COMPLEX NUMBERS

COMPLEX NUMBERS

- In the early days of modern mathematics, people were puzzled by equations like this one:

$$x^2 + 1 = 0$$

- The equation looks simple enough, but in the sixteenth century people had no idea how to solve it. This is because to the common-sense mind the solution seems to be without meaning:

$$x = \pm\sqrt{-1}$$

- For this reason, mathematicians dubbed $\sqrt{-1}$ an imaginary number. We abbreviate this by writing "*i*" in its place, that is:

$$\boxed{i = \sqrt{-1}}$$

COMPLEX NUMBERS

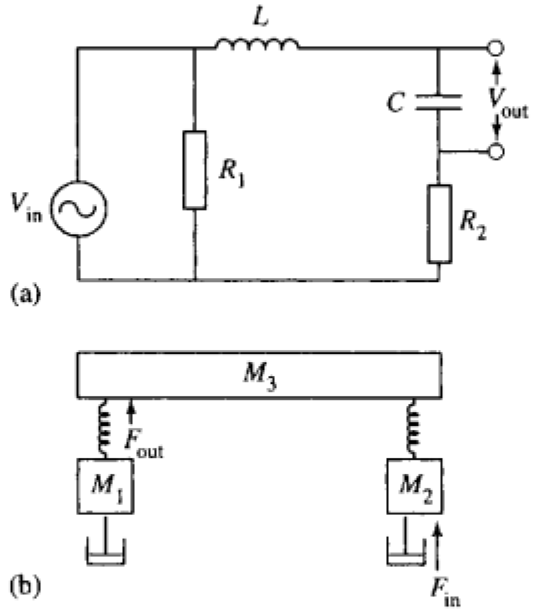

Figure 10.2 *(a) An electrical system made up of resistors, capacitors, and inductors with voltage as input and output. (b) A mechanical system made up of masses, springs, and dampers. The input is the external force and the output is tension in the spring.*

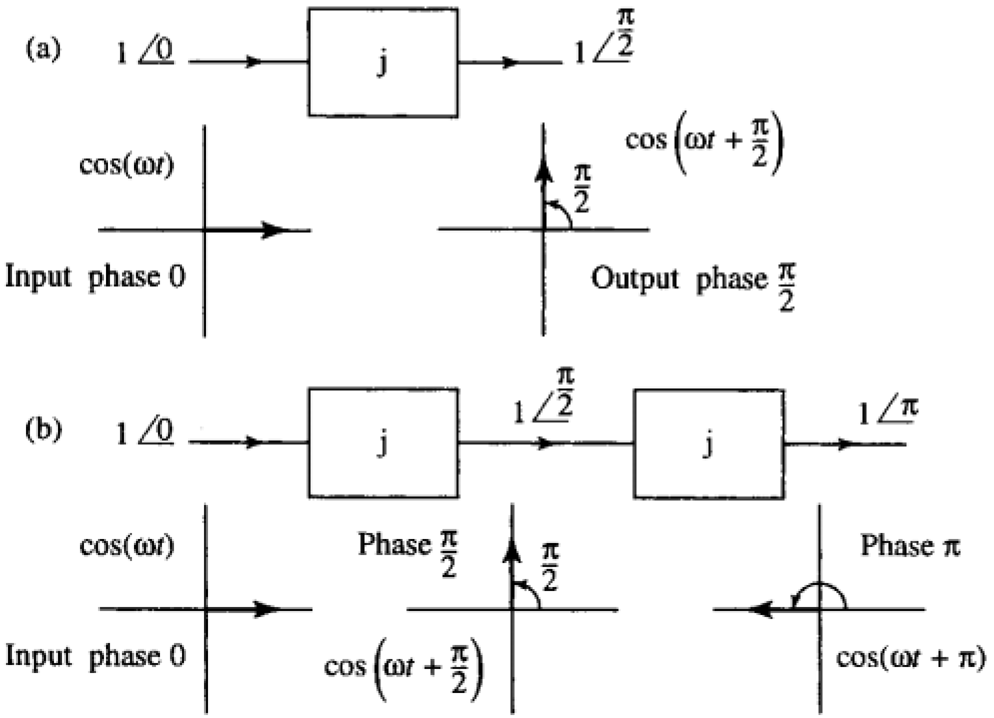

Figure 10.3 *(a) A system which produces a phase shift of $\pi/2$, that is, rotates a phasor by $\pi/2$. This may be represented as a multiplication by j. (b) A system consisting of two sub-systems, both of which produce a phase shift of $\pi/2$ giving a combined shift of π. As a phase shift of π inverts a wave, that is, $\cos(\omega t + \pi) = -\cos(\omega t)$ this is equivalent to multiplication by -1. Hence, $\mathrm{j} \times \mathrm{j} = -1$.*

