Reconstructing a simple 3D object using phase-shift coherence holography

Generation of a Fourier transform coherence hologram

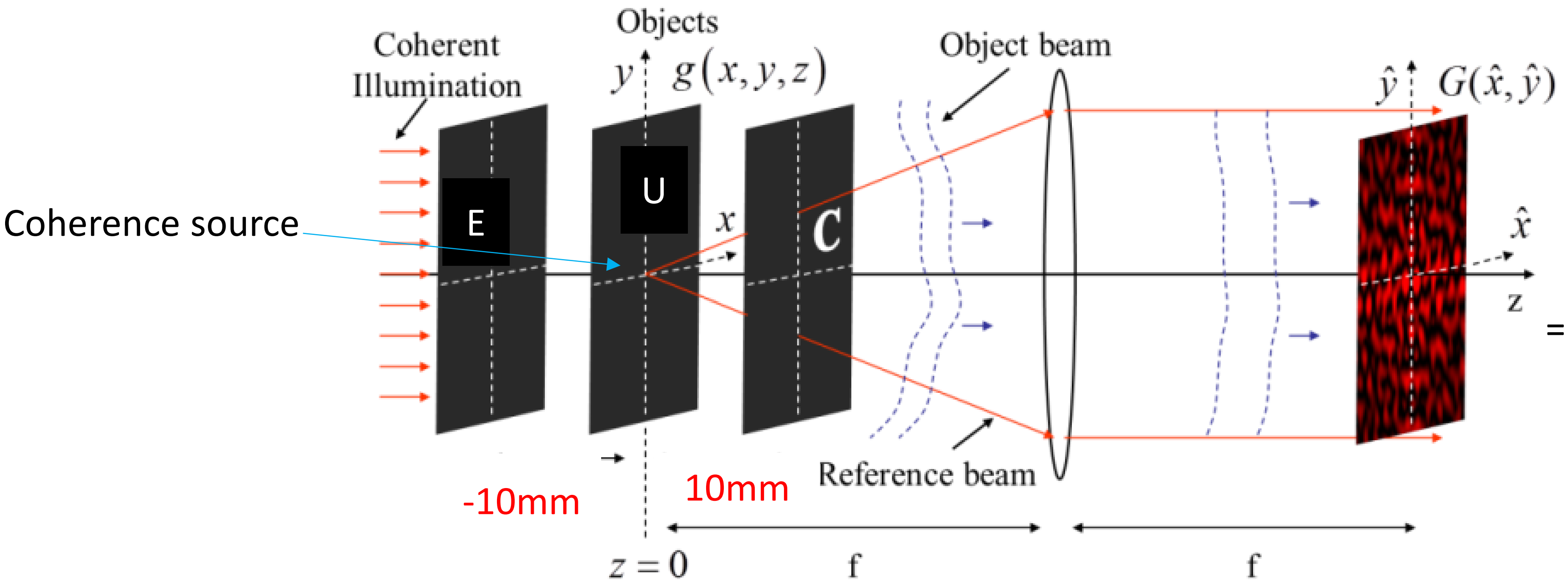


Figure 1

Recorded Hologram

$$= \left| \begin{matrix} reference\ beam\ (plane\ wave) \\ +object\ beam \end{matrix} \right|^2$$

Object beam at hologram plane:

$$G(\hat{x},\hat{y}) \propto \int \underbrace{\left\{ \iint g(x,y,z) \exp\left[-i \frac{2\pi}{\lambda f}(x\hat{x}+y\hat{y}) \right] dxdy \right\}}_{\text{Fourier Transform (Based on Fraunhofer diffraction)}} \overbrace{\exp\left[-ik_z(\hat{x},\hat{y})z \right] dz}^{\text{Defocus and propagation of the angular spectrum}}, \quad (1)$$

$$k_z(\hat{x},\hat{y}) = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{\hat{x}}{f}\right)^2-\left(\frac{\hat{y}}{f}\right)^2}.$$

Phase-shift holograms

- A set of **phase-shift holograms** are generated **numerically** by giving **known phase shifts** to the object spectrum

$$G(\hat{x},\hat{y};m)=G(\hat{x},\hat{y})\exp(i\,2\pi m/N), \qquad (2)$$

- There is a term $|G(\hat{x},\hat{y})|^2$ that is the source of unwanted autocorrelation image→ **numerically** eliminate this term

Intensity transmittance of the hologram

- Unlike conventional hologram, the intensity (rather than amplitude) transmittance of the hologram was made proportional to the interference fringe pattern

$$
\begin{aligned}
H(\hat{x},\hat{y};m) &\propto |G(\hat{x},\hat{y})| + \frac{1}{2}G(\hat{x},\hat{y})\exp\left(i\,\frac{2m\pi}{N}\right) + \frac{1}{2}G^{*}(\hat{x},\hat{y})\exp\left(-i\,\frac{2m\pi}{N}\right) \\
&\propto |G(\hat{x},\hat{y})|\left\{1+\cos\left[\Phi(\hat{x},\hat{y})+\frac{2m\pi}{N}\right]\right\}
\end{aligned}
\tag{3}
$$

Incoherent term

Incoherence illumination

$$
\underbrace{\sqrt{H(\hat{x},\hat{y};m)}\,\boxed{\exp\left[i\,\Phi_{R}(\hat{x},\hat{y})\right]}}_{\text{The transmitted wave}}
\tag{4}
$$

