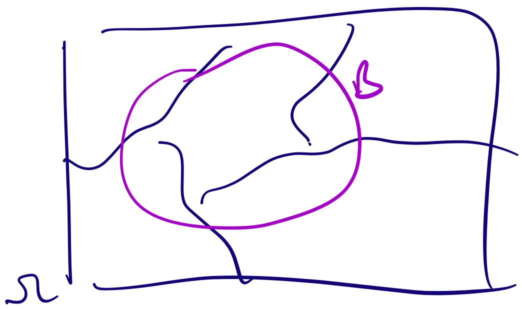
Law of Total Probability

$\Omega = \bigcup_{i=1}^{n} A_i$, $\{A_i\}_{i=1}^{n}$ partitions

so $A_i \cap A_j = \phi$

for all $i \neq j$



$$P(B) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i)$$

(1) H_1 = flipping heads in 1st flip

F = fair coin

$$P(H_1) = P(H_1 \mid F) P(F) + P(H_1 \mid \bar{F}) P(\bar{F})$$

$$= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}$$

Suppose we have 3 fair coins

$$P(H_2 \mid H_1) = P(H_2) = 1/2$$

↳ Obs. H_1 will not change the prob. of observing H_2

→ H_1 & H_2 are statistically independent (s.i)

chain rules.

① $P(A \cap B) = P(A \mid B)\, P(B)$

② $P(A \cap B) = P(B \mid A)\, P(A)$

[Remember: $P(A \cap B) = P(B \cap A)$]

Let $A, B \in \mathcal{F}$:

If A is s.i. of B: $P(A \mid B) = P(A)$

$$P(A \cap B) = P(A \mid B)\, P(B) = P(A)\, P(B)$$

$$\boxed{\text{If two events } A, B \text{ are s.i.: } P(A \cap B) = P(A)\, P(B)}$$

If A is s.i. of $B \Rightarrow$ is B s.i. of A?

- If A is s.i. of B: $P(A|B) = P(A)$
- Using definition of conditional probability:

$$P(B|A) \overset{\text{by def.}}{=} \frac{P(B \cap A)}{P(A)}$$

$$\overset{\text{chain rule}}{=} \frac{P(A|B)\,P(B)}{P(A)}$$

$$\overset{A \text{ is s.i. of } B}{=} \frac{P(A)\,P(B)}{P(A)}$$

$$= P(B)$$

$\boxed{\Rightarrow \text{ If } A \text{ is s.i. of } B, \text{ then } B \text{ is s.i. of } A}$

If A and B are s.i. events:

1. A and $\bar{B}$ are also s.i.
2. $\bar{A}$ and B are also s.i.
3. $\bar{A}$ and $\bar{B}$ are also s.i.

② A fair 6-sided die. Roll it twice. Prob. of obs. a 1 or 2 on the top face on either roll.

$E_i \equiv$ obs. a 1 or 2 on top face of roll i.

$$P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

E_1 & E_2 s.i.

$$= P(E_1) + P(E_2) - P(E_1)P(E_2)$$

$$= \frac{2}{6} + \frac{2}{6} - \frac{2}{6} \cdot \frac{2}{6}$$

② Take 2

$$P(E_1 \cup E_2) = 1 - P(\overline{\overline{E_1 \cup E_2}})$$

De Morgan's Law

$\overline{E_1 \cup E_2} = \bar{E_1} \cap \bar{E_2}$

$$= 1 - P(\bar{E_1} \cap \bar{E_2})$$

if E_1 & E_2 are s.i., then $\bar{E_1}$ & $\bar{E_2}$ are s.i.

$$= 1 - P(\bar{E_1})P(\bar{E_2})$$

$$= 1 - \frac{4}{6} \cdot \frac{4}{6} = \frac{5}{9}$$

Conditionally Independent Events

$A, B, C \in \mathcal{F}$

Given some event C, A and B said to be conditionally independent ($\not\equiv$ s.i.) if:

$$P((A \cap B) \mid C) = P(A \mid C)\, P(B \mid C)$$

If A and B are s.i. $\Longrightarrow$ $\not\Longleftarrow$ A and B are conditionally independent for any other event C in sample space

Example (3):

Magician has 2 coins $<$ fair, 2-headed

$H_i \equiv$ obs. heads in flip i.

$F \equiv$ fair coin

1. $P(H_1 \cap H_2) = P(H_1 \cap H_2 \mid F)\,P(F) + P(H_1 \cap H_2 \mid \bar{F})\,P(\bar{F})$

Law of total prob.

$= P(H_1 \mid F)\,P(H_2 \mid F)\,P(F) + P(H_1 \mid \bar{F})\,P(H_2 \mid \bar{F})\,P(\bar{F})$

when conditioned on fair coin, obs. H_1 is independent of H_2!
$\Rightarrow H_1$ & H_2 are conditionally independent.