DEFINITION

A **complex number** z is a number of the form $\boxed{x+iy}$

where

x is the real part and y the imaginary part, written as $x = \mathbf{Re}\, z$, $y = \mathbf{Im}\, z$.

i is called the imaginary unit $\boxed{i=\sqrt{-1}}$

If $x = 0$, then $z = iy$ is a pure imaginary number.

The **complex conjugate** of a complex number, $z = x + iy$, denoted by z^* , is given by

$$z^* = x - iy.$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

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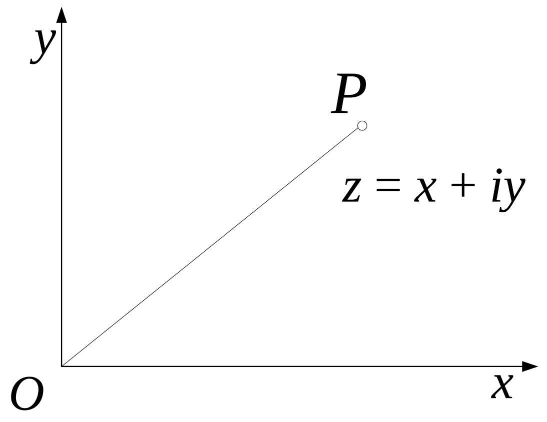
COMPLEX PLANE

- A complex number can be plotted on a plane with two perpendicular coordinate axes
  - The horizontal x-axis, called the real axis
  - The vertical y-axis, called the imaginary axis



The complex plane

Represent $z = x + jy$ geometrically as the point $P(x,y)$ in the x-y plane, or as the vector $\overrightarrow{OP}$ from the origin to $P(x,y)$.

x-y plane is also known as the complex plane.

POLAR COORDINATES

With $x = r\cos\theta, \quad y = r\sin\theta$

z takes the polar form: $z = r(\cos\theta + j\sin\theta)$

r is called the absolute value or **modulus** or **magnitude** of *z* and is denoted by |z|.

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{zz^*}$$

Note that : $zz^* = (x + jy)(x - jy)$

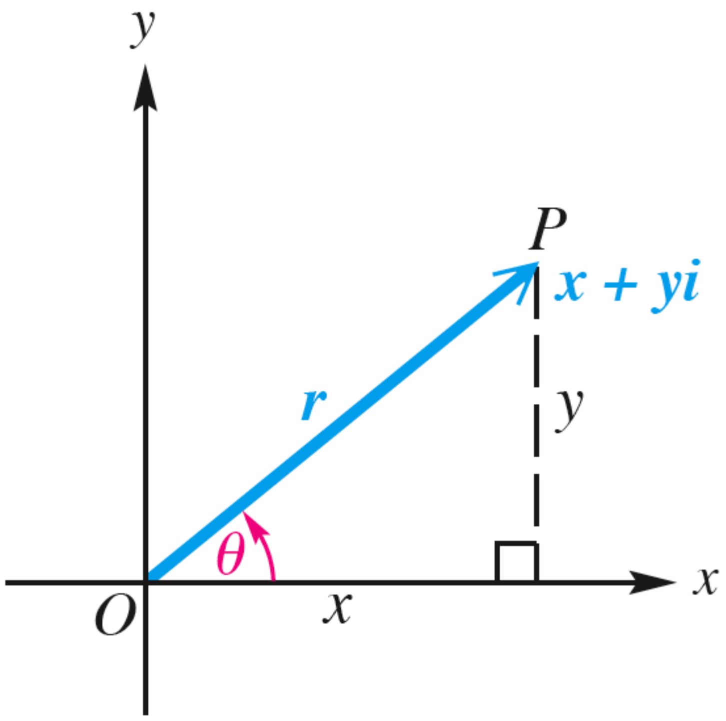
$$= x^2 + y^2$$

TRIGONOMETRIC FORM FOR COMPLEX NUMBERS

- We modify the familiar coordinate system by calling the horizontal axis the real axis and the vertical axis the imaginary axis.
- Each complex number $a + bi$ determines a unique position vector with initial point (0, 0) and terminal point (a, b).