Incoherence Illumination

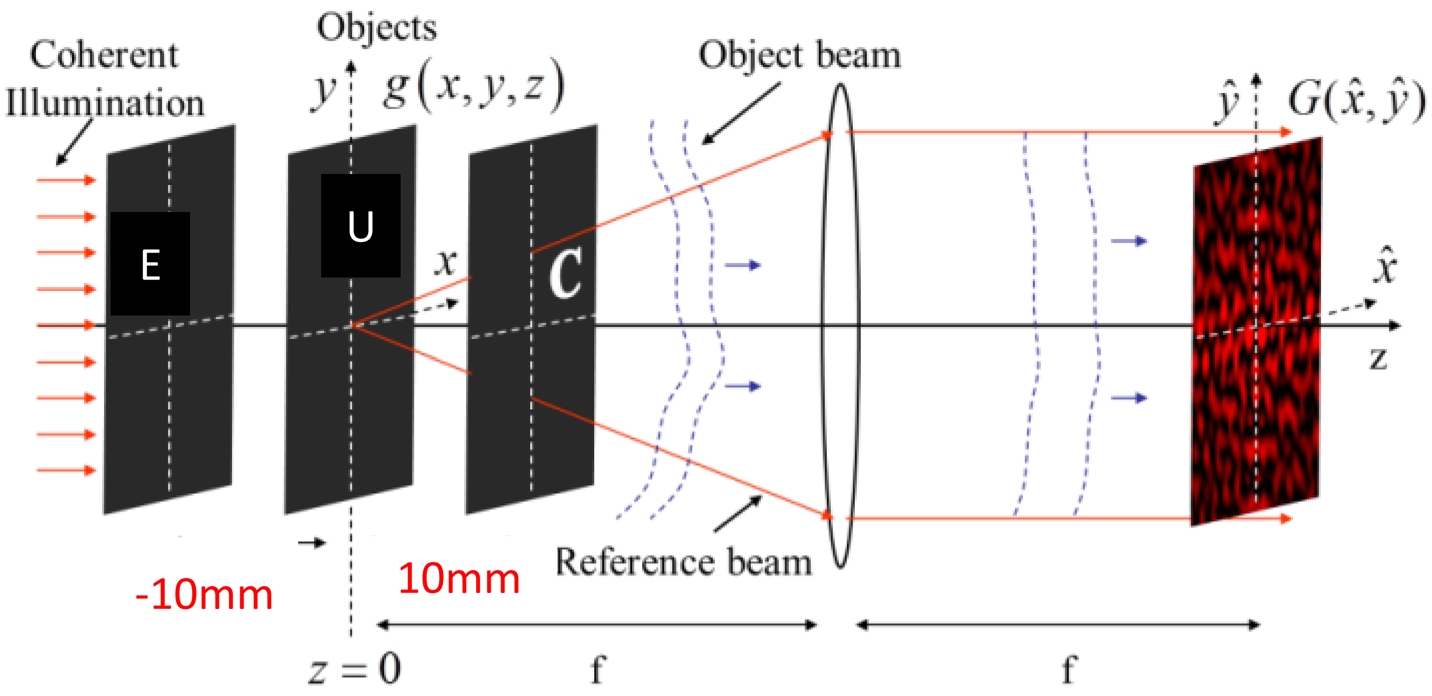


Figure 1

$$u(x,y,z;m)=\iint \underbrace{\sqrt{H(\hat{x},\hat{y};m)}\,\overset{\text{Incoherent term}}{\boxed{\exp\left[i\Phi_R(\hat{x},\hat{y})\right]}}}_{\text{The transmitted wave}}\exp\left[ik_z(\hat{x},\hat{y})z\right]\exp\left[i\frac{2\pi}{\lambda f}(x\hat{x}+y\hat{y})\right]d\hat{x}d\hat{y} \quad (4)$$

Coherence function- used to reconstruct the object

$$\Gamma(\Delta x,\Delta y;\Delta z)=\left\langle u^*(x_1,y_1;z_1)\,\mathrm{u}(x_2,y_2;z_2)\right\rangle$$

:Correlation of 2 signals

$$=\iint\iint \sqrt{H(\widehat{x_1},\widehat{y_1})}\,\sqrt{H(\widehat{x_2},\widehat{y_2})}\left\langle \exp\left[-i\Phi_R(\hat{x}_1,\hat{y}_1)\right]\exp\left[i\Phi_R(\hat{x}_2,\hat{y}_2)\right]\right\rangle$$

$$\times\exp\left[-ik_z(\hat{x}_1,\hat{y}_1)z_1\right]\exp\left[ik_z(\hat{x}_2,\hat{y}_2)z_2\right]$$

$$\times\exp\left[-i\frac{2\pi}{\lambda f}(x_1\hat{x}_1+y_1\hat{y}_1)\right]\exp\left[i\frac{2\pi}{\lambda f}(x_2\hat{x}_2+y_2\hat{y}_2)\right]d\hat{x}_1d\hat{y}_1d\hat{x}_2d\hat{y}_2$$

$$=\iint H(\widehat{x_1},\widehat{y_1})\,\exp\left[ik_z(\hat{x}_1,\hat{y}_1)\Delta z\right]\exp\left[i\frac{2\pi}{\lambda f}(\hat{x}_1\Delta x+\hat{y}_1\Delta y)\right]d\hat{x}_1d\hat{y}_1,$$

IFFT of the Intensity transmittance hologram

where < > denotes ensemble average, $\Delta x=x_2-x_1$, $\Delta y=y_2-y_1$, $\Delta z=z_2-z_1$, and use has been made of the assumption $\underbrace{\left\langle \exp\left[-i\Phi_R(\hat{x}_1,\hat{y}_1)\right]\exp\left[i\Phi_R(\hat{x}_2,\hat{y}_2)\right]\right\rangle=\delta(\hat{x}_1-\hat{x}_2,\hat{y}_1-\hat{y}_2)}_{\text{Definition of impulse response}}$ for an ideal rotating ground glass.

