Frequency Domain Measurements

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Measurements

- Linear Spectrum
- Autopower Spectrum
- Crosspower Spectrum
- Frequency Response Function
  - H1, H2, Hv
- Coherence
- Principle Component Analysis

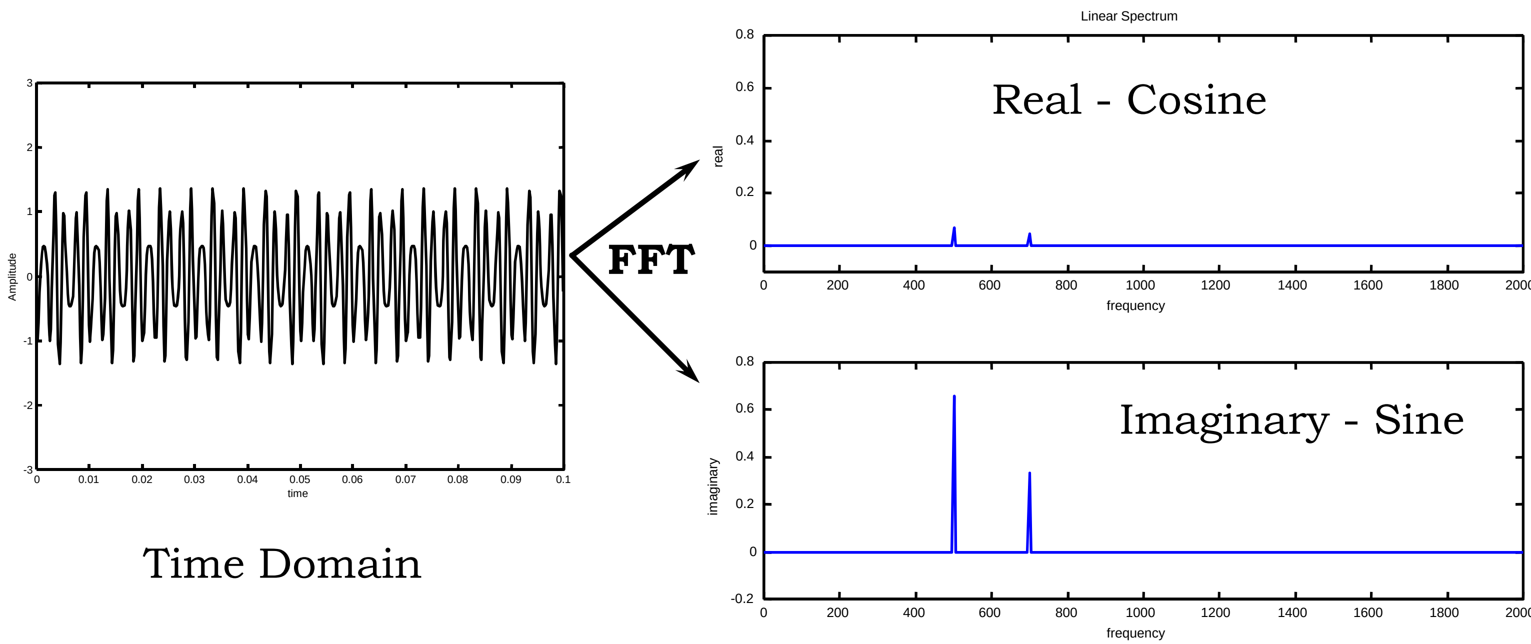
Linear Spectrum

- The basic function of the Frequency Domain is the Linear Spectrum, $G_x(\omega)$. It is defined as the Fourier transform of a time history.

- The Linear Spectrum is a complex valued function. The real part indicates the frequency content of the time history that is co-sinusoidal with respect to the start of the time history.

- The imaginary part indicates the frequency content of the time history that is sinusoidal with respect to the start of the time history.

Linear Spectrum



Time Domain

Frequency Domain

Autopower Spectrum

- The Auto Power Spectrum is defined as the Linear Spectrum, G_x, multiplied by its complex conjugate.

$$G_{xx}(\omega) = G_x^*(\omega)\, G_x(\omega)$$

- The result is a real valued function equivalent to the magnitude of the Linear Spectrum squared.
- The autopower does NOT have phase!