RELATIONSHIPS AMONG *X*, *Y*, *R*, AND θ

- $$x = r\cos\theta$$
  $$y = r\sin\theta$$
  $$r = \sqrt{x^2 + y^2}$$
  $$\tan\theta = \frac{y}{x}, \quad \text{if } x \neq 0$$



TRIGONOMETRIC (POLAR) FORM OF A COMPLEX NUMBER

- The expression

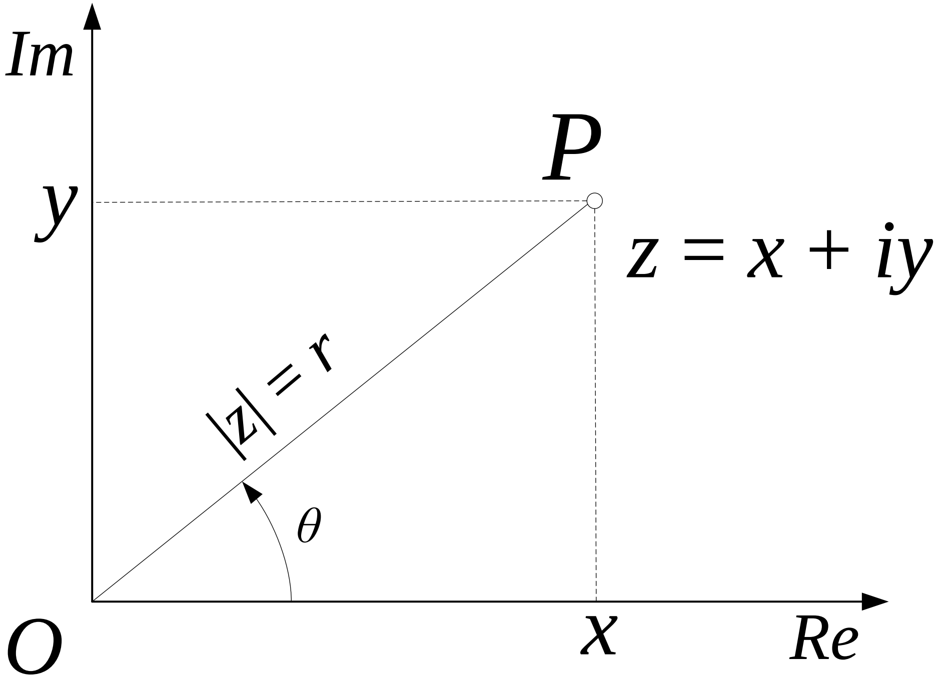
$$r(\cos\theta + i\sin\theta)$$

is called the **trigonometric form** or (**polar form**) of the complex number *x* + *yi*. The expression cos θ + *i* sin θ is sometimes abbreviated cis θ.
Using this notation

$$r(\cos\theta + i\sin\theta) \text{ is written } r \text{ cis } \theta.$$

COMPLEX PLANE

Complex plane, polar form of a complex number



Geometrically, $|z|$ is the distance of the point z from the origin while θ is the directed angle from the positive x-axis to OP in the above figure.

From the figure,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

COMPLEX NUMBERS

- θ is called the **argument** of z and is denoted by arg z. Thus,

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right) \qquad z \neq 0$$

For $z = 0$, θ is undefined.

- A complex number $z \neq 0$ has infinitely many possible arguments, each one differing from the rest by some multiple of 2π. In fact, arg z is actually

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \pm 2n\pi, \quad n = 0,1,2,\ldots$$

- The value of θ that lies in the interval $(-\pi, \pi]$ is called the **principle argument** of z ($\neq 0$) and is denoted by Arg z.

EULER FORMULA – AN ALTERNATE POLAR FORM

The polar form of a complex number can be rewritten as :

$$z = r(\cos\theta + j\sin\theta) = x + jy$$

$$= re^{j\theta}$$

This leads to the complex exponential function :

$$e^{z} = e^{x+jy} = e^{x}e^{jy}$$

$$= e^{x}\left(\cos y + j\sin y\right)$$

Further leads to :

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}\right\}$$

EULER FORMULA

- Remember the well-known Taylor Expansions :

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots$$

EULER FORMULA

Now, let's look at $e^{i\theta}$. The power series expansion of this function is given by

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots \\
&= 1 + i\theta - \frac{1}{2}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \cdots
\end{aligned}$$

