the equation for the image of y = constant in the w-plane:

$$v = 2y\sqrt{y^2 + u} \tag{33}$$

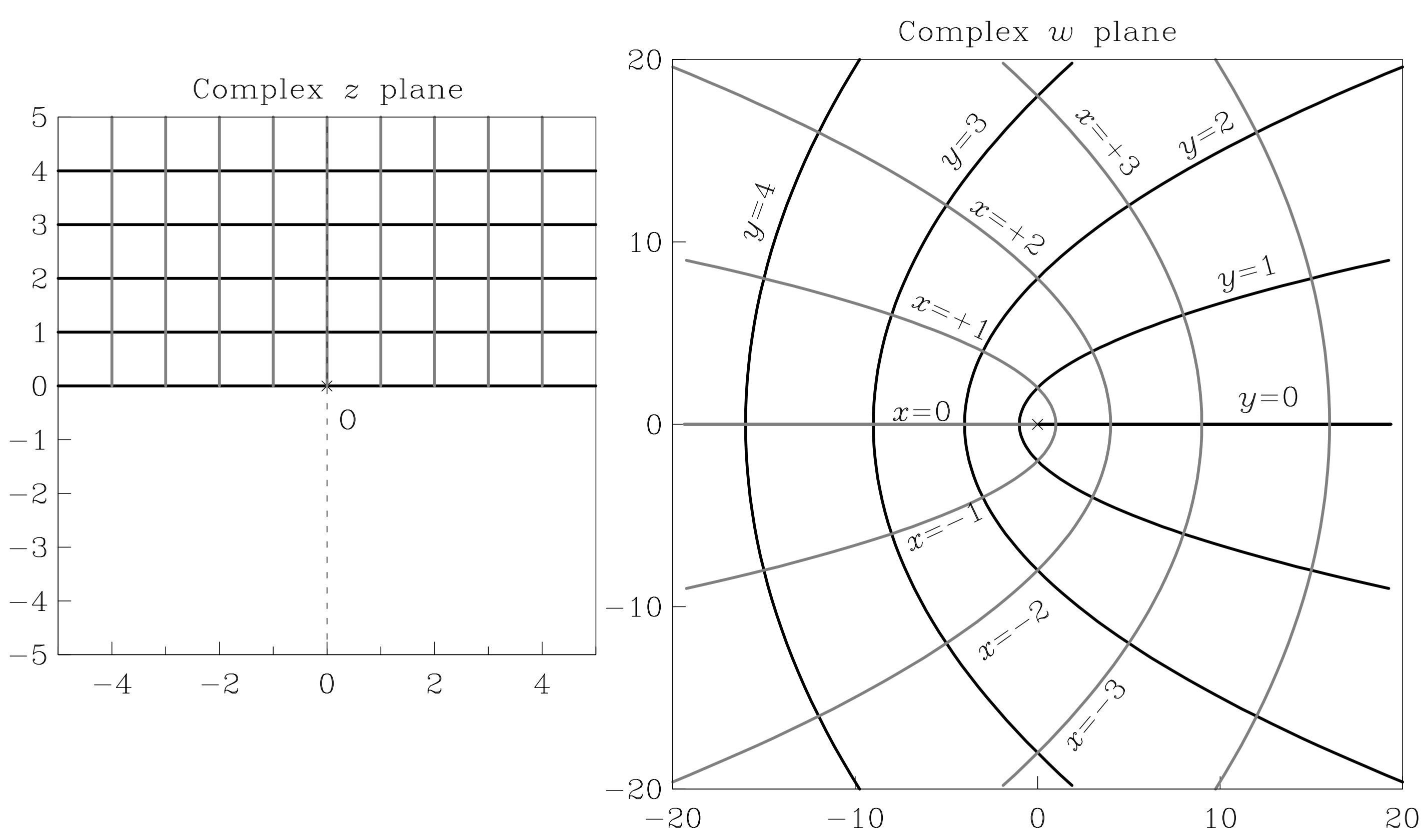
which describes a family of parabolas with horizontal axes, as shown in black in the figure. And of course, we can find the vertical grid lines in the z-plane in the same way:

$$v = 2x\sqrt{x^2 - u} \tag{34}$$

shown in gray below. We see that only the half plane $y \geq 0$ gets mapped into the whole w-plane; this is a choice, of course, necessitated to avoid two points in z landing at the same w. The real x axis is folded at $z = 0$ back onto itself.