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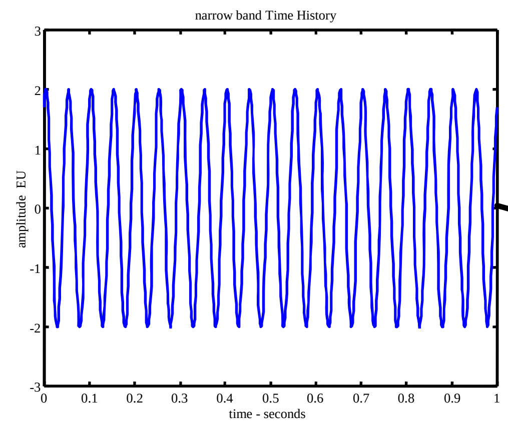
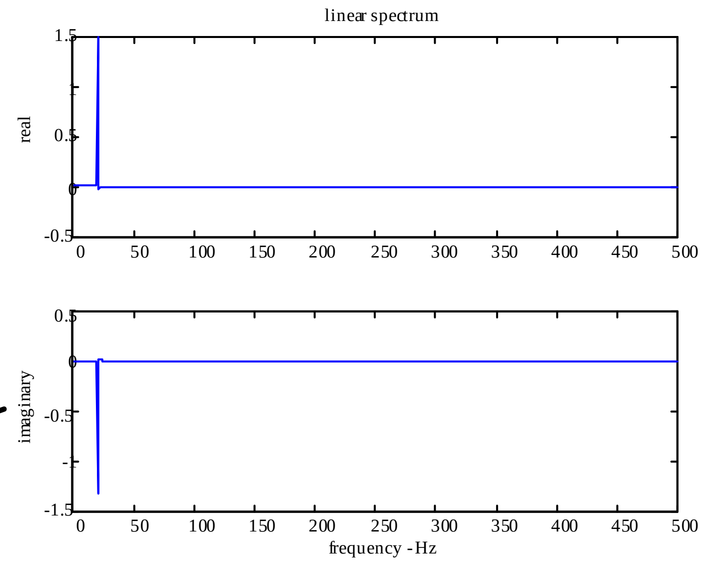
Autopower Spectrum

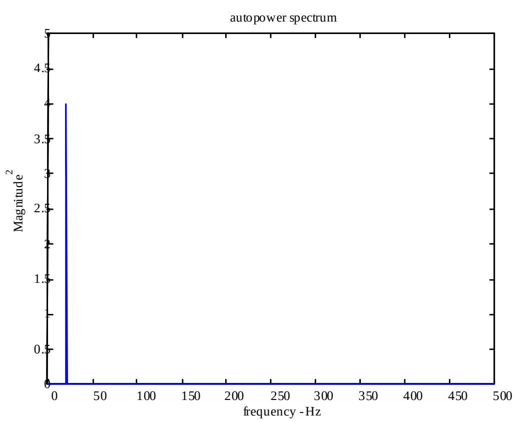
- Narrowband Data - Autopower Spectra
  - Narrowband implies only a few frequencies present!



FFT



Autopower

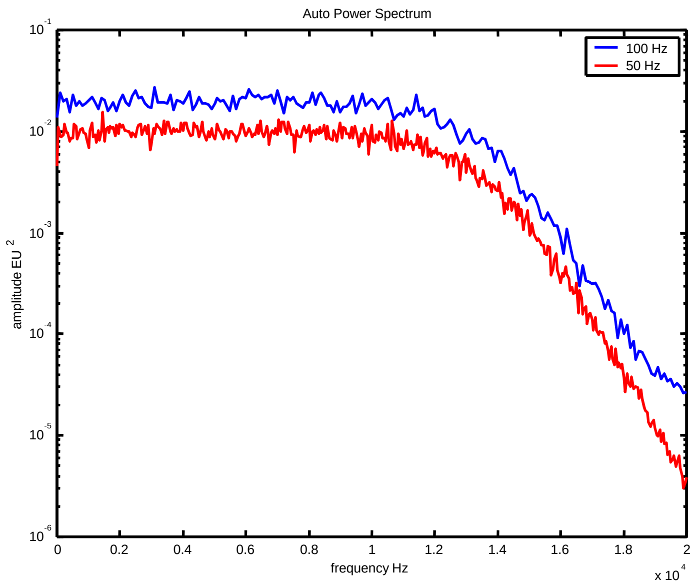


Power Spectral Density

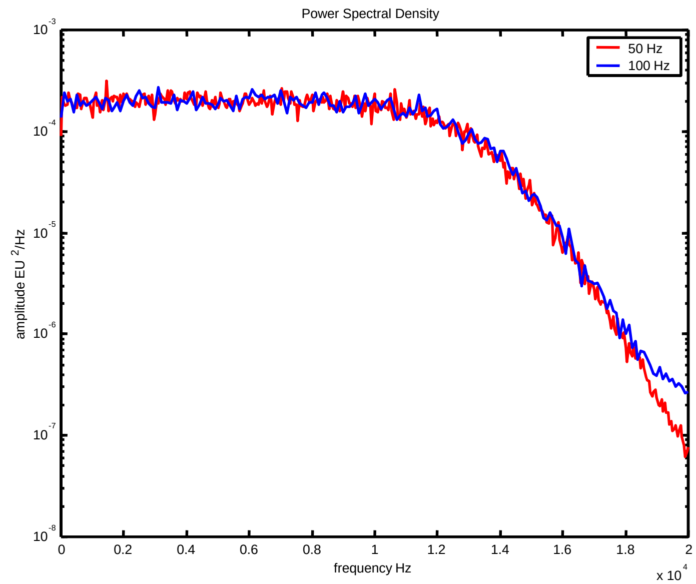
- Power Spectral Density is defined as the Power (magnitude squared) per unit frequency.
- The units of the function are (units squared/Hz.).
- The function is computed by normalizing (dividing) the Autopower Spectrum with respect to the effective bandwidth of the measurement, Δf.
- A PSD is used as a convenient function for calculating total power in a frequency band or for comparing broadband spectra that were computed with different resolution.