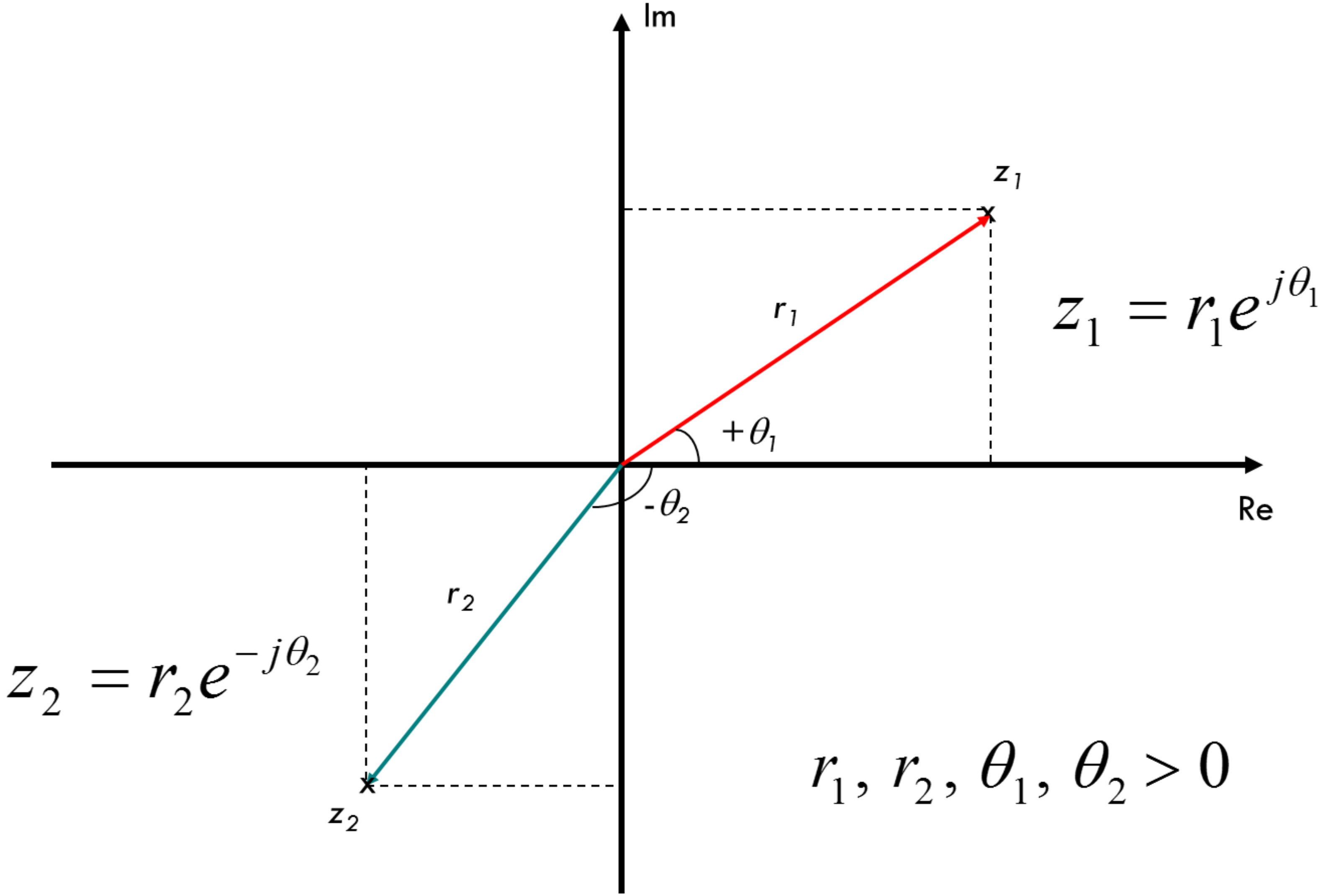
(Now group terms—looking for sin and cosine)

$$\begin{aligned}
&= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + \left(i\theta - i\frac{1}{3!}\theta^3 + i\frac{1}{5!}\theta^5 + \cdots\right) \\
&= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

- So, we can conlude that : $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

GRAPHIC REPRESENTATION



EXAMPLE

A complex number, $z = 1 + j$, has a magnitude $|z| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

and argument : $\angle z = \tan^{-1}\left(\frac{1}{1}\right) + 2n\pi = \left(\frac{\pi}{4} + 2n\pi\right)\text{rad}$