Now imagine that we have a solution to $\nabla^2 V = 0$ in the z-plane expressed as a complex function $V = \mathrm{Re}\, g(z)$; then $\mathrm{Re}\, g(z^{'\S})$ is a solution over in the w-plane, and the boundary values, say on $y = 0$ map onto the image points of $y = 0$, which is the top and bottom sides of $v = 0$, $u \geq 0$. For more details of this amazing technique see the references given earlier. The masterpiece of conformal mapping is the **Schwarz-Christoffel transform**, which maps the half plane $\mathrm{Im}\, z \geq 0$ into the interior (or exterior) of an arbitrary polygon in w! See the reference to the paper by myself and Klitgord given in the first section.

These transforms are called conformal, because, except at the singular points (like the origin in this example) angles between lines in z are preserved between the



corresponding image lines in w. Observe how the images of the grid remain orthogonal in the figure. We have no space to prove this here.

Exercises

4.1 The common Taylor series expansion for the natural log is centered on the point $z = 1$, and is often written:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

Where is the singularity that controls the radius of convergence? Choose a suitable point z_1 within the disk of convergence of this series at which to evaluate the log and as many of its derivatives as you may need, in order calculate ln(i/2) from the Taylor series. Sum the new series to obtain a value with at least 4 significant figures of accuracy. Minimize the amount of numerical effort required. Explain your reasoning.

4.2 Map makers often use conformal transformations. The first and most famous of these is Mercator's projection:

$$x = \phi, \quad y = \tfrac{1}{2}\ln\left[\frac{1+\sin\lambda}{1-\sin\lambda}\right]$$

where ϕ and λ are longitude and latitude coordinates (in radians) of points on a (spherical) globe. Now suppose we write $z = x + iy$; any analytic function $f : \mathbb{C} \to \mathbb{C}$ (or, if you prefer, $w = f(z)$) maps the Mercator plane into another conformal projection of the same set of points. What function f maps the Mercator plane into a stereographic projection? First find the function that maps the south pole to the point at infinity; then introduce parameters to describe the general stereographic projection.
From data provided, use Matlab to draw (a) a Mercator map of the world; (b) a stereographic map, using the f you have determined; (c) another conformal map of your own devising. Provide listings of the Matlab functions.
Take care to make your maps look presentable.

5. Integration in the Complex Plane

Now we come to the most celebrated part of the theory, and one where applications may occur to you. To differentiate we just followed the ordinary real-variable recipe. Because we are in a plane there are several possible choices, for example, we might want to evaluate an integral of $f(z)$ over an area of the complex plane. In fact the most fruitful approach is as a line integral and usually not over any old path, but over closed contours. So let us examine the simplest case, in which there is a simple domain C in the plane, which we will consider both to be a set of points in $\mathbb{R}^2$ and in $\mathbb{C}$. Within C the function f is analytic at every point and on the boundary ∂C. We will evaluate

$$I = \int_{\partial C} f(z)\, dz = \int_{\partial C} [u(x,y) + \mathrm{i}v(x,y)]\,(dx + \mathrm{i}dy)\ . \tag{36}$$

We will use a result from vector calculus to help us here. Recall **Stoke's Theorem**: if C is a smooth surface with boundary ∂C then

$$\int_C \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \int_{\partial C} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \tag{37}$$

where $\mathbf{F}$ is a smooth vector-valued function defined on a surface, $\hat{\mathbf{n}}$ is the surface normal, $\hat{\mathbf{t}}$ is a unit vector tangent to the boundary ∂C. The path must be traversed counterclockwise, which means $\hat{\mathbf{t}}\times\hat{\mathbf{n}}$ must point away from the C. Let us use Stoke's Theorem on the real part of (36):

$$\mathrm{Re}\, I = \int_{\partial C} [u(x,y)\,dx - v(x,y)\,dy]\ . \tag{38}$$

We would like to identify (38) with the line integral on the right of (37). So we say

$$\mathbf{F}(x,y) = \hat{\mathbf{x}}u(x,y) - \hat{\mathbf{y}}v(x,y)\ . \tag{39}$$