Power Spectral Density

Two measurements have different Δf's



Autopower - Different Amplitudes!



PSD - Same Amplitude!

Crosspower Spectrum

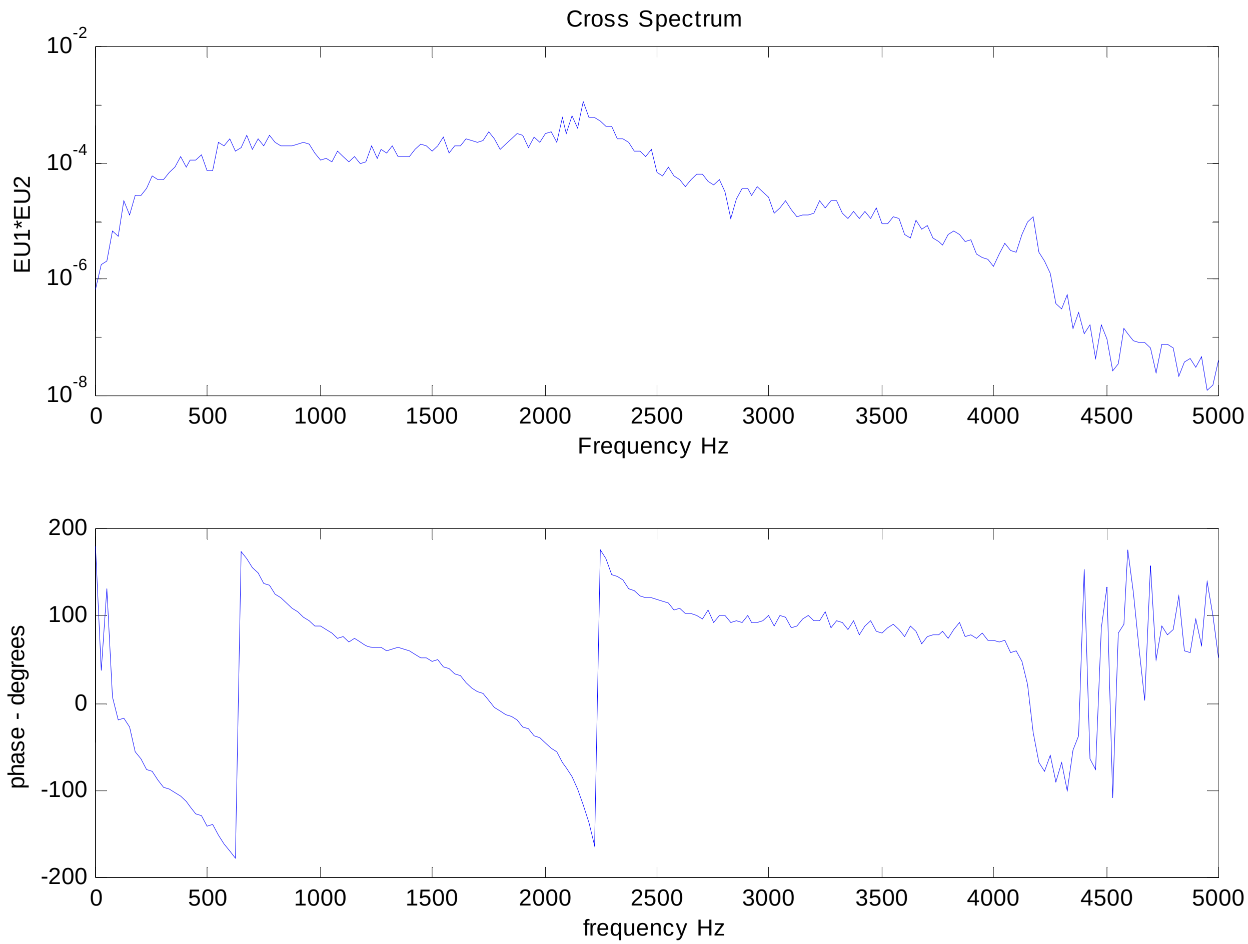
- The Cross Power Spectrum is defined as the Linear Spectrum of signal y(t), $G_y(\omega)$, multiplied by the complex conjugate of the Linear Spectrum of signal x(t), $G_x(\omega)$.

$$G_{xy}(\omega) = G_x^*(\omega) G_y(\omega)$$

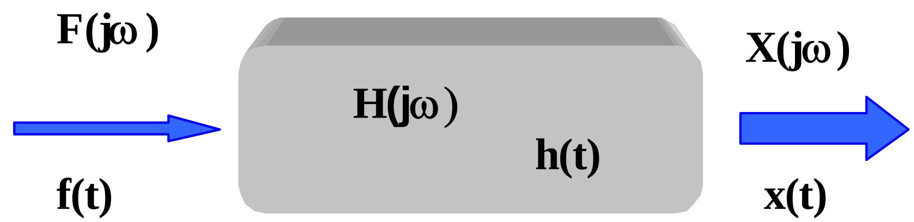
- The result is a complex valued function indicating the mutual power and relative phase between signals x(t) and y(t).