Adding phase-shift

$$\Gamma(\Delta x,\Delta y,\Delta z;m)=\left\langle u^*(x_1,y_1,z_1;m)u(x_2,y_2,z_2;m)\right\rangle$$

$$=\iint H(\hat{x},\hat{y};m)\exp\left[ik_z(\hat{x},\hat{y})\Delta z\right]\exp\left[i\frac{2\pi}{\lambda f}(\hat{x}\Delta x+\hat{y}\Delta y)\right]d\hat{x}d\hat{y}$$

(7)

Sagnac Interferometer- CCD captures the interference of this interferometer $\langle |u(x_1,y_1,z_1;m)+u(x_2,y_2,z_2;m)|^2\rangle$

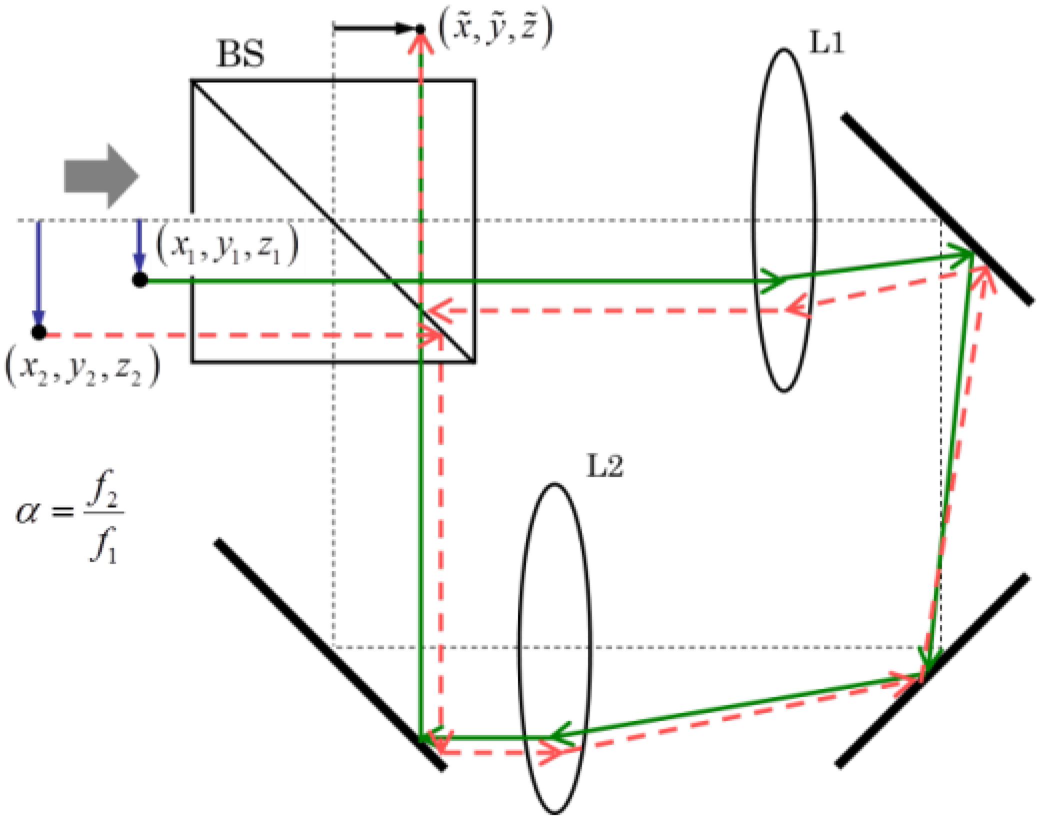


Fig. 3. Radial and axial shearing Sagnac common path interferometer.

$$I(\Delta x,\Delta y,\Delta z;m) = \langle |u(x_1,y_1,z_1;m)+u(x_2,y_2,z_2;m)|^2\rangle$$
$$= 2\Gamma(0,0,0;m)+2\,\mathrm{Re}[\Gamma(\Delta x,\Delta y,\Delta z;m)],$$

$$\Delta x = x_2 - x_1,\ \Delta y = y_2 - y_1 \qquad \Delta z = z_2 - z_1$$

$$\Delta\mathbf{r} = (\Delta x,\Delta y,\Delta z) = (m_x^{-1}\tilde{x}, m_y^{-1}\tilde{y}, m_z^{-1}\tilde{z})$$

$$m_x = m_y = -(\alpha - \alpha^{-1})^{-1} \qquad m_z = (\alpha^2 - \alpha^{-2})^{-1}$$

The equation of CCD image

$$I(\Delta x, \Delta y, \Delta z; m) = \left\langle \left| u(x_1, y_1, z_1; m) + u(x_2, y_2, z_2; m) \right|^2 \right\rangle$$
$$= \underbrace{2\Gamma(0,0,0;m)}_{\text{Background}} + 2\,\mathrm{Re}\left[\underbrace{\Gamma(\Delta x, \Delta y, \Delta z; m)}_{\text{Coherence function}}\right], \quad (5)$$

Background

Coherence function

$$\Gamma(0,0,0;m) = \left\langle u^*(x_1, y_1, z_1; m)\, u(x_1, y_1, z_1; m) \right\rangle$$
$$= \left\langle u^*(x_2, y_2, z_2; m)\, u(x_2, y_2, z_2; m) \right\rangle \quad (6)$$
$$= \iint H(\hat{x}, \hat{y}; m)\, d\hat{x}\, d\hat{y}$$