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot 1 \cdot \frac{1}{2} = \frac{5}{8}$

2. Are H_1 & H_2 s.i.?

If they are, then we want to show that:

$P(H_1 \cap H_2) = P(H_1)\,P(H_2)$

$P(H_1) = P(H_1 \mid F)\,P(F) + P(H_1 \mid \bar{F})\,P(\bar{F})$

$= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}$

$P(H_2) = \frac{3}{4}$

$P(H_1)P(H_2) = \frac{9}{16} \neq \frac{5}{8} = P(H_1 \cap H_2)$

$\Rightarrow$ not s.i. even though they are conditionally indep.

(do by themselves)
3. $P(H_1 \cap H_2 \cap H_3)$

$$= P((H_1 \cap H_2 \cap H_3) \mid F)\, P(F) + P(H_1 \cap H_2 \cap H_3 \mid \bar{F})\, P(\bar{F})$$

Law of total prob.

$$= P(H_1 \mid F)\, P(H_2 \mid F)\, P(H_3 \mid F)\, P(F) + P(H_1 \mid \bar{F})\, P(H_2 \mid \bar{F})\, P(H_3 \mid F)\, P(\bar{F})$$

$$= \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + (1)^3 \cdot \frac{1}{2}$$

$$= \frac{9}{16}$$

4. $$P(H_3 \mid H_1 \cap H_2) = \frac{P(H_3 \cap H_2 \cap H_1)}{P(H_1 \cap H_2)}$$

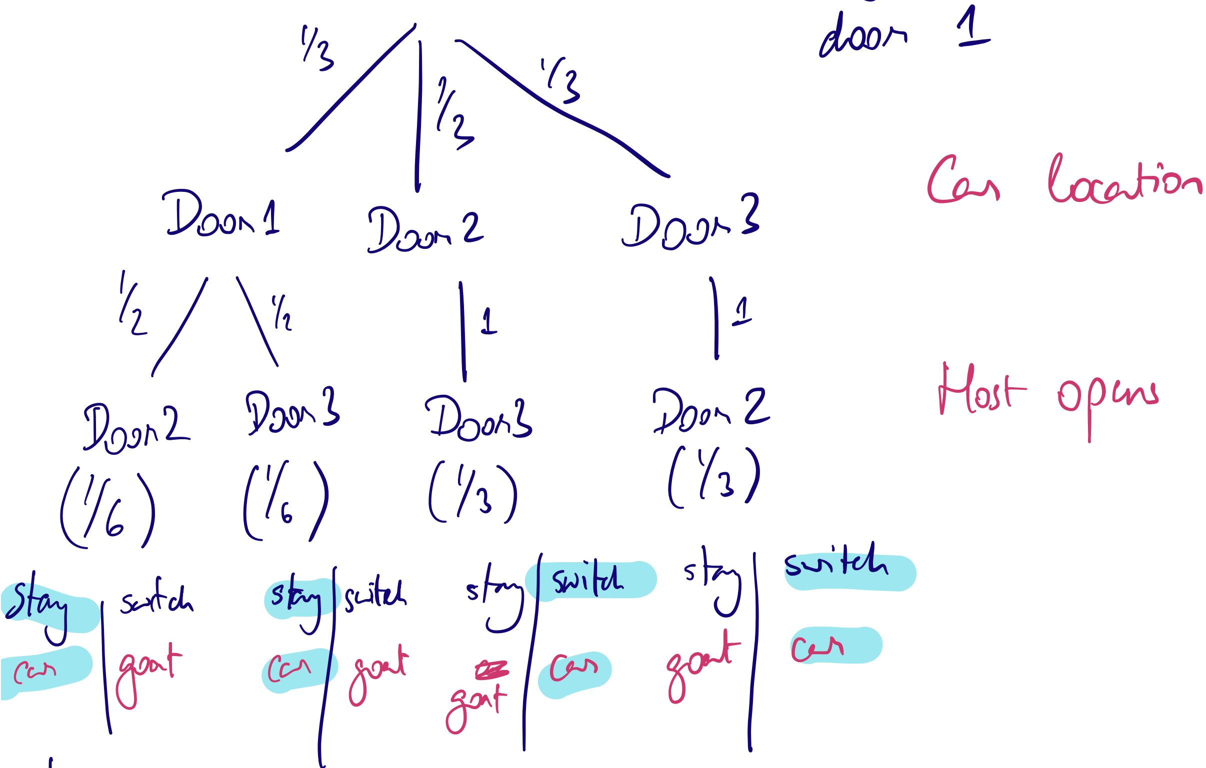
$$= \frac{P(H_1 \cap H_2 \cap H_3)}{P(H_1 \cap H_2)}$$

$$= \frac{9/16}{5/8} = 9/10$$

Example 4

Monty Hall Problem

w.l.o.g., Say participant ~~opens~~ picks door 1



stay ⇒ $1/3$ ⇒ car

switch ⇒ $2/3$ ⇒ car

Host is revealing info when he reveals the goat because he cannot choose your door & he cannot choose the door with the car.

$$\left.\begin{array}{l} H \to \text{switch} \\ T \to \text{stay} \end{array}\right\} \quad \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}$$