We turn now to the left side of (37). Since the plane $\mathbb{R}^2$ is flat, the unit normal $\hat{\mathbf{n}}$ is a constant vector in the third orthogonal direction of $\mathbb{R}^3$ — we must not call it the z direction for obvious reasons! And so we need only that component of the curl of $\mathbf{F}$:

$$\hat{\mathbf{n}}\cdot\nabla\times\mathbf{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{40}$$

$$= -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\ . \tag{41}$$

You will see at once from the second of the Cauchy-Riemann relations that this expression vanishes identically at every point in C. And so the line integral around the boundary is zero. There is no need to do any more work to evaluate $\mathrm{Im}\, I$, because we can apply this result to the function $g(z) = \mathrm{i}f(z)$. Therefore we have established the important result that, if C is any simple domain in $\mathbb{C}$ within which $f(z)$ is analytic, then

$$\int_{\partial C} f(z)\, dz = 0\ . \tag{42}$$

Our proof is a bit weak because we needed some smoothness of ∂C to get the vector $\hat{\mathbf{t}}$, but that is in fact unnecessary.

This remarkable result may seem to make integration in the complex plane rather dull until you recall that every interesting analytic function has singularities. What happens if ∂C encloses a singularity? Let us look at the integration around a pole of order n; put the pole at $z = 0$ and make the path a circle radius R, counter-clockwise again.

$$I_n = \oint \frac{dz}{z^n} \quad n = 1, 2, \cdots \; . \tag{43}$$

We use the polar representation $z = R\,\mathrm{e}^{\mathrm{i}\theta}$ and note that on the contour $dz = \mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}\,d\theta$ because R is constant. Hence

$$I_n = \int_0^{2\pi} \frac{\mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}}{R^n\,\mathrm{e}^{\mathrm{i}n\theta}}\,d\theta = \frac{\mathrm{i}}{R^{n-1}} \int_0^{2\pi} d\theta\,\mathrm{e}^{\mathrm{i}(n-1)\theta} \,. \tag{44}$$

For $n > 1$ we must integrate a sinusoidal function over a period or a multiple period; all those integrals vanish. Only when $n = 1$ is the answer nonzero. So

$$I_n = \begin{cases} 2\pi\mathrm{i}, & n = 1 \\ 0, & n > 1 \,. \end{cases} \tag{45}$$

The result applies to *any shape contour enclosing the pole* not just a circle centered on it. This is because (42) allows us to deform a contour path in manner we please, provided the deformed line encloses a region where $f(z)$ is analytic throughout. In the figure we insert two circular arcs Γ_2 and Γ_3 on which the integration is running in opposite directions. Then clearly

$$\int_{\Gamma_2} = -\int_{\Gamma_3} \,. \tag{46}$$

Thus

$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \,. \tag{47}$$

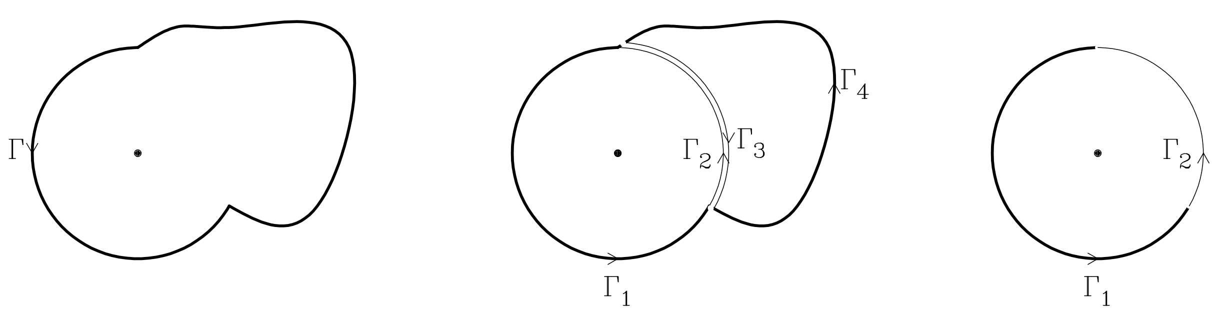
But the circuit $\Gamma_3 \cup \Gamma_4$ encloses no singularities so by (42)

