Complex Variables SIO239 2003

0. Motivation

We are going to spend a few lectures on the theory of analytic functions of a complex variable. There are three principal areas where this theory intersects geophysics. The first is through Fourier analysis, often the treatment of time-invariant linear systems. The output $y(t)$ of such a system in response to an input $x(t)$ can always be written a **convolution**:

$$y(t) = \int_{-\infty}^{\infty} g(t-s)\, x(s)\; ds \tag{1}$$

and the Fourier transform of this is

$$\tilde{y}(f) = \tilde{g}(f)\, \tilde{x}(f) \tag{2}$$

where f is the frequency and $\tilde{g}$ is called the **response function** or **transfer function** of the system. In (2) $\tilde{x}$, $\tilde{y}$ and $\tilde{g}$ are complex-valued functions; one interpretation is that $\tilde{g}$ is a complex factor that multiplies the amplitude $\tilde{x}$ of a sinusoidal input and since

$$\tilde{g} = |\tilde{g}|\, \mathrm{e}^{\mathrm{i}\Phi} \tag{3}$$

there is a factor of $|\tilde{g}|$ gain in amplitude and a phase shift of Φ in the output $\tilde{y}$ caused by the linear system.

An example of this idea is that a seismometer represents a linear system transferring ground acceleration to recorded voltage; here the question of calibrating the seismometer comes down to finding $\tilde{g}$ for a wide range of frequencies. Electrical engineers will know that tremendous simplification results in describing $\tilde{g}$ if we discuss the behavior of $\tilde{g}(f)$ for **complex** frequencies, even though it is impossible to shake the ground with a complex-frequency signal. See Berger, et al., 'Seismic system calibration, 2. Cross-spectral calibration using random binary signals', *Bull. Seism. Soc. Am.*, 69, 271-288, 1979.

A similar example is the **magnetotelluric sounding** problem; in (1) x would be the horizontal magnetic field as a time series recorded at a site and y the orthogonal horizontal electric field. Estimates $\tilde{g}$ from recordings allows one to pose an inverse problem for the electrical conductivity profile beneath the site, a problem beautifully solved by posing the equations in complex f. See Parker, 'The inverse problem of electromagnetic induction: Existence and construction of solutions based upon incomplete data', *J. Geophys. Res.*, 85, 4421-4428, 1980. We will briefly describe a remarkable result concerning every physical linear time-invariant system, the **Kronig-Kramers relations**.

Another quite different area where complex variables used to be very popular is the solution of electrostatic and fluid flow problems in two dimensions. An ideal fluid and an electrostatic field obey **Laplace's equation** and, as we will see, the real part and the imaginary of any analytic function in the complex plane also satisfies that equation too. This enables us to provide closed form solutions to some quite complex flow and field geometries, for example the flow over an aircraft wing; see Chapter 6 in

Keener, 'Principles of Applied Mathematics', Addison-Wesley, 1987. These days few people are satisfied with 2-D approximations and huge computer codes are used to get better solutions but less elegantly. For a geophysical example see Parker and Klitgord, 'Magnetic upward continuation from an uneven track', *Geophysics*, 37, 4, 662-668, 1972.

The third application of complex variables, often related to the first, is in the evaluation of integrals. We will see how complex variable theory can sometimes evaluate explicitly complicated definite integrals that you imagine could never be expressed in terms of elementary functions. This is an old-fashioned pastime for which the modern generation probably has little patience. But very frequently one can express approximate solutions to differential equations as integrals and one needs to know the behavior of these functions: here again complex variables can prove invaluable. And sometimes, complex analysis offers a way to evaluate a real integral numerically that may be quite refractory if one insists on a purely real setting.