The purpose of adding phase-shift to the holograms

From (3) and (7)

$$\Gamma(\Delta x,\Delta y,\Delta z;m)=\tilde{g}(\Delta x,\Delta y,\Delta z)+\frac{1}{2}g(\Delta x,\Delta y,\Delta z)\exp\left(i\,\frac{2m\pi}{N}\right)+\frac{1}{2}g^{*}(-\Delta x,-\Delta y,-\Delta z)\exp\left(-i\,\frac{2m\pi}{N}\right) \quad (8)$$

where

$$\tilde{g}(\Delta x,\Delta y,\Delta z)=\iint\left|G(\hat{x},\hat{y})\right|\exp\left[ik_{z}(\hat{x},\hat{y})\Delta z\right]\exp\left[i\,\frac{2\pi}{\lambda f}(\hat{x}\Delta x+\hat{y}\Delta y)\right]d\hat{x}d\hat{y}\,. \quad (9)$$

CCD image

Vary sinusoidally -> Sin-fit Algorithm

$$I(\Delta x,\Delta y,\Delta z;m)=\mathrm{Re}\Big\{2\tilde{g}(0,0,0)+2\tilde{g}(\Delta x,\Delta y,\Delta z)+g(0,0,0)\exp\left(i\,\frac{2m\pi}{N}\right)+g(\Delta x,\Delta y,\Delta z)\exp\left(i\,\frac{2m\pi}{N}\right)+g^{*}(0,0,0)\exp\left(-i\,\frac{2m\pi}{N}\right)+g^{*}(-\Delta x,-\Delta y,-\Delta z)\exp\left(-i\,\frac{2m\pi}{N}\right)\Big\} \quad (10)$$

The purpose of adding phase-shift to the holograms

$I(\Delta x, \Delta y, \Delta z, m) = \{|g(\Delta x, \Delta y, \Delta z)|\cos[\theta(\Delta x, \Delta y, \Delta z)] + |g(-\Delta x, -\Delta y, -\Delta z)|\cos[\theta(-\Delta x, -\Delta y, -\Delta z)]\}\cos[\frac{2m\pi}{N}]$

$+\{-|g(\Delta x, \Delta y, \Delta z)|\sin[\theta(\Delta x, \Delta y, \Delta z)] - |g(-\Delta x, -\Delta y, -\Delta z)|\sin[\theta(-\Delta x, -\Delta y, -\Delta z)]\sin[\frac{2m\pi}{N}]$

$+C$

$= \mathbf{A}\cos[\frac{2m\pi}{N}] + \mathbf{B}\sin[\frac{2m\pi}{N}] + \mathbf{C}$

3 Unknowns: **x=** (A, B, C)^T → need at least 3 equations or 3 CCD images (y_1, y_2, y_3) with different phase-shifts

$\rightarrow A^2 + B^2 = |g(\Delta x, \Delta y, \Delta z)|^2 + |g(-\Delta x, -\Delta y, -\Delta z)|^2$

$\frac{B}{A} = -\tan(\theta(\Delta x, \Delta y, \Delta z))$-$\tan(-\theta(\Delta x, \Delta y, \Delta z))$

$$\mathbf{y} = (y_1, \ldots, y_N)^T. \quad (3)$$

Then, $\mathbf{y}$ obeys the linear set of equations

$$\mathbf{y} = \mathbf{D}\mathbf{x} \quad (4)$$

where $\mathbf{D}$ is the $N \times 3$ matrix

$$\mathbf{D} = \begin{pmatrix} \cos\omega t_1 & \sin\omega t_1 & 1 \\ \cos\omega t_2 & \sin\omega t_2 & 1 \\ \vdots & \vdots & \vdots \\ \cos\omega t_N & \sin\omega t_N & 1 \end{pmatrix}. \quad (5)$$

Equation (4) is an overdetermined (i.e. $N > 3$) set of linear equations, with the least-squares solution $\hat{\mathbf{x}}$ (in general, $\hat{}$ denotes an estimate) given by [3]

$$\hat{\mathbf{x}} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{y}. \quad (6)$$

The Reconstructed object

$$\Gamma(\Delta x, \Delta y, \Delta z; m) = \langle u^*(x_1, y_1, z_1; m) u(x_2, y_2, z_2; m) \rangle$$
$$= \iint H(\hat{x}, \hat{y}; m) \exp[ik_z(\hat{x}, \hat{y}) \Delta z] \exp\left[i \frac{2\pi}{\lambda f}(\hat{x}\Delta x + \hat{y}\Delta y)\right] d\hat{x} d\hat{y}$$

(7)