$$\int_{\Gamma_3} + \int_{\Gamma_4} = 0 \,. \tag{48}$$

Thus from (47) we see that

$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2} = \oint \,. \tag{49}$$

This argument applies to the distortion of the path of any contour integral in the



complex plane: the path can be distorted in any continuous manner without changing the value of the integral, provided that in so doing no singularities (including branch cuts) are crossed during the distortion procedure. The distortion of the path of integration lies at the heart of a very useful approximation method called **Saddle-Point Intergration** in which we move the path of integration off the real axis, where the integral was to be evaluated, to make it pass over a local stationary point, where one can evaluate a convenient approximation for it.

It is easy to show from these results that if ∂C encloses z_0, and there are no singularities of f within the contour, then

$$\int_{\partial C} \frac{f(z)}{z - z_0}\, dz = 2\pi \mathrm{i}\, f(z_0) \tag{50}$$

which is the famous **Cauchy Integral Formula**. Let us quickly prove this important result. We write $f(z) = f(z_0) + f(z) - f(z_0)$ under the integral; then

$$\int_{\partial C} \frac{f(z)}{z - z_0}\, dz = \int_{\partial C} [f(z_0) + f(z) - f(z_0)] \frac{dz}{z - z_0} \tag{51}$$

$$= \int_{\partial C} \frac{f(z_0)}{z - z_0}\, dz + \int_{\partial C} \frac{f(z) - f(z_0)}{z - z_0}\, dz \tag{52}$$

$$= 2\pi \mathrm{i} f(z_0) + \int_{\partial C} \frac{f(z) - f(z_0)}{z - z_0}\, dz\ . \tag{53}$$

We need to show the integral in (53) vanishes; it will if the integrand is analytic throughout C, and it is, except possibly at $z = z_0$. But $f(z)$ is analytic everywhere within C, so we can find a locally convergent Taylor series valid in a small disk around the point z_0; then the integrand of the remaining integral in (53) is

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{[f(z_0) + (z - z_0) f'(z_0) + \cdots\ ] - f(z_0)}{z - z_0} \tag{54}$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(n+1)!}\, f^{(n+1)}(z_0)\ . \tag{55}$$

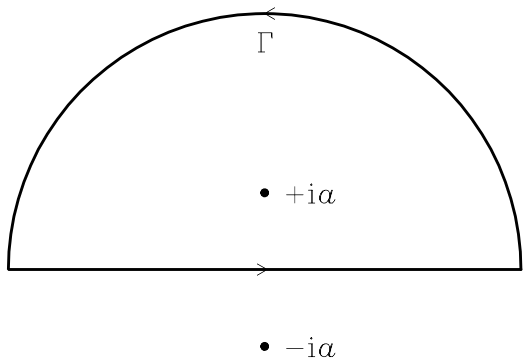
The function (55) is evidently analytic at $z = z_0$ because it has a convergent power series about that point — (55) converges faster than the series for $f'(z)$. Thus the integral in (53) is over an integrand that is analytic throughout C and so the value of the integral is zero. We are done.

There are scores of applications of this formula. For example, if a function has only poles in the upper half plane ($\mathrm{Im}\, z > 0$) and dies away for large values of z there, we can find the integral along the real axis just by picking up the contributions from the order-one poles. Here is not quite the simplest illustration of that process. With $k, a > 0$ we evaluate

$$N = \int_{-\infty}^{\infty} \frac{\cos kx}{a^2 + x^2}\, dx\ . \tag{56}$$

There are two order-one poles of the integrand, at $z = \pm \mathrm{i} a$, so we draw a closed contour Γ that captures one of them, as shown. Then the Cauchy formula delivers

$$\int_\Gamma \left[\frac{\cos kz}{z+\mathrm{i}a}\right]\frac{dz}{z-\mathrm{i}a} = \frac{2\pi\mathrm{i}\cos \mathrm{i}ka}{2\mathrm{i}a} = \frac{\pi\cosh ka}{a}\ . \tag{57}$$