1. Complex Numbers and Simple Functions

You are of course familiar with the idea that real numbers can be generalized into complex numbers by introducing the imaginary unit $\sqrt{-1}$; then if $x, y \in \mathbb{R}$, the complex number z is written

$$z = x + \mathrm{i}y \ . \tag{4}$$

The complex numbers share the same arithmetic operations and obey the same axioms as the reals and behave like them in almost every way, with the additional rule that $\mathrm{i}^2 = -1$. The traditional **magnitude** of a complex number is its Euclidean distance from the origin: we will assume (4) unless explicitly stated otherwise; then $|z| = \sqrt{x^2 + y^2}$. Often $|z|$ is called the **modulus** of z. And you will remember another simple operation: the **complex conjugate** of z, written here always as $z^* = x - \mathrm{i}y$. Some authors use $\bar{z}$. Again clearly if $z = z^*$, then $y = -y$ which means $y = 0$ and so z is real. Equally obvious is $z\,z^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2 = |z|^2$. You will also be familiar with the notation that y is the **imaginary part** of z and x is its **real part**. I will write

$$x = \mathrm{Re}\, z; \quad y = \mathrm{Im}\, z \ . \tag{5}$$

I will denote the set of complex numbers by $\mathbb{C}$, so $z \in \mathbb{C}$. Because it is natural to plot a complex number $z = x + \mathrm{i}y$ as a point with coordinates (x, y), $\mathbb{C}$ is also called the **complex plane.**

Just to be absolutely complete, here are the arithmetic operations for complex numbers, spelled out in terms of the real and imaginary parts:

$$\begin{aligned} z_1 + z_2 &= x_1 + x_2 + \mathrm{i}(y_1 + y_2) \\ z_1 - z_2 &= x_1 - x_2 + \mathrm{i}(y_1 - y_2) \\ z_1 z_2 &= x_1 x_2 - y_1 y_2 + \mathrm{i}(x_1 y_1 + x_2 y_2) \\ \frac{z_1}{z_2} &= \frac{x_1 x_2 + y_1 y_2 - \mathrm{i}(x_1 y_1 - x_2 y_2)}{x_2^2 + y_2^2} \ . \end{aligned}$$

Of course computers perform these complicated manipulations effortlessly; naturally *matlab* has complex arithmetic, and so does Fortran, but C, that darling of programmers, does not.

With this elementary stuff out of the way we can begin to discuss functions of complex variables. Real functions defined using only the ordinary real arithmetic operations will have complex analogs without any difficulty. Hence we know immediately what is mean by the functions

$$f(z) = x^5 - 10z + 137 \quad \text{or} \quad g(z) = \frac{2+z}{1+7z-8z^2}\ . \tag{6}$$

On the real axis (where $\mathrm{Im}\, z = 0$) these functions will obviously give their real-valued counterparts. Infinite power series of real numbers can be similarly defined and convergence examined in essentially the same way. For example, if e^x is defined by its real power series, we can define its complex analog by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}\ . \tag{7}$$

The idea of the limit, needed to define an infinite sum, requires a notion of distance between two points and this is taken to be the magnitude $|z_1 - z_2|$. But some real-variable functions do not have extensions into the complex world. For example, the Heaviside step function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases} \tag{8}$$

because there isn't a natural ordering of $\mathbb{C}$. Quite arbitrarily one might define the complex analog of H by

$$H(z) = \begin{cases} 0, & \mathrm{Re}\, z < 0 \\ 1, & \mathrm{Re}\, z \ge 0 \end{cases} \tag{9}$$

but as we shall see in a moment extensions like this are usually worthless. One can usefully regard the real and imaginary parts of z as components of a 2-vector, a position in the real plane $\mathbb{R}^2$, which is of course the complex plane. Similarly, it is possible to regard the result of a complex valued function as components of another 2-vector and then the complex function is a mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$. For example, we might define

$$f(x + iy) = x^2 - 2ie^{3xy}\ . \tag{10}$$

Like (9), this function lacks a number of important properties that we come to regard as essential in complex analysis, and functions like (10) are not of much interest. Enormous rewards follow if we study a special class of complex-valued functions.

Exercise

1.1 From what we have discussed so far show that $z = R\, e^{i\theta}$, where $R = |z|$ and $\theta \in \mathbb{R}$ is an angle in the complex plane. What angle is it?