- Effect of Averaging:
  - The components of x(t) and y(t) that are not correlated are attenuated with averaging regardless of trigger condition.

Crosspower Spectrum



Frequency Response Functions (FRFs)



- The Frequency Response Function is defined as the ratio of the output to the input of a system, expressed in the frequency domain.
- The FRF and the Transfer Function are not the same!
  - The FRF is the imaginary axis of the Transfer Function.

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FRF

- For structural dynamics, this is usually defined as:

$$H(\omega) = \frac{response(\omega)}{input(\omega)} = \frac{motion(\omega)}{force(\omega)}$$

- Dynamic Compliance $\quad H(\omega) = \dfrac{X(\omega)}{F(\omega)}$ $\quad$ displacement

- Mobility $\quad H(\omega) = \dfrac{\dot{X}(\omega)}{F(\omega)}$ $\quad$ velocity

- Inertance $\quad H(\omega) = \dfrac{\ddot{X}(\omega)}{F(\omega)}$ $\quad$ acceleration

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FRF relationship to Impulse Response

- The response is related to the force through the FRF in the frequency domain, and the impulse response function in the time domain.

Frequency Domain

$$X(\omega) = H(\omega)F(\omega)$$

Impulse Response Function

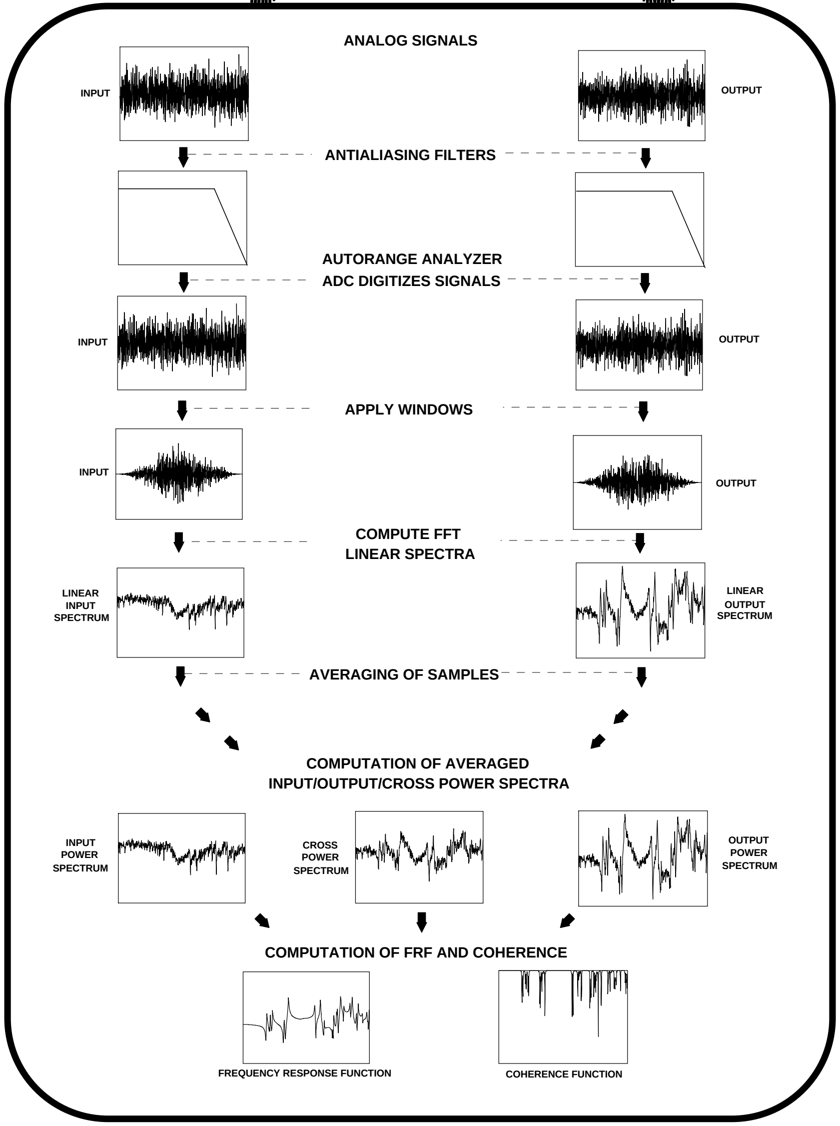
$$h(t) = F^{-1}\left[H(\omega)\right]$$

Time Domain Response

$$x(t) = h(t) \otimes f(t)$$

Convolution

Review of FRF Measurement Process



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Computing the FRF using Linear Spectrum