To identify this with N in (56) we would like to say that the integral along the curved piece of the contour vanishes, as the size grows. Unfortunately that simply isn't true; you can see that $\cos kz$ gets very big, growing exponentially as we move up the imaginary axis. So the next trick is to realize that we can replace the cosine in (56) with $\mathrm{e}^{\mathrm{i}x}$ because the imaginary part of the integrand is odd and therefore makes no contribution to the final integral; even if it did we could simply take the real part of the complex integral at the end. The new closed contour integral is

$$\int_\Gamma \left[\frac{\mathrm{e}^{\mathrm{i}kz}}{z+\mathrm{i}a}\right]\frac{dz}{z-\mathrm{i}a} = \frac{2\pi\mathrm{i}\,\mathrm{e}^{-ka}}{2\mathrm{i}a} = \frac{\pi\,\mathrm{e}^{-ka}}{a}\ . \tag{58}$$

Now as you go up the imaginary axis, the exponential term shrinks (surprise!) exponentially and we may guess that on a large semicircle the contribution to the integral becomes more and more negligible as the radius grows. Then (58) gives the answer for N. There is a general result called **Jordan's Lemma** which says that, if $f(z)$ has only poles in the upper half plane, and $f(z)\to 0$ uniformly as $|z|\to\infty$ for $0\le\theta\le\pi$, and $m>0$, then

$$\int_{S(R)} \mathrm{e}^{\mathrm{i}mz}\ f(z)\ dz\ \to 0,\ \ \text{as } R\to\infty \tag{59}$$

where $S(R)$ is the semicircular arc radius R in the upper half plane.

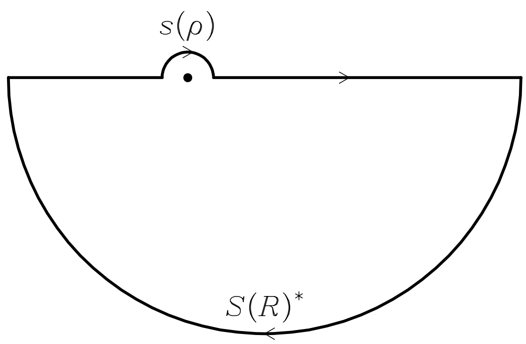
A profound application of this result can be shown now. Recall at the beginning of this set of notes I mentioned the response function of a linear, time-invariant system. We take the Fourier transform of (2), when the input is a delta-function in time, which has a unit spectrum:

$$y(t) = \int_{-\infty}^{\infty} \mathscr{g}(f)\,\mathrm{e}^{2\pi\mathrm{i}ft}\ df\ \ . \tag{60}$$

A realizable linear system cannot produce an output before there is input; thus for $t<0$ it must be that $y(t)=0$. Consider $g(f)$ as a function of complex frequency f; suppose there are poles (and possibly other singularities) in the lower half plane. Further assume the response at very large (complex) frequency vanishes. Let us look at $t<0$; complete the contour in the lower half plane with the semicircular arc $S(R)^*$. We see from Jordan's lemma (applied upside-down) that the arc contribution is zero, and so the real integral (60) equals the contour integral, which in turn will have contributions from the poles below the real axis. But these must be zero, or the system would be responding before the input. This shows that the response function has no singularities in the lower half of the complex plane, a result of importance in its own right.

When $t > 0$, the system responds and the results are the contributions from the poles (and branch cut integrals, which we haven't discussed yet). Next, closing the contour in the lower plane as before, write the Cauchy integral formula for a frequency f_0 with negative imaginary part:

$$2\pi i\, \mathcal{G}(f_0) = \int_\Gamma \frac{\mathcal{G}(f)}{f - f_0}\, df \ . \tag{61}$$



Suppose that f_0 is moved onto the real axis, and the contour is deformed to move out of the way, with a semi-circular detour $s(\rho)$, radius ρ. Then we can write the piece along the real axis, which is in any case the only nonvanishing part,

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{f_0-\rho} + \int_{s(\rho)} + \int_{f_0+\rho}^{\infty} \ . \tag{62}$$