2. Differentiation and Analytic Functions

In real analysis, a function f is said to be **analytic** at a point x_0 if there is an open neighborhood containing x_0 in which f can be expressed as a convergent power series; this means for some ε

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{, for all } |x - x_0| < \varepsilon, \text{ with } 0 < \varepsilon \,. \tag{11}$$

The same definition, with complex numbers everywhere (except for ε which must still be real), works for complex functions.

But another, and it turns out equivalent, definition is frequently used and is often easier to check out. First we must define the **derivative** of a complex function: the derivative of f at z is defined by

$$\frac{df}{dz} = \lim_{z' \to z} \frac{f(z') - f(z)}{z' - z} \tag{12}$$

with the vital proviso: *the value must be independent of the path on which z' approaches z in the complex plane.* Let us see what this means and how it might fail. Let $f(z) = u(z) + \mathrm{i}v(z)$ where u and v are real-valued functions. In (12) we will write $z' = z + \Delta z$ with the $\Delta z = \Delta x + \mathrm{i}\Delta y$. Then performing a Taylor series expansion of u and v we find

$$\frac{f(z') - f(z)}{z' - z} = \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{13}$$

$$= \frac{(\partial u/\partial x + \mathrm{i}\partial v/\partial x)\Delta x + (\partial u/\partial y + \mathrm{i}\partial v/\partial y)\Delta y}{\Delta x + \mathrm{i}\Delta y} + o\,|\Delta z| \tag{14}$$

$$= \frac{\partial u/\partial x + \mathrm{i}\partial v/\partial x}{1 + \mathrm{i}\dfrac{\Delta y}{\Delta x}} \left[1 + \mathrm{i}\frac{\Delta y}{\Delta x}\, \frac{\partial v/\partial y - \mathrm{i}\partial u/\partial y}{\partial u/\partial x + \mathrm{i}\partial v/\partial x} \right] + o\,|\Delta z| \,. \tag{15}$$

It will be clear that unless the factor multiplying $\Delta y/\Delta x$ in the square bracket is exactly one, the ratio in (12) defining the complex derivative will depend on the slope of the line of approach of z' towards z. So if the derivative in (12) is to be uniquely defined we must have:

$$\frac{\partial v/\partial y - \mathrm{i}\,\partial u/\partial y}{\partial u/\partial x + \mathrm{i}\partial v/\partial x} = 1 \tag{16}$$

which becomes after equating real and imaginary parts, the famous **Cauchy-Riemann** relationships:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \,. \tag{17}$$

Notice that all the Cauchy-Riemann relationships tells us is that the derivative exists and is uniquely defined at z. An analytic function must have a first derivative, so the C-R relationships are necessary conditions for analyticity. But more is needed for f to be analytic at z; *the additional demand is merely that df/dz must be continuously differentiable at z.* What this means is that (17) not only holds exactly at z but at all points in an open set around z.

As we said earlier an analytic function has a convergent power series development valid in a neighborhood around the point. (We haven't proved that yet.) So in the complex plane, mere existence of a continuous first derivative at a point guarantees the existence of all derivatives and the validity of a local Taylor series. This is completely different from the corresponding state of affairs for real variables, as Exercise 2.4 will demonstrate.

Because almost all the familiar elementary functions and solutions to differential equations have local power series, they are analytic, except perhaps at certain points (or lines), the **singularities** of the function. Thus e^z, $\sin z$, $\tan z$, $J_n(z)$ are analytic, and so are all polynomials and rational functions. But some simple functions of z are not analytic and so functions that include them may not be analytic either (but not always, as the exercises will show!). For example $f(z) = z^*$ is not an analytic function: in this case $u(z) = x$ and $v(z) = -y$, so $\partial u/\partial x = 1$, while $\partial v/\partial y = -1$ and then obviously the C-R relations are violated for every possible z, not just a subset of singular points in the plane. Similarly $\mathrm{Re}\, z$, $\mathrm{Im}\, z$ and $|z|$ are not analytic.