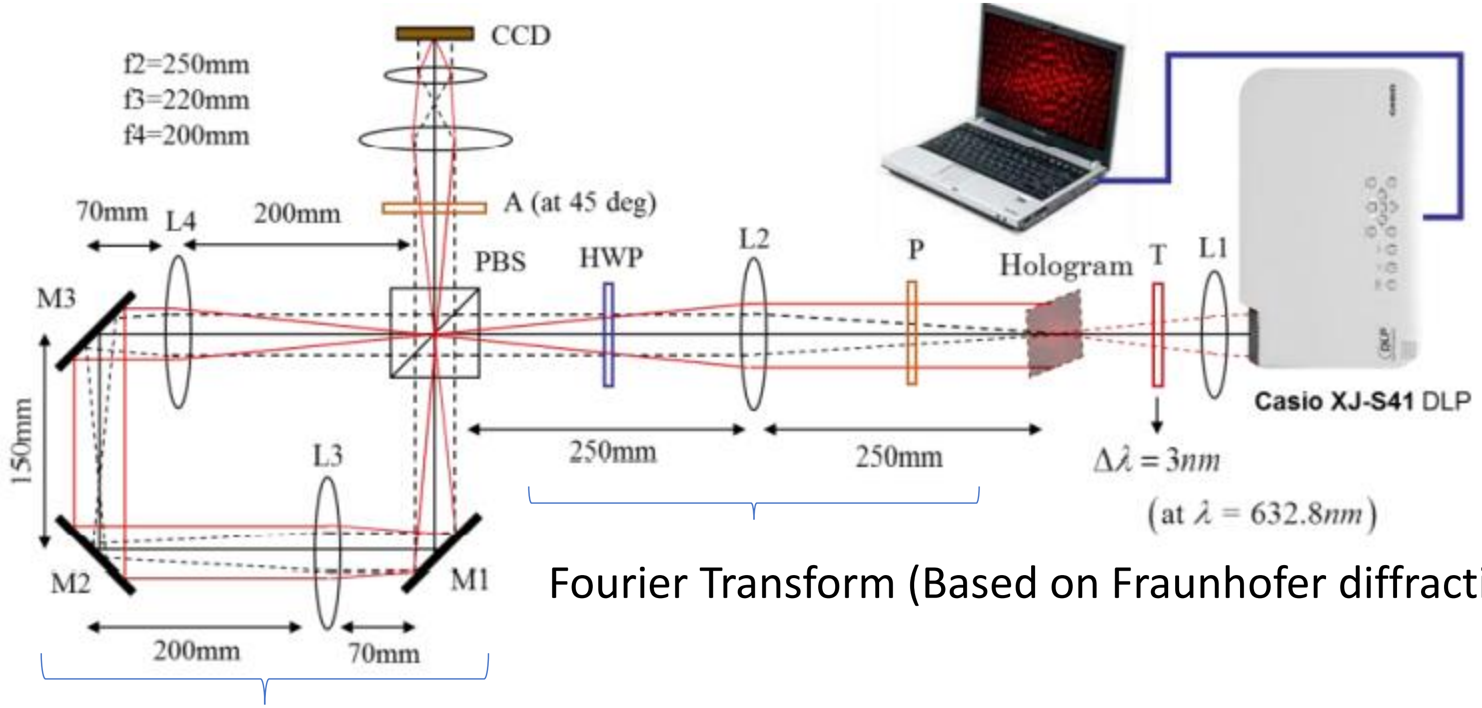
$$\Delta x = x_2 - x_1, \Delta y = y_2 - y_1 \qquad \Delta z = z_2 - z_1$$

$$\Delta \mathbf{r} = (\Delta x, \Delta y, \Delta z) = (m_x^{-1}\tilde{x}, m_y^{-1}\tilde{y}, m_z^{-1}\tilde{z})$$

$$m_x = m_y = -(\alpha - \alpha^{-1})^{-1} \qquad m_z = (\alpha^2 - \alpha^{-2})^{-1}$$

The reconstructed object at $(\alpha^2 - \alpha^{-2})^{-1} \times z_{\text{object}}$ plane is scale by a factor $(\alpha - \alpha^{-1})^{-1}$, where z_{object} is the z coordinate of the 3 planes of the 3D object

The setup to record the CCD images



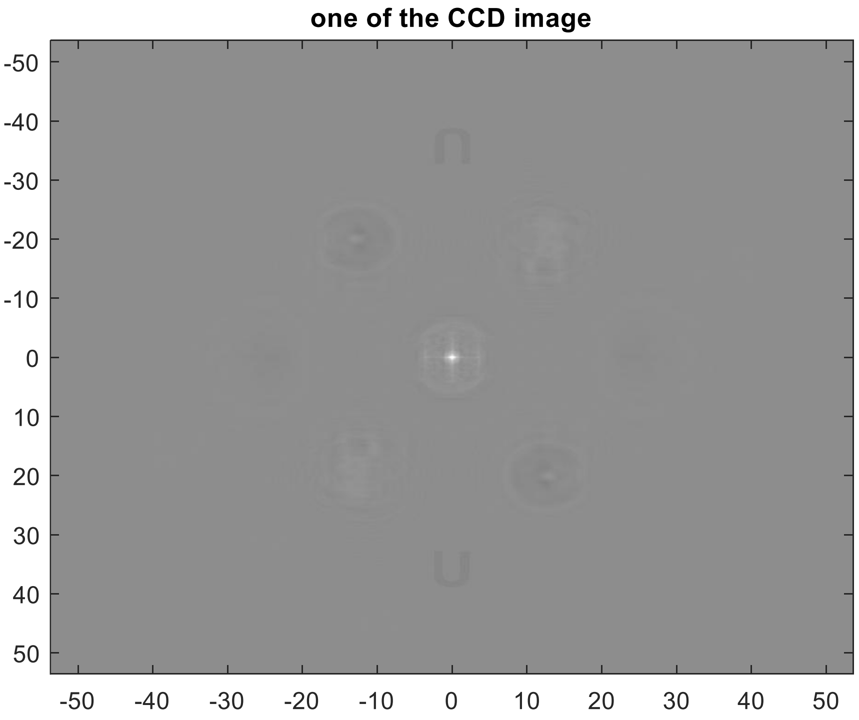
Fourier Transform (Based on Fraunhofer diffraction)