Take the limit as $\rho \to 0$. A short calculation will show that one gets *exactly half the contribution of a pole*, with opposite sign, because of the reversed direction. One usually writes the limit

$$\lim_{\rho \to 0} \int_{-\infty}^{f_0-\rho} + \int_{f_0+\rho}^{\infty} = P\int_{-\infty}^{\infty} \tag{63}$$

and this is called the **principal part** of the integral. Putting this all together gives us

$$i\pi\, \mathcal{G}(f_0) = P\int_{-\infty}^{\infty} \frac{\mathcal{G}(f)}{f - f_0}\, df \ . \tag{64}$$

Now take the real and imaginary parts and we get the **Kronig-Kramers** relations:

$$\mathrm{Re}\, \mathcal{G} = +\mathcal{H}\,[\mathrm{Im}\, \mathcal{G}]; \quad \mathrm{Im}\, \mathcal{G} = -\mathcal{H}\,[\mathrm{Re}\, \mathcal{G}] \tag{65}$$

where $\mathcal{H}$ is called a **Hilbert transform**:

$$\mathcal{H}\,[y](t_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{y(t)}{t - t_0}\, dt \ . \tag{66}$$

What (66) is telling us that the real and imaginary parts of a transfer function are not independent — far from it: we can determine one from the other. One consequence of this is that if a medium (seismic, optical) exhibits physical dispersion, there must be an accompanying attenuation, which can be calculated via one of the K-K relationships.

We have so far discussed what happens only when one encloses simple a pole (a pole of order unity) within the contour of integration. The generalization to higher order poles is called **Cauchy's Residue Theorem**, which is equation (68). Suppose at the point z_0 the function f has a singularity of the form:

$$f(z) = g(z) + \frac{c_1}{z - z_0} + \frac{c_2}{(z - z_0)^2} + \cdots + \frac{c_N}{(z - z_0)^N} \tag{67}$$

where g is analytic at z_0. Then in integrating around this point only the simple pole contributes and we find from (50) that:

$$\int_{\partial C} f(z)\,\Delta z = 2\pi i c_1 \qquad (68)$$

The number c_1 is called the **residue** of the pole. It is easy to generate a formula for c_1 when we know N:

$$\mathrm{Res}(z_0) = \lim_{z \to z_0} \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)] \qquad (69)$$

Obviously if ∂C encloses several poles, to find the integral we sum the residues and multiply by $2\pi i$.

The behavior with a branch cut is less simple. Consider a straight cut of finite length joining two branch points and a contour surrounding it but no other singularities. Then the integral can be shrunk right onto the cut; the path of integration goes in one direction on one side and the opposite direction on the other and therefore the final result is just *the line integral from one branch point to the other of the discontinuity on the cut.* Sometimes this can be used to evaluate interesting integrals as you will discover in the exercises that follow.

Exercises

5.1 Use contour integration to evaluate the integral:

$$M = \int_{-\infty}^{\infty} \frac{\sin ax}{x}\,dx\,, \quad a > 0\,.$$

Hence evaluate the Fourier transform of the integrand.

5.2 By choosing one of the representations of $\sqrt{1-z^2}$ that we considered in section 3, use contour integration to evaluate the integral:

$$L = \int_{-1}^{+1} \sqrt{1-x^2}\,dx\,.$$

Hint: first shrink an enclosing contour down onto the cut; then expand the contour to a very large circle, centered on the origin.

7. Saddle-Point Integration

A very useful approximation for integrals, particularly Fourier integrals at large parameter (large frequency or wavenumber), arises because analytic functions have a peculiar property: at any point where the function f is analytic, its real part is never a local maximum or a local minimum, and neither is its imaginary part. So local maxima and minima can only happen at singularities. We show this as follows. A local maximum can only arise at a place where the derivative of f vanishes, say z_0. Perform the Taylor series about this point:

$$f(z_0+z) = f(z_0) + \tfrac{1}{2} z^2 f''(z_0) + O(z^3) \tag{70}$$

Assuming for the moment that the second derivative does not vanish, we see the behavior around the point z_0 is like z^2 times a complex constant. Let us look at the real part of f; an identical argument works for the imaginary part. Writing $z = |z| e^{i\theta}$, and $f'' = g\, e^{i\phi}$ then