Exercises

In all the following it will be the case that $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

2.1 Show that none of the following functions is an analytic function of a complex variable:

$$|z|^2, \ \ \mathrm{Re}\, z, \ \ \mathrm{i}\,\mathrm{Im}\, z, \ \ x^2 - \mathrm{i}y^2,$$

and neither is the functions defined by (10).

2.2 Determine which of the following functions is analytic in z. For those that are, give a complex power series development about some nonsingular point.

$$f_1(z) = -\,\mathrm{Im}\, z + \mathrm{i}\,\mathrm{Re}\, z$$

$$f_2(z) = \ln|z| + \mathrm{i}\tan^{-1}(\mathrm{Im}\, z/\mathrm{Re}\, z)$$

$$f_3(z) = x^4 + y^4 + 2\mathrm{i}xy(x^2 + y^2)\ .$$

2.3 Show that at $z = 0$ the function $f(z) = z\,|z|^2$ satisfies the Cauchy-Riemann relations. Is this function analytic at $z = 0$?

2.4 Consider the real function of a real variable given by

$$g(x) = \begin{cases} \exp(-1/x^2), & |x| > 0 \\ 0, & x = 0\ . \end{cases}$$

Show that g has derivatives of all orders at $x = 0$ and is arbitrarily differentiable in the neighborhood of the origin. Evaluate the derivatives of g at $x = 0$. What is the corresponding Taylor series about the origin? Is it a convergent series? Is g an analytic function of a real variable at $x = 0$? Is g analytic at $x = \varepsilon$, where $\varepsilon > 0$?

2.5 A convergent Fourier series defines a function for all real x:

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

The function switches discontinuously between $\pm\pi/4$, except for the isolated points $x = n\pi$ with n an integer. Is this function analytic at $x = \pi/2$? If real x is replaced by complex z, at what set of points in $\mathbb{C}$ is the series convergent? Can f be analytically continued away from this set of points?

3. Singularities, Branch Points and Cuts

You are already familiar with the idea that one can add the point at infinity to the complex plane for the purposes of studying the behavior of a function. So an analytic function f is singular or analytic at infinity if the function $g(z) = f(1/z)$ is singular or analytic at $z = 0$. With the point at infinity included, it turns out there is only one complex function that is analytic in the whole plane: f = constant! Every other analytic function has at least one singular point. Functions that are analytic *except at infinity* are called **entire** functions. (Some authors use the word **integral**, but this seems an unnecessarily confusing name!) Examples include all polynomials, e^z, $\sin z$, $\cos z$, Bessel functions of the first kind $J_n(z)$.

One class of singularity is the **isolated singularity**, a simple point where f isn't analytic; at every other point in an open set surrounding an isolated singularity, the function is still analytic. The simplest of these is the **pole of order** n: if f has a pole of order n at z_0 then

$$z \to z_0, \quad f(z) \to \frac{b}{(z - z_0)^n}, \quad b \neq 0 \; . \tag{18}$$

The most important kind of pole, as we will see, is the pole of order 1.

Another kind of isolated singularity is called an **essential singularity**. This is the behavior one gets near the origin for the function $e^{1/z}$. The contrast with a pole is that, while the magnitude $|f|$ grows as you approach a pole from whatever direction, an essential singularity can appear to be finite if you approach on one path (for example, along the negative real axis), infinite on another, or any value between. There is a theorem that says in the neighborhood of an essential singularity, the function takes on every possible complex value, *except for one number!* We won't be dealing with this kind in our brief tour.

A more complicated sort of singularity than the isolated sort is the **branch point**. To illustrate this singularity let us consider the square root function; fundamentally the idea is to find a function s that satisfies the equation:

$$s(z)^2 = z \; . \tag{19}$$

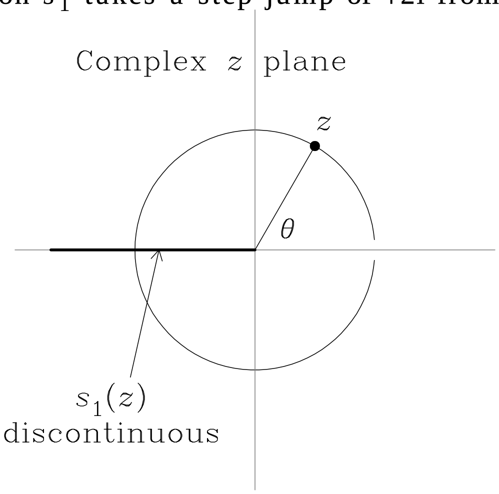
We already know that on the real axis $x^{1/2}$ cannot be differentiated at $x = 0$, so there is going to be a singularity of some sort at $z = 0$, at least, but it is not one of the kinds we have met so far. New questions arise because (19) does not have a single solution: there are infinitely many functions in the complex plane (or on the real line for that matter) that will solve (19), and to make progress we must choose one of them to work with. Which one we choose will often depend on the application. For real variables the choice is often presented as one between $+\sqrt{x}$ and $-\sqrt{x}$, (this is the conventional oversimplification) but with complex variables the matter is more complicated.

Suppose we write

$$z = R\, e^{i\theta}, \quad \text{and } s(z) = z^{1/2} = \sqrt{R}\; e^{i\theta/2} \; . \tag{20}$$

Certainly this definition solves (19), but it is unsatisfactory unless we say something more about the allowed values of θ, because it gives two values for every point when $z \neq 0$. To see this consider the point $z = 1$; then $R = 1$, but θ can be 0 or 2π. Then (20) says $s(z)$ is either +1 or −1. While many authors entertain the notion of a multiplyvalued function, to modern mathematics this concept is not logical: s is a function — it is a rule that assigns a single complex number in its domain to another complex number in its range, and the range is not a complex number pair, but a single element of $\mathbb{C}$. To break the ambiguity, *we must choose a rule to give us a single value for s for each z*. Therefore, to be concrete, let us say that in (20) $-\pi < \theta \le \pi$. Then to every z there is only one θ and hence only one value of s. And now imagine taking a circuit around the origin at a unit distance, evaluating (20) under the condition on θ:

$$s_1(z) = s_1(\cos\theta + \mathrm{i}\sin\theta) = \cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta, \quad -\pi < \theta \le \pi\ . \tag{21}$$



As we move from just below the negative real axis ($\theta = -\pi-\varepsilon$) to being on it ($\theta = \pi$), the function s_1 takes a step jump of +2i from −i to +i. And there is a similar step discontinuity (jump $= 2\mathrm{i}|z|^{\frac{1}{2}}$) at every point on the negative real axis. Clearly $s_1(z)$ cannot be analytic on the negative real axis; there is a line singularity there, which is called a **branch cut**. At every other point, except $z = 0$, our function s_1 is analytic. But recall this was a particular choice we made to assign a unique θ to each z. Another choice will give another function; for example let $0 < \theta \le 2\pi$. A new function results, with its branch cut extending down the positive real axis. The line can be at any angle; it could be chosen not to be straight; if we were perverse the transition might not be confined to a line, but diffused over a region, but that would extend the region of singularity, and ideally we would like the function to be "as analytic as possible." What these different solutions to (19) share is that the origin is always a singular point, and this is called a **branch point** of the square root function. There is another branch point, too, for every solution — the point at infinity. Check this.

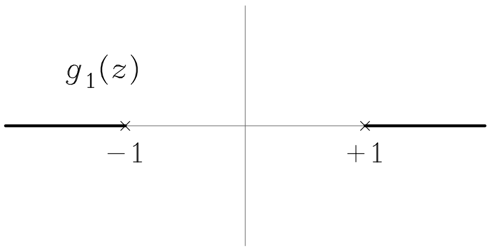
So what we see is that there is no single function *square root* in the complex plane, nor is it simply a matter of choosing a sign; the different square root functions differ by the location and shape of a branch cut, a line of singularity, extending from one branch point at the origin to another at infinity; conventionally the cut is straight. Compilers or other computer programs that support complex arithmetic will give you only a single choice for square root. Find out where FORTRAN and Matlab put their cuts for square root.