Interferometer

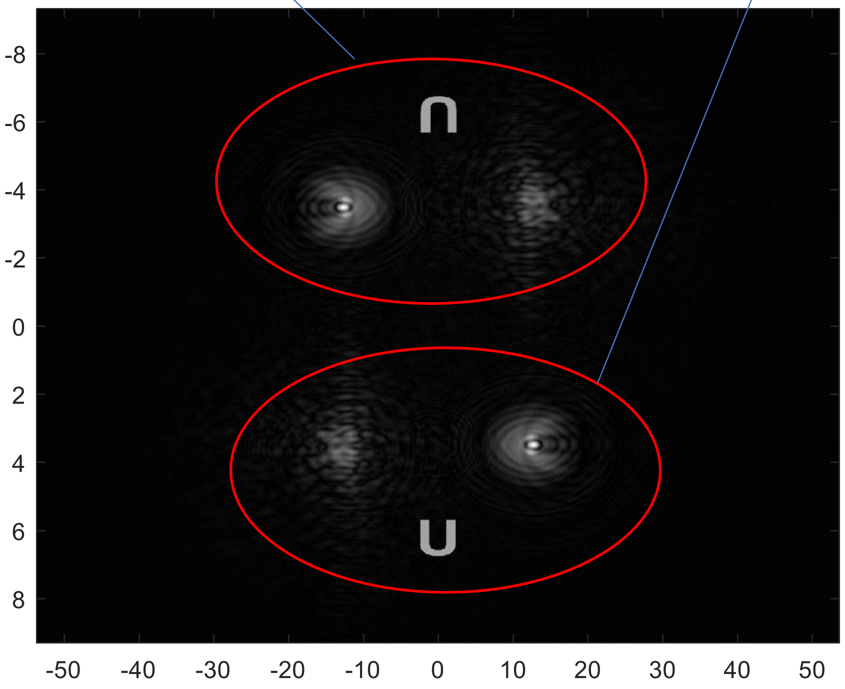
Results at $(\alpha^2 - \alpha^{-2})^{-1} \times z_{\text{object}}$ plane, $z_{\text{object}} = 0$