- Linear Spectrum Formulation
  - Advantage:
    - Simple
  - Disadvantage:
    - Synchronous averaging required
    - Uncorrelated noise is difficult to remove
    - Coherence function is not calculated

$$G_X(\omega) = H(\omega) G_F(\omega)$$

$$H_{XF}(\omega) = \frac{G_X(\omega)}{G_F(\omega)}$$

H1 Formulation for FRF's

- H1 Formulation
  - Advantage:
    - Asynchronous averaging (no consistent trigger required)
    - Computed from basic measurement functions
    - Uncorrelated components attenuated in the crosspower spectrum
  - Disadvantage:
    - **Uncorrelated components remain in the input autopower spectrum**

$$G_X(\omega) = H(\omega) G_F(\omega)$$

$$G_F^*(\omega) G_X(\omega) = H G_F^*(\omega) G_F(\omega)$$

$$G_{FX}(\omega) = H(\omega) G_{FF}(\omega)$$

$$\frac{G_{FX}(\omega)}{G_{FF}(\omega)} = H_1(\omega)$$

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H2 Formulation for FRF's

- H2 Formulation
  - Advantage:
    - Asynchronous averaging (no consistent trigger required)
    - Computed from basic measurement functions
    - Uncorrelated components attenuated in the crosspower spectrum
  - Disadvantage:
    - **Uncorrelated components remain in the response autopower spectrum**

$$G_X(\omega) = H(\omega) G_F(\omega)$$

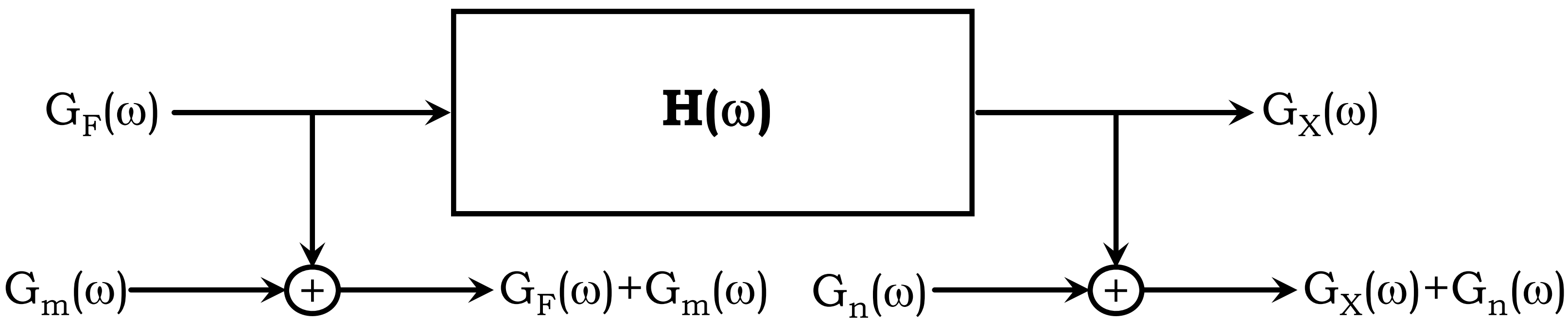
$$G_X^*(\omega) G_X(\omega) = H G_X^*(\omega) G_F(\omega)$$

$$G_{XX}(\omega) = H(\omega) G_{XF}(\omega)$$

$$\frac{G_{XX}(\omega)}{G_{XF}(\omega)} = H_2(\omega)$$

Computing the FRF with noise

- Most of the time conditions are less than ideal. This leads to the presence of noise on our signals.



$G_F(\omega)$ = Input Spectrum

$G_X(\omega)$ = Output Spectrum

$G_m(\omega)$ = Uncorrelated Input Noise Spectrum

$G_n(\omega)$ = Uncorrelated Output Noise Spectrum

Computing the FRF with noise

- Desired Measurement

$$G_X(\omega) = H(\omega) G_F(\omega)$$

- Actual Measurement

$$\left[G_X(\omega) + G_n(\omega)\right] = H(\omega)\left[G_F(\omega) + G_m(\omega)\right]$$

Linear Spectrum with noise

- Using linear spectrums to compute the FRF with noise results in the following.

$$G_X(\omega)+G_n(\omega)=H(\omega)\left[G_F(\omega)+G_m(\omega)\right]$$

$$H(\omega)=\frac{G_X(\omega)+G_n(\omega)}{G_F(\omega)+G_m(\omega)}$$

- This FRF will be biased by the ratio of the noise spectrums.

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H1 Formulation with noise

- Computing the FRF with H1 with noise results in the following.

$$G_X(\omega)+G_n(\omega)=H(\omega)\left[G_F(\omega)+G_m(\omega)\right]$$

$$\left[G_F^*(\omega)+G_m^*(\omega)\right]\left[G_X(\omega)+G_n(\omega)\right]=H(\omega)\left[G_F^*(\omega)+G_m^*(\omega)\right]\left[G_F(\omega)+G_m(\omega)\right]$$

$$G_{FX}(\omega)+G_{Fn}(\omega)+G_{mX}(\omega)+G_{mn}(\omega)=H_1(\omega)\left[G_{FF}(\omega)+G_{Fm}(\omega)+G_{mF}(\omega)+G_{mm}(\omega)\right]$$

- Assuming the noise spectrums are uncorrelated with each other and averaging results in:

$$G_{FX}(\omega)=H_1\left[G_{FF}(\omega)+G_{mm}(\omega)\right]$$

$$H_1(\omega)=\frac{G_{FX}(\omega)}{G_{FF}(\omega)+G_{mm}(\omega)}$$

- Biased low if $G_{mm}(\omega)\neq 0$!