In more complicated functions involving several branch points there are more ways to choose the cuts. Consider the family of functions associated with

$$g(z) = (1 - z^2)^{\frac{1}{2}}. \tag{22}$$

There are two branch points where the function behaves like $(z - z_0)^{\frac{1}{2}}$; one is at $z = +1$ the other at $z = -1$. Cuts will radiate from these two singularities. But is infinity a branch point too? No, because for small z the function $g(1/z)$ tends to i/z which is pole, not a branch point. This shows that it is not necessary for the cut line to go to infinity at all; so where can it go? For the sake of simplicity let us adopt a convention that $\sqrt{z}$ is the function that has its cut running in a straight line from the origin along the negative real axis — in the discussion above $-\pi < \theta \le \pi$. Then if we write (22) with this square root:

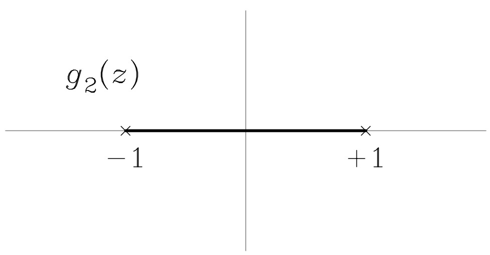
$$g_1(z) = \sqrt{1 - z^2} = \sqrt{1+z}\,\sqrt{1-z}\,. \tag{23}$$



We see the first factor puts its cut from -1 out along the negative real axis, while the other extends its cut from $+1$ on the positive real axis, as illustrated in the figure to the left. In the rest of the plane g_1 is analytic. Next suppose we rewrite (22) in a different way, as follows:

$$g_2(z) = \mathrm{i}\, z \sqrt{1 - z^{-2}} = \mathrm{i}\, z \sqrt{1 + 1/z}\,\sqrt{1 - 1/z}\,. \tag{24}$$

Now we have a function which is analytic for large z and at the point at infinity too, because our chosen square root is analytic at $z = 1$. To find where the cut of the first square-root factor goes just ask what set of z gives values of the argument on the negative real axis: answer, z is real, on the interval $(-1, 0)$; similarly, the second root gives a cut on $(0, +1)$. So this version of (22) has a single cut running along the real axis between the branch points as shown in the figure above.



You should be able to work out how to handle the branch points and cuts from all the powers z^ν when ν is not an integer. The only other common branch point is the one associated with the logarithm. There is a singularity of every log function at $z = 0$, because even for real variables log fails to be differentiable there; there is another at $z = \infty$. The branch point arises exactly in the same way as we found for square root — the ambiguity of the angle in the representation of z in polar form:

$$z = R\, \mathrm{e}^{\mathrm{i}\theta}, \quad \text{then } \log(z) = \ln R + \mathrm{i}\,\theta\,. \tag{25}$$

If we choose to make $-\pi < \theta \le \pi$, the branch cut of log runs down the negative real axis and the jump is exactly $2\mathrm{i}\pi$ everywhere on the cut. Clearly other choices for the cut are possible; but there are always branch points at $z = 0$ and ∞.

Exercises

3.1 Use Matlab to construct a pretty graphic illustrating the real and imaginary parts of the square root function in the complex plane with the cut running down the negative imaginary axis. Clarity and artistic merit are factors in this question.

3.2 Evaluate the function $g_1(z)$ in (23) on the line $z = x + \mathrm{i}y$, where $-\infty < x < \infty$ and

$y \to 0^+$. Write down an explicit expression. Also graph it. Repeat for $g_2(z)$ in (24). What is the jump in $g_2(z)$ across the cut: when z is on the real axis and $-1 \le \mathrm{Re}\, z \le +1$?

3.3 Write a Matlab M-file to evaluate one member of the family of functions:

$$g(z) = (z^2 - 1)^{1/2}.$$

Find the function with a branch cut from +1 running "vertically" in the positive imaginary direction, and one from −1 running "downward" in the negative imaginary direction. Provide a listing of your code. Answer question 3.2 for this function.