$$\mathrm{Re}\,[f(z_0+z) - f(z_0)] = \mathrm{Re}\,(\tfrac{1}{2}|z|^2 e^{2i\theta} |g| e^{i\phi}) + O(z^3) \tag{71}$$

$$= \tfrac{1}{2}|g|\,|z|^2 \cos(2\theta+\phi) + O(z^3) \tag{72}$$

and we see that the real part of $f(z_0)$ cannot be a local maximum or minimum because by choosing θ appropriately we can get a value larger or smaller than this. If $f''(z_0)$ vanishes, a similar argument applies to the first nonvanishing term in the Taylor series.

If the second derivative does not vanish, we have behavior at z_0 called a **saddle point**. Approaching the point z_0 along an appropriate direction, $\mathrm{Re}\, f$ exhibits a local minimum, while crossing through z_0 on the perpendicular path, there is a local maximum. With exception of the case $f(z_0) = 0$, the magnitude $|f|$ also presents a saddle point at z_0 because, as you may recall, $\mathrm{Re} \ln f = \ln |f|$ which is analytic except where f vanishes. So, local maxima and minima are all at singularities; sites of vanishing first derivative are saddle points.

What does this have to do with integration? Consider for the moment the normal situation of an integral of $f(z)$ on the real line. Suppose the function f has a single saddle point in the upper half of the complex plane. We can distort the path of integration to go through the saddle point. Assume for the moment this can be done without crossing a pole or branch point; then the value of the integral remains the same, if the path returns to the real axis. Approaching the saddle point in the right orientation will make the integrand on the path go through a local maximum there. We may be able to check that this is the dominant contribution to the integral. Then we approximate the integral by making a local Gaussian function approximation there: we say near z_0 the Taylor series for $\ln z$ can be approximated by the quadratic terms

$$f(z) = \exp(\ln f(z)) = \exp[g(z_0) + \tfrac{1}{2}(z - z_0)^2 g''(z_0)] \tag{73}$$

Now we integrate this approximation along a path that causes the steepest decay away from the saddle; this is often called the **path of steepest descent.** On that line the function f varies like $\exp -\tfrac{1}{2}|g''(z_0)|\, t^2$, a Gaussian, and the integral is then

$$\int f(z)\, dz = \exp[g(z_0)] \int \exp[\tfrac{1}{2} g''(z_0)(z - z_0)^2] \tag{74}$$

$$= f(z_0)\sqrt{\frac{2\pi}{-g''(z_0)}} \tag{75}$$

This is the saddle point approximation for the integral. The approximation is a proper asymptotic approximation in a parameter k for large k, if the integrand can be written in the form $\exp(kg(z))$. Then the error of the approximation tends to zero for large k; otherwise we don't have such a guarantee. See Bender and Orszag (1978).

In practice things can often be more complicated. There may be several places where $f'(z) = 0$ and the distorted path may have to visit all of them, are maybe only a subset, to collect all the contributions. The path may have to dragged over a pole, and then the contribution from the pole will have to be included. But as a way of getting the high-frequency behavior of an integral, the method is very reliable, if not always highly accurate.

Let us perform a couple of examples. The first one is very simple and really does not make use of complex variables at all. Recall (29) the Γ function, which we rewrite:

$$\Gamma(k+1) = \int_0^\infty z^k \, e^{-z} \, dz \tag{29a}$$

We are interested in large k. If we absorb the power into the exponent we don't have an integrand of the form $\exp(kg(z))$, but let us proceed anyhow and see how things work. For large real k the integrand has a single maximum, which in the complex z plane is a saddle point on the real axis. Write

$$\Gamma(k+1) = \int_0^\infty \exp g(t) \, dt \tag{76}$$

where

$$g(z) = -z + k \ln z, \qquad g'(z) = -1 + \frac{k}{z}, \qquad g''(z) = -\frac{k}{z^2} \tag{77}$$

Then the saddle point is at the point z_0, where $g(z_0) = 0$, which is obviously given by $z_0 = k$. Then $g''(z_0) = -1/k$. Plugging these results into (75) we have

$$\Gamma(k+1) \to k^k \, e^{-k} \sqrt{\frac{2\pi}{k^{-1}}} = \sqrt{2\pi}\, k^{k+1/2} e^{-k} \tag{78}$$