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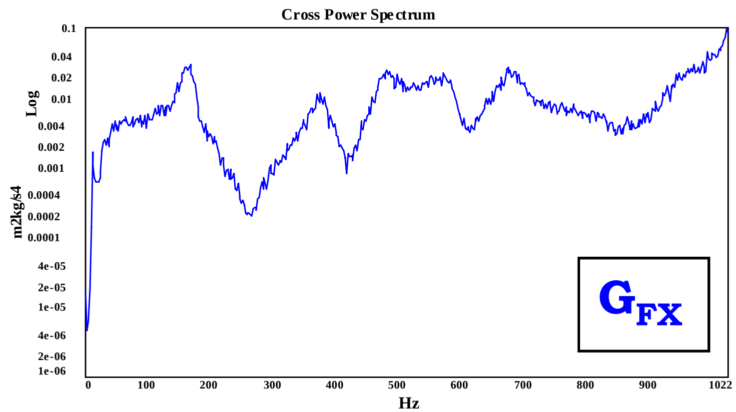
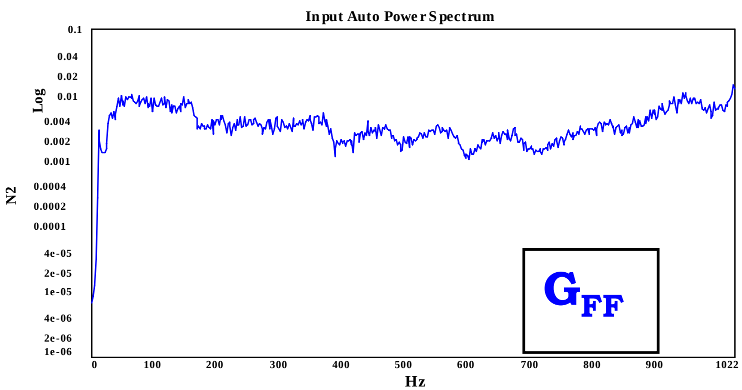
H1 Computation





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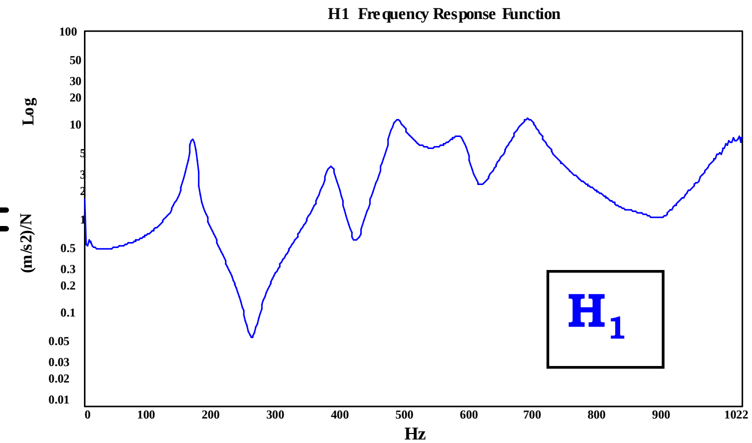

H2 Formulation with noise

➤ Computing the FRF with H2 with noise results in the following.

$$G_X(\omega)+G_n(\omega)=H(\omega)[G_F(\omega)+G_m(\omega)]$$

$$[G_X^*(\omega)+G_n^*(\omega)][G_X(\omega)+G_n(\omega)]=H(\omega)[G_X^*(\omega)+G_n^*(\omega)][G_F(\omega)+G_m(\omega)]$$

$$G_{XX}(\omega)+G_{Xn}(\omega)+G_{nX}(\omega)+G_{nn}(\omega)=H_1(\omega)[G_{XF}(\omega)+G_{Xm}(\omega)+G_{nF}(\omega)+G_{nm}(\omega)]$$

➤ Assuming the noise spectrums are uncorrelated with each other and averaging results in:

$$G_{XF}(\omega)=H_2[G_{XX}(\omega)+G_{nn}(\omega)]$$

$$H_1(\omega)=\frac{G_{XX}(\omega)+G_{nn}(\omega)}{G_{XF}(\omega)}$$

➤ Biased high if $G_{nn}(\omega)\neq 0$!