Hence its principal argument is : $\operatorname{Arg} z = \pi / 4$ rad

Hence in polar form : $z = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \sqrt{2}e^{j\frac{\pi}{4}}$

EXAMPLE

A complex number, $z = 1 - j$, has a magnitude

$$|z| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

and argument :
$$\angle z = \tan^{-1}\left(\frac{-1}{1}\right) + 2n\pi = \left(-\frac{\pi}{4} + 2n\pi\right)\text{rad}$$

Hence its principal argument is : $Arg\ z = -\dfrac{\pi}{4}$ rad

Hence in polar form :

$$z = \sqrt{2}e^{-j\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right)$$

In what way does the polar form help in manipulating complex numbers?

EXAMPLE

What about $z_1=0+j$, $z_2=0-j$, $z_3=2+j0$, $z_4=-2$?

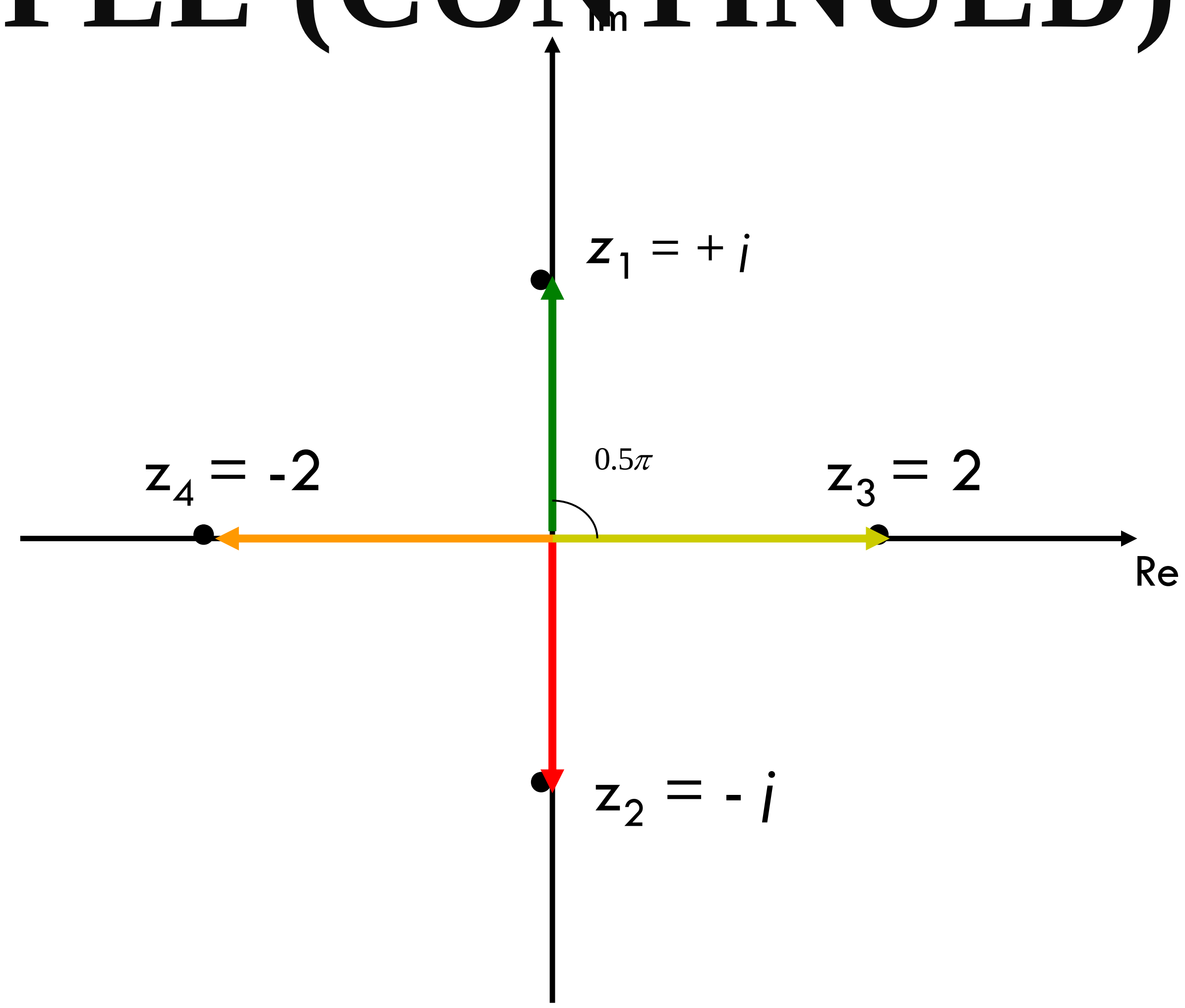
$$\begin{aligned} z_1 &= 0 + j1 \\ &= 1e^{j0.5\pi} \\ &= 1\angle 0.5\pi \end{aligned}$$