The conjugate object's amplitude

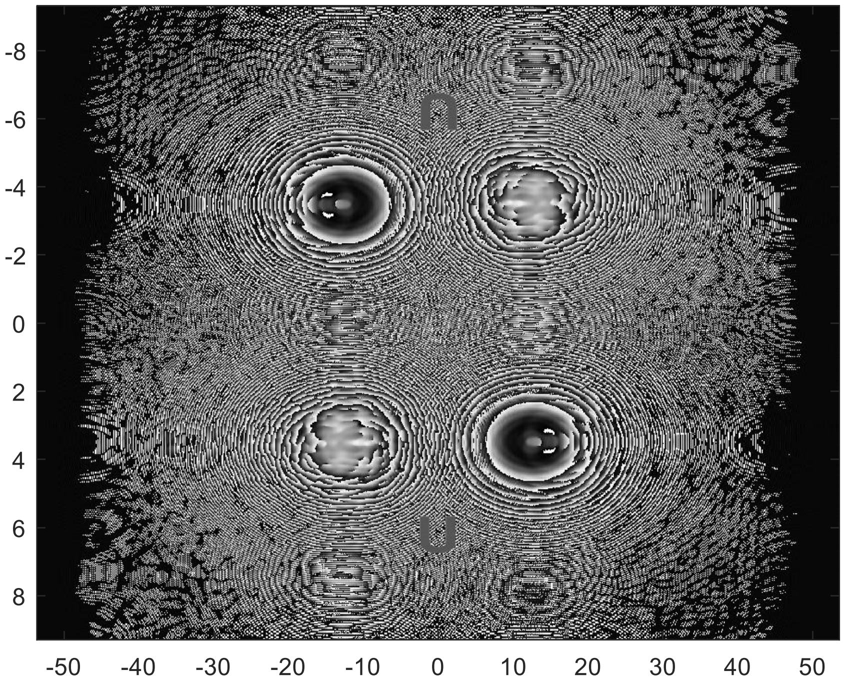
The object's amplitude



One of the CCD images

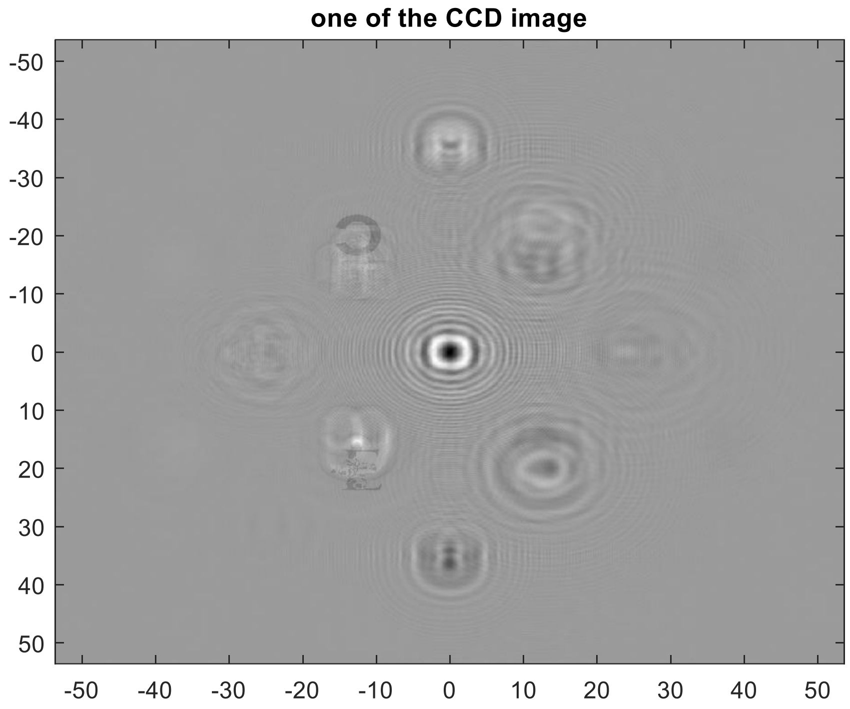


Reconstructed object's amplitude from IEEE paper

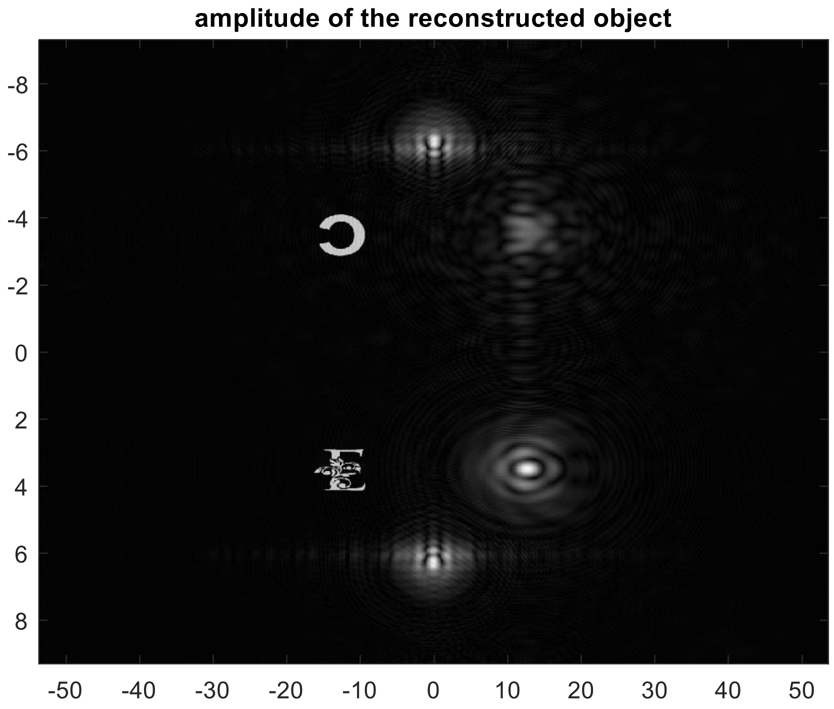


Reconstructed object's phase from IEEE paper

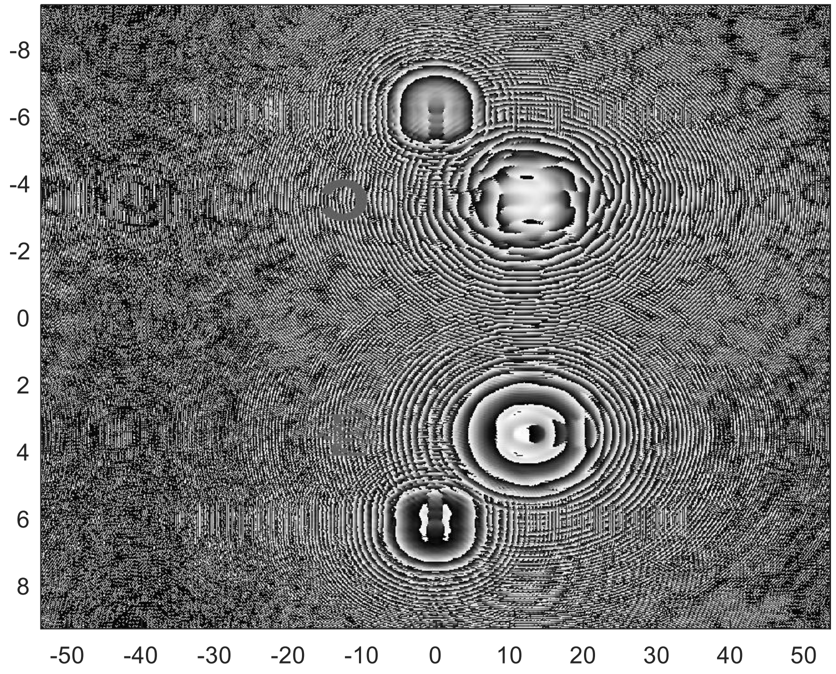
Results at $(\alpha^2 - \alpha^{-2})^{-1} \times z_{object}$ plane, $z_{object} = -10\text{mm}$