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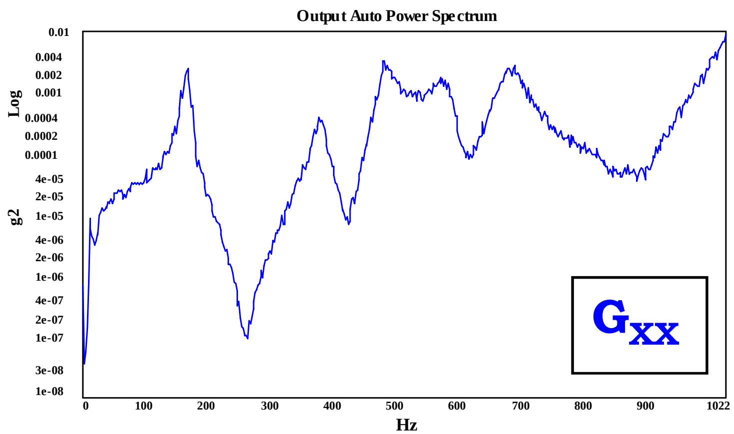
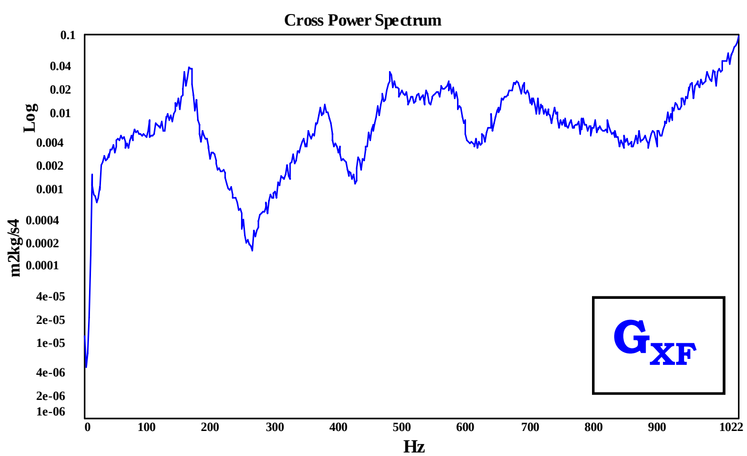
H2 Computation





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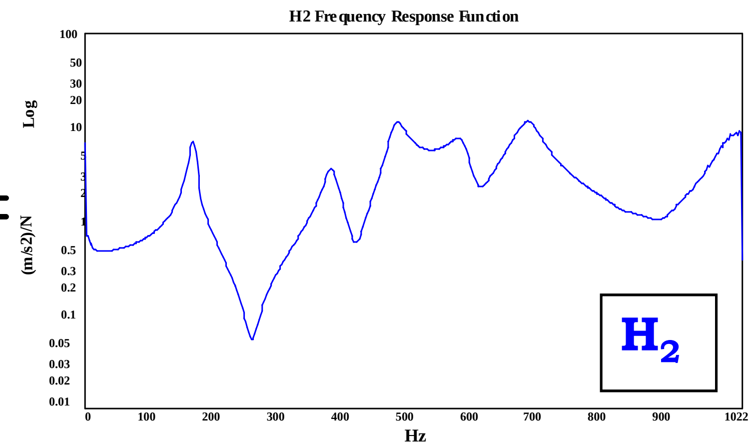


H_v FRF Computation

- H_v minimizes the noise on the input and the output!
- H_1 and H_2 gave the following equations:

$$G_{FX}(\omega) = H(\omega)\left[G_{FF}(\omega) + G_{mm}(\omega)\right] \qquad G_{XX}(\omega) + G_{nn}(\omega) = H(\omega) G_{XF}(\omega)$$

- Forming an eigenvalue problem to minimize the noise give:

$$\begin{bmatrix} G_{XX}(\omega) & G_{XF}(\omega) \\ G_{FX}(\omega) & G_{FF}(\omega) \end{bmatrix} \begin{Bmatrix} -1 \\ H(\omega) \end{Bmatrix} = -\varepsilon \begin{bmatrix} G_{nn}(\omega) & 0 \\ 0 & G_{mm}(\omega) \end{bmatrix} \begin{Bmatrix} -1 \\ H(\omega) \end{Bmatrix}$$

- The eigenvector associated with the smallest eigenvalues gives H_v.

Coherence

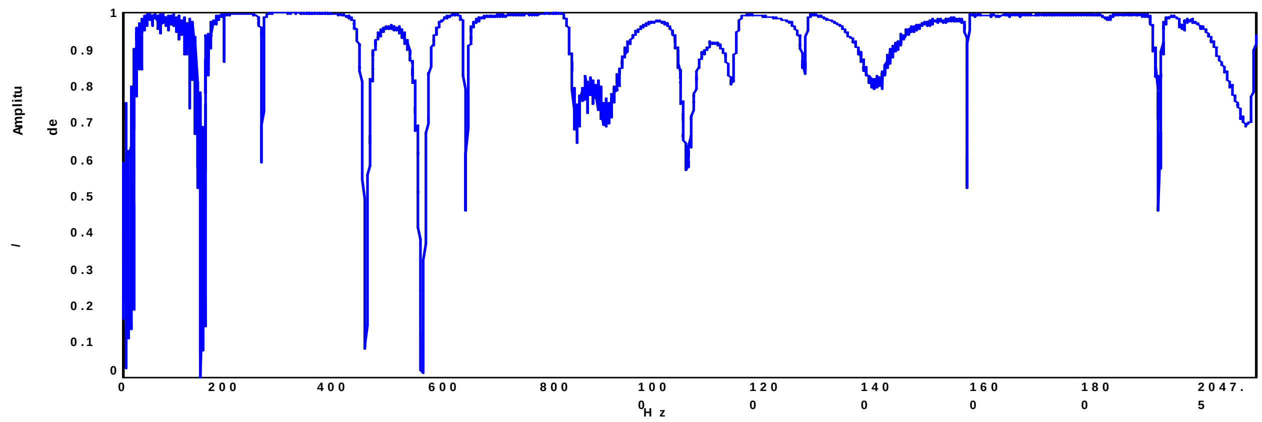
- Coherence is defined as the fraction of output that is linearly related to the input.