$$\begin{aligned} z_2 &= 0 - j1 \\ &= 1e^{-j0.5\pi} \\ &= 1\angle -0.5\pi \end{aligned}$$

$$\begin{aligned} z_3 &= 2 + j0 \\ &= 2e^{j0} \\ &= 2\angle 0 \end{aligned}$$

$$\begin{aligned} z_4 &= -2 + j0 \\ &= 2e^{-j\pi} \\ &= 2\angle -\pi \end{aligned}$$

EXAMPLE (CONTINUED)



EXAMPLE

- Express 2(cos 120° + *i* sin 120°) in rectangular form.
- $$\cos 120^\circ = -\frac{1}{2}$$
  $$\sin 120^\circ = \frac{\sqrt{3}}{2}$$
  $$2(\cos 120^\circ + i\sin 120^\circ) = 2\left(-\frac{1}{2}, i\frac{\sqrt{3}}{2}\right)$$
  $$= -1 + i\sqrt{3}$$
- Notice that the real part is negative and the imaginary part is positive, this is consistent with 120 degrees being a quadrant II angle.

CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM

- Step 1 Sketch a graph of the number $x + yi$ in the complex plane.
- Step 2 Find r by using the equation $r = \sqrt{x^2 + y^2}$.
- Step 3 Find θ by using the equation $\tan\theta = \frac{y}{x}, x \neq 0$ choosing the quadrant indicated in Step 1.

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

- For complex numbers a + bi and c + di,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

- Examples

$(4 - 6i) + (-3 + 7i)$	$(10 - 4i) - (5 - 2i)$
$= [4 + (-3)] + [-6 + 7]i$	$= (10 - 5) + [-4 - (-2)]i$
$= 1 + i$	$= 5 - 2i$

MULTIPLICATION OF COMPLEX NUMBERS

- For complex numbers $a + bi$ and $c + di$,

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$

- The product of two complex numbers is found by multiplying as if the numbers were binomials and using the fact that $i^2 = -1$.

EXAMPLES: MULTIPLYING

- (2 – 4i)(3 + 5i)

$$= 2(3) + 2(5i) - 4i(3) - 4i(5i)$$
$$= 6 + 10i - 12i - 20i^2$$
$$= 6 - 2i - 20(-1)$$
$$= 26 - 2i$$

(7 + 3i)2

$$= 7^2 + 2(7)(3i) + (3i)^2$$
$$= 49 + 42i + 9i^2$$
$$= 49 + 42i + 9(-1)$$
$$= 40 + 42i$$

ARITHMETIC OPERATIONS IN POLAR FORM

- The representation of z by its real and imaginary parts is useful for addition and subtraction.

- For multiplication and division, representation by the polar form has apparent geometric meaning.

Suppose we have 2 complex numbers, z_1 and z_2 given by :

$$z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$$

$$z_2 = x_2 - jy_2 = r_2 e^{-j\theta_2}$$

$$\left.\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 - jy_2) \\ &= (x_1 + x_2) + j(y_1 - y_2) \end{aligned}\right\}$$

Easier with normal form than polar form

$$\left.\begin{aligned} z_1 z_2 &= \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{-j\theta_2}\right) \\ &= r_1 r_2 e^{j(\theta_1 + (-\theta_2))} \end{aligned}\right\}$$

Easier with polar form than normal form

magnitudes multiply!

phases add!

For a complex number $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - (-\theta_2))} = \frac{r_1}{r_2} e^{j(\theta_1 + \theta_2)}$$

phases subtract!

magnitudes divide!

$$\left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2}$$

$$\angle z = \theta_1 - (-\theta_2) = \theta_1 + \theta_2$$

EXERCISES

- Let $z = x + iy$ and $w = u + iv$ be two complex variables. Prove that :

$$\overline{z+w} = \overline{z} + \overline{w}$$

- Find z^2 if $z = (2+i)/[4i-(1+2i)]$.

- Write $(2-i)^4$ in the standard form $a+ib$.