One of the CCD images

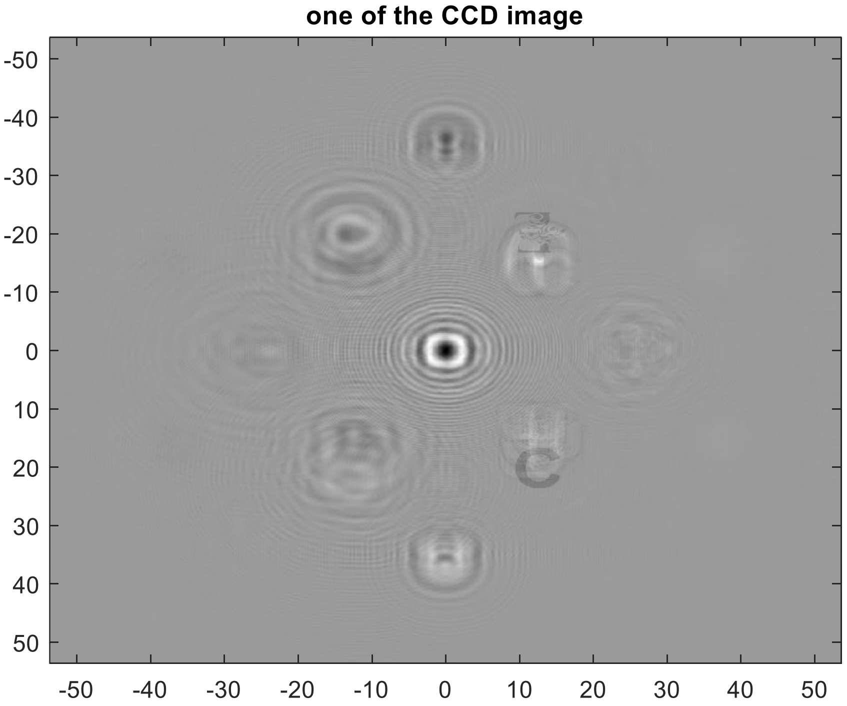


Reconstructed object's amplitude from IEEE paper

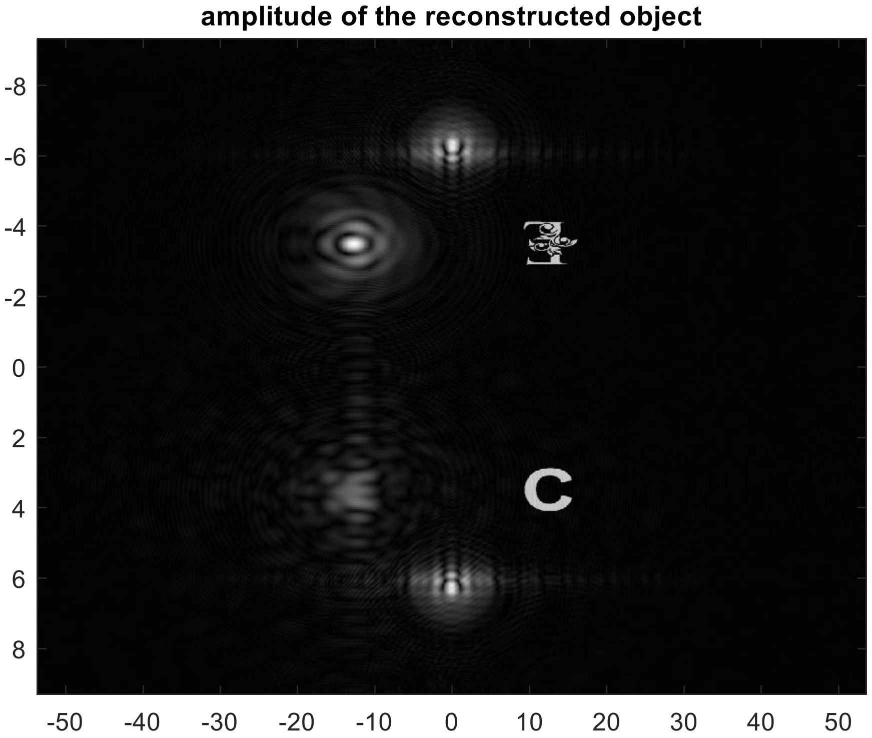


Reconstructed object's phase from IEEE paper

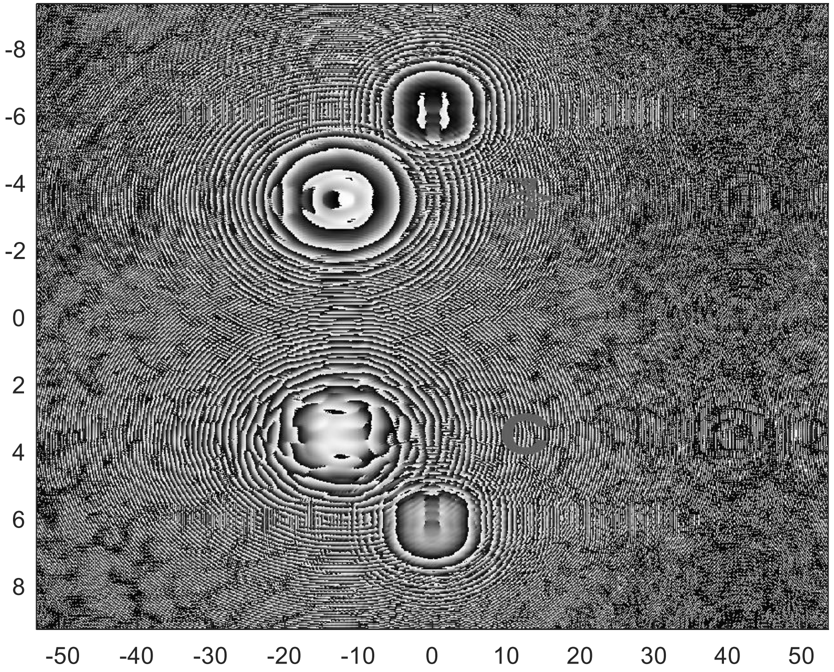
Results at $(\alpha^2 - \alpha^{-2})^{-1} \times z_{\text{object}}$ plane, $z_{\text{object}} = 10\text{mm}$



One of the CCD images



Reconstructed object's amplitude from IEEE paper



Reconstructed object's phase from IEEE paper

References

- (1) D. N. Naik, T. Ezawa, R. K. Singh, Y. Miyamoto, and M. Takeda, “Coherence holography by achromatic 3-D field correlation of generic thermal light with an imaging Sagnac shearing interferometer”, 2012 Optical Society of America
- (2) P. H¨andel, Senior Member, IEEE , “Properties of the IEEE-STD-1057 four parameter sine wave fit algorithm”, IEEE January 2001
- (3) D. N. Naik, T. Ezawa, Y. Miyamoto, and M. Takeda, “Real-time coherence holography”, 2010 Optical Society of America