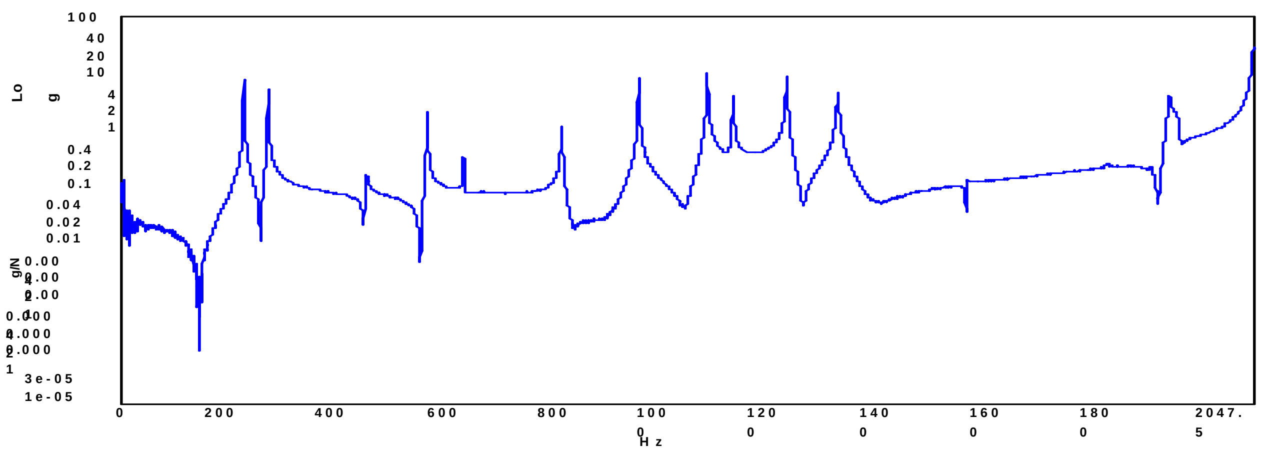
$$\gamma_{FX}^{2}(\omega)=\frac{G_{FX}(\omega)G_{XF}(\omega)}{\left[G_{FF}(\omega)+G_{mm}(\omega)\right]\left[G_{XX}(\omega)+G_{nn}(\omega)\right]}$$

$$0\le\gamma_{FX}^{2}(\omega)\le 1$$

- If the coherence function is less than 1, the following may be true:
  - The system relating the input and the output is not linear.
  - There are unmeasured inputs to the system.
  - There is no output from the system.
  - Bias errors such as noise or leakage are present in the measurement.

Coherence Example





Coherence Values – Function of # of averages

90% confidence limits on the measurement of the amplitude |H| and phase ϕ of transfer functions, as a function of the measured value of coherence and the number of averages.

Measured value of coherence function	Number of Averages 16	32	64	128	256
0.2	+ 5.2 - 14.6 (± 54)	+ 3.8 - 7.1 (± 34)	+ 2.8 - 4.2 (± 23)	+ 2.1 - 2.7 (± 16)	+ 1.5 - 1.8 (± 11)
0.3	+ 4.2 - 8.4 (± 38)	+ 3.1 - 4.8 (± 25)	+ 2.2 - 3.0 (± 17)	+ 1.6 - 2.0 (± 12)	+ 1.2 - 1.4 (± 8)
0.4	+ 3.5 - 6.0 (± 30)	+ 2.6 - 3.6 (± 20)	+ 1.8 - 2.3 (± 14)	+ 1.3 - 1.6 (± 10)	+ 1.0 - 1.1 (± 7)
0.5	+ 3.0 - 4.5 (± 24)	+ 2.1 - 2.8 (± 16)	+ 1.5 - 1.9 (± 11)	+ 1.1 - 1.3 (± 8)	+ 0.8 - 0.9 (± 5)
0.6	+ 2.5 - 3.5 (± 19)	+ 1.8 - 2.2 (± 13)	+ 1.3 - 1.5 (± 9)	+ 0.9 - 1.0 (± 6)	+ 0.7 - 0.7 (± 4)
0.7	+ 2.1 - 2.7 (± 15)	+ 1.5 - 1.7 (± 10)	+ 1.0 - 1.2 (± 7)	+ 0.7 - 0.8 (± 5)	+ 0.5 - 0.6 (± 4)
0.8	+ 1.6 - 2.0 (± 12)	+ 1.1 - 1.3 (± 8)	+ 0.8 - 0.9 (± 6)	+ 0.6 - 0.6 (± 4)	+ 0.4 -0.4 (± 3)
0.9	+ 1.1 - 1.3 (± 8)	+ 0.8 - 0.8 (± 5)	+ 0.5 - 0.6 (± 4)	+ 0.4 - 0.4 (± 3)	+ 0.3 - 0.3 (± 2)

For each entry, the first two digits are the upper and lower bounds on |H|, in dB.
Digits in parentheses are the bounds on ϕ, in degrees.