- Prove that :

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

COMPLEX ANALYSIS

- In the early days, all of this probably seemed like a neat little trick that could be used to solve obscure equations, and not much more than that.
- It turns out that an entire branch of analysis called complex analysis can be constructed, which really supersedes real analysis.
- For example, we can use complex numbers to describe the behavior of the electromagnetic field.
- Complex numbers are often hidden. For example, as we'll see later, the trigonometric functions can be written down in surprising ways like:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

AXIOMS SATISFIED BY THE COMPLEX NUMBERS SYSTEM

- These axioms should be familiar since their general statement is similar to that used for the reals.
- We suppose that *u*, *w*, *z* are three complex numbers, that is, *u*, *w*, *z* ∈ C, then these axioms follow:

$$z + w \quad \text{and} \quad zw \in \mathbb{C} \qquad \text{(closure law)}$$

$$z + w = w + z \qquad \text{(commutative law of addition)}$$

$$u + (w + z) = (u + w) + z \qquad \text{(associative law of additio)}$$

$$zw = wz \qquad \text{(commutative law of multiplication)}$$

$$u(wz) = (uw)z \qquad \text{(associative law of multiplication)}$$

$$u(w + z) = uw + uz \qquad \text{(distributive law)}$$

AXIOMS SATISFIED BY THE COMPLEX NUMBERS SYSTEM

$$|z_1 z_2| = |z_1||z_2|$$

$$|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3|\dots|z_n|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

A relationship called the *triangle inequality* deserves special attention:

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$|z_1 + z_2 + \dots + z_n| \le |z_1| + |z_2| + \dots + |z_n|$$

$$|z_1 + z_2| \ge |z_1| - |z_2|$$

$$|z_1 - z_2| \ge |z_1| - |z_2|$$

Also note that $w\bar{z} + \bar{z}w = 2\,\mathrm{Re}(z\bar{w}) \le 2|z||w|$.

DE MOIVRE'S THEOREM

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

DE MOIVRE’S THEOREM

- De Moivre’s theorem is about the powers of complex numbers and a relationship that exists to make simplifying a complex number, raised to a power, easier.
- The resulting relationship is very useful for proving the trigonometric identities and finding roots of a complex number.

DE MOIVRE’S THEOREM

- If $r_1 = (\cos\theta_1 + i\sin\theta_1)$ is a complex number, and if *n* is any real number, then

$$\left[r(\cos\theta_1 + i\sin\theta_1)\right]^n = r^n(\cos n\theta + i\sin n\theta).$$

- In compact form, this is written

$$\left[r\ \text{cis}\,\theta\right]^n = r^n(\text{cis}\ n\theta).$$

EXAMPLE: FIND (–1 – I)^5 AND EXPRESS THE RESULT IN RECTANGULAR FORM.

$$-1-i=\sqrt{2}\left(\cos 225+i\sin 225\right)$$

- First, find trigonometric notation for –1 – i
- Theorem

$$\left(-1-i\right)^5=\left[\sqrt{2}\left(\cos 225^\circ+i\sin 225^\circ\right)\right]^5$$

$$=\left(\sqrt{2}\right)^5\left[\cos(5\cdot 225^\circ)+i\sin(5\cdot 225^\circ)\right]$$

$$=4\sqrt{2}\left(\cos 1125^\circ+i\sin 1125^\circ\right)$$

$$=4\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$$

$$=4+4i$$

*N*TH ROOTS

- For a positive integer n, the complex number $a + bi$ is an nth root of the complex number $x + yi$ if

$$(a+bi)^n = x+yi.$$

*N*TH ROOT THEOREM

- If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\left(\cos\alpha + i\sin\alpha\right) \quad \text{or} \quad \sqrt[n]{r}\ \text{cis}\ \alpha,$$

- where

$$\alpha = \frac{\theta + 360^\circ \cdot k}{n} \ \text{or}\ \alpha = \frac{\theta}{n} + \frac{360^\circ \cdot k}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

EXAMPLE: SQUARE ROOTS

- Find the square roots of $1+\left(\sqrt{3}\right)i$
- Trigonometric notation: $1+\left(\sqrt{3}\right)i=2\left(\cos 60+i\sin 60\right)$

$$\left[2\left(\cos 60+i\sin 60\right)\right]^{\frac{1}{2}}=2^{\frac{1}{2}}\left[\cos\left(\frac{60}{2}+k\cdot\frac{360}{2}\right)+i\sin\left(\frac{60}{2}+k\cdot\frac{360}{2}\right)\right]$$

$$=\sqrt{2}\left[\cos\left(30+k\cdot 180\right)+i\sin\left(30+k\cdot 180\right)\right]$$