You may recall that for positive integers $\Gamma(k+1) = k!$ and that (78) is a famous approximation for factorial known as **Stirling's approximation.** To see how good this approximation is consider the following table:

k	$k!$	Eqn (78)
1	1	0.922137
2	2	1.91900
3	6	5.83621
5	120	118.01917
10	3628800	3598695.6
20	2.43290×10^{18}	2.422787×10^{18}

As you see the approximation is gradually improving with increasing k, at least in the relative error.

Let us look for our second example at an integral we already know how to do exactly, the integral

$$N = \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2 + x^2}\, dx\ . \tag{79}$$

Again the integrand is not in the form $\exp k\, g(x)$ and so true asymptotic convergence may not be obtained. We proceed in the usual way: write

$$N = \int_{\Gamma} e^{g(z)}\, dz\,, \quad \text{where} \ \ g(z) = ikz - \ln(a^2 + z^2) \tag{80}$$

where Γ is a path initially running along the real axis, but which we will move later out into the complex z plane. Then

$$g'(z) = ik - \frac{2z}{a^2 + z^2} \quad \text{and} \quad g''(z) = \frac{2z^2 - 2a^2}{(a^2 + z^2)^2} \tag{81}$$

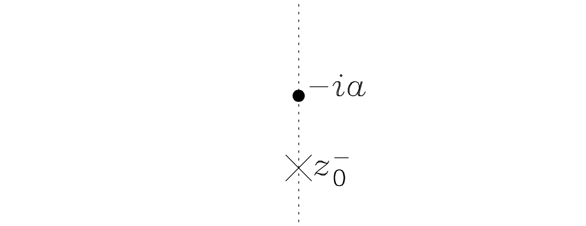
To find the saddle points we must solve the equation $g(z_0) = 0$, which leads to a quadratic equation and two roots, both on the imaginary axis:

$$z_0^+ = i(\sqrt{a^2 + k^{-2}} - k^{-1}) \quad \text{and} \quad z_0^- = -i(\sqrt{a^2 + k^{-2}} + k^{-1}) \tag{82}$$

As shown in the diagram z_0^+ lies above the real axis, but inside the pole at $z = ia$, while z_0^- lies on the far side of the pole at $-ia$. If we wish to distort the path down to z_0^- we would cross the pole, but if we move it up onto z^+, the integral remains unchanged, making our life much easier. So we do that. Now the algebra becomes tedious and so I will just quote the result of plugging z_0^+ into (75). To make the results more digestible we assume k is large and perform a series development:

$$N \ ƒ\, e^{\sqrt{1}}\S\pi \frac{-e^{-ka}}{a} \left[1 - \frac{1}{4k^2a^2} + \frac{1}{24k^3a^3} + \cdots \right]. \tag{83}$$

In comparing this result to the correct answer $\pi e^{-ka}/a$ we notice that the leading factor is a messy constant whose value is 3.4068 and not π as we would expect! So no matter how large k becomes there is an irreducible error of about 8.4 percent. Hence we do not have a proper asymptotic formula, but merely a reasonably good approximation.



7. References

Bender, C. M., and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill, New York, 1978.

Berger, J., Agnew, D. C., Parker, R. L., and Farrell, W. E., 'Seismic system calibration, 2. Cross-spectral calibration using random binary signals', *Bull. Seism. Soc. Am.*, 69, 271-88, 1979.

Carrier, Krook, and Pearson, *Functions of a Complex Variable*, 1966. Keener, J. P., *Principles of Applied Mathematics*, Addison-Wesley, 1987.

Morse, P. M., and Feshbach, H., *Methods of Theoretical Physics*, 1953

Parker, R. L., 'The inverse problem of electromagnetic induction: Existence and construction of solutions based upon incomplete data', *J. Geophys. Res.*, 85, 4421-8, 1980.

Parker. R. L., and Klitgord, K. D., 'Magnetic upward continuation from an uneven track', *Geophysics*, 37, 4, 662-8, 1972.