- For $k = 0$, root is $\sqrt{2}\left(\cos 30+i\sin 30\right)$
- For $k = 1$, root is $\sqrt{2}\left(\cos 210+i\sin 210\right)$

EXAMPLE: FOURTH ROOT

- Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.
- Write in trigonometric form.

$$-8+8i\sqrt{3}=16 \text{ cis } 120^\circ$$

- Here r = 16 and θ = 120°. The fourth roots of this number have absolute value

$$\sqrt[4]{16}=2.$$

$$\alpha=\frac{120^\circ}{4}+\frac{360^\circ \cdot k}{4}=30^\circ+90^\circ \cdot k$$

EXAMPLE: FOURTH ROOT CONTINUED

- There are four fourth roots, let k = 0, 1, 2 and 3.

$$k = 0 \qquad \alpha = 30^\circ + 90^\circ \cdot 0 = 30^\circ$$

$$k = 1 \qquad \alpha = 30^\circ + 90^\circ \cdot 1 = 120^\circ$$

$$k = 2 \qquad \alpha = 30^\circ + 90^\circ \cdot 2 = 210^\circ$$

$$k = 3 \qquad \alpha = 30^\circ + 90^\circ \cdot 3 = 300^\circ$$

- Using these angles, the fourth roots are

$$2 \text{ cis } 30^\circ, \qquad 2 \text{ cis } 120^\circ, \qquad 2 \text{ cis } 210^\circ, \qquad 2 \text{ cis } 300^\circ$$

EXAMPLE: FOURTH ROOT CONTINUED

Written in rectangular form

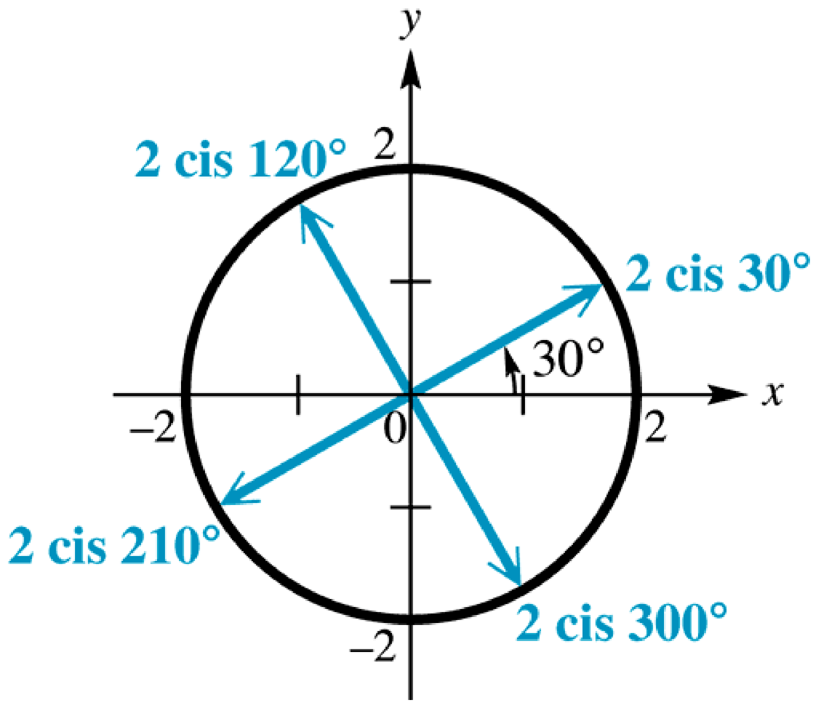
$$\sqrt{3}+i$$

$$-1+i\sqrt{3}$$

$$-\sqrt{3}-i$$

$$1-i\sqrt{3}$$

The graphs of the roots are all on a circle that has center at the origin and radius 2.



HOMEWORK

1. Show that $\cos z = \cos x \cosh y - i \sin x \sinh y$.
2. Show that $\sin^{-1} z = -i \ln(iz \pm \sqrt{1-z^2})$.
3. Find the fourth roots of 2.

Please do the homework on a paper. This exercise should be submitted on Thursday, August 12th 2013 before the class begins.

NEXT AGENDA

- Limit, Functions, and Continuity of Complex Variables.