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1. Question 1

   (a) $f(x) = -3x^2 + 7$
   $f(-1) = 4$
   $f(1) = 4$

   Obviously, $1 \in \mathbb{R} \wedge -1 \in \mathbb{R}$, so both are within the domain of f. This goes against the definition of a bijection, since two different elements in the domain of f have the same image. In order to rectify this, the domain of f should be $\{x \in \mathbb{R} : x \geq 0\}$. The range would also have to be modified to be $\{y \in \mathbb{R} : y \geq 7\}$, since this function will never be less than 7.

   Inverse: $x = -3y^2 + 7$
   $\frac{7-x}{3} = y^2$
   $y = \sqrt{\frac{7-x}{3}}$
   $f^{-1}(x) = \sqrt{\frac{7-x}{3}}$
   This is, of course, given the modified domain and range.

   (b) $f(x) = \frac{x+2}{x+2}$
   $f(-2) = DNE$

   In order for f to be a bijection, every element in its domain must have an image in its range. Since $f(-2)$ is undefined, the conditions are not satisfied. Here, the domain of f could be modified to be $\{x \in \mathbb{R} : x \neq -2\}$. The range could be modified to be $\{y \in \mathbb{R} : y \neq 1\}$, since this function, by definition, can never be equal to 1.

   Inverse: $x = \frac{y+1}{y+2}$
   $x * (y+2) = y + 1$
   $xy + 2x = y + 1$
   $2x - 1 = y - xy$
   $2x - 1 = y(1-x)$
   $f^{-1}(x) = \frac{2x-1}{1-x}$

   (c) $f(x) = x^5 + 1$
   This function is a bijection, since every element in the domain has exactly one unique image.
   Inverse: $x = y^5 + 1$
   $x - 1 = y^5$
   $f^{-1}(x) = \sqrt[5]{x-1}\ \{y \in \mathbb{R}\}\ \{x \in \mathbb{R}\}$

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2. Question 2

   $f(x) = ax + b$

   $g(x) = cx + d$

   $\{a, b, c, d \in \mathbb{R}\}$

   $f \circ g = a(cx + d) + b$

   $g \circ f = c(ax + b) + d$

   $a(cx + d) + b = c(ax + b) + d$

   $acx + ad + b = cax + cb + d$

   $ad + b = cb + d$

   $(f \circ g = g \circ f) \leftrightarrow (ad + b = cb + d)$

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3. Question 3

Proof by Cases:

In each case, $x = n + q$

Case 1: $0 \leq q < \frac{1}{3}$

$$\begin{array}{ll} 3x = 3n + 3q & \\ \lfloor 3x \rfloor = 3n & \text{Because } 0 \leq 3n < 1 \\ \left\lfloor x + \frac{1}{3} \right\rfloor = n & x + \frac{1}{3} = n + \frac{1}{3} + q \text{ and } 0 \leq \frac{1}{3} + q < 1 \\ \left\lfloor x + \frac{2}{3} \right\rfloor = n & x + \frac{2}{3} = n + \frac{2}{3} + q \text{ and } 0 \leq \frac{1}{3} + q < 1 \\ \lfloor x \rfloor = n & \end{array}$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + n = 3n = \lfloor 3x \rfloor$

Case 2: $\frac{1}{3} \leq q < \frac{2}{3}$

$$\begin{array}{ll} 3x = 3n + 3q & 3x = (3n + 1) + (3q - 1) \\ \lfloor 3x \rfloor = 3n + 1 & 0 \leq 3q - 1 < 1 \\ \left\lfloor x + \frac{1}{3} \right\rfloor = n & x + \frac{1}{3} = n + \frac{1}{3} + q \text{ and } 0 \leq q - \frac{1}{3} < 1 \\ \left\lfloor x + \frac{2}{3} \right\rfloor = \left\lfloor n + \frac{2}{3} + q \right\rfloor & \\ \left\lfloor x + \frac{2}{3} \right\rfloor = n + 1 & \end{array}$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + n + 1 = 3n + 1 = \lfloor 3x \rfloor$

Case 3: $\frac{2}{3} \leq q < 1$

$$\begin{array}{ll} 3x = 3n + 3q & 3x = (3n + 2) + (3q - 2) \\ \lfloor 3x \rfloor = 3n + 2 & 0 \leq 3q - 2 < 1 \\ \left\lfloor x + \frac{1}{3} \right\rfloor = n + 1 & x + \frac{1}{3} = n + 1 + (q - \frac{2}{3}) \text{ and } 0 \leq q - \frac{2}{3} < 1 \\ \left\lfloor x + \frac{2}{3} \right\rfloor = n + 2 & x + \frac{2}{3} = n + 1 + (q - \frac{1}{3}) \text{ and } 0 \leq q - \frac{2}{3} < 1 \end{array}$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + 1 + n + 1 = 3n + 2 = \lfloor 3x \rfloor$

Since all three cases satisfy the initial statement, it is correct.

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4. Question 4

   (a) $a_n = n^5$
   $a_0 = 0$
   $a_1 = 1$
   $a_2 = 32$
   $a_3 = 243$

   (b) $a_n = n^2 + n$
   $a_0 = 0$
   $a_1 = 2$
   $a_2 = 6$
   $a_3 = 12$
   $a_4 = 20$
   As this sequence continues, it is clear that the difference between each term increases by two each time. The recurrence relation can then be seen to be

   $$a_n = a_{n-1} + 2^n$$

   with the initial condition being that $a_0 = 0$

   (c) $a_n = n + (-1)^n$
   $a_0 = 1$
   $a_1 = 0$
   $a_2 = 3$
   $a_3 = 2$
   $a_4 = 5$
   $a_5 = 4$
   This sequence simply switches each pair of items. Every even number becomes the next odd number, and every odd number becomes the previous even number. The recurrence relation is then very straightforward.

   $$a_n = a_{n-2} + 2$$

   With the initial conditions being $a_0 = 1, a_1 = 0$

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5. Question 5

   (a) $P(2) := 2! < 2^2$

   (b) $2 * 1 < 2 * 2$
   $2 < 4$

   (c) $\exists k \in \mathbb{Z}_{\geq 0} : P(k)$

   (d) $P(k) \rightarrow P(k+1)$

   (e) $P(k+1) := (k+1)! < (k+1)^{(k+1)}$
   Assume $P(k)$ is true.
   $(k+1)! = k! * (k+1)$
   $(k+1)! < k^k * (k+1)$
   This is true by definition of the inductive hypothesis.
   $k^k < (k+1)^k$
   Due to the inequality, we can substitute in this term to the previous relationship of $(k+1)!$
   $(k+1)! < (k+1)^k * (k+1)$
   $(k+1)! < (k+1)^{(k+1)}$

   (f) This inequality is true only when $n > 1$, because the base case would not be true for $n = 1$. If the base case does not hold, then the inductive reasoning cannot be used to prove that it is true $\forall n$. However, since $P(k) \rightarrow P(k+1)$, and we know that $P(2)$ holds, then we know that $P(2) \rightarrow P(3) \rightarrow P(4)$ and so on.

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6. Question 6

   Proof by induction

   $P(n) := 3|n^3 + 2n$, when n is a positive integer.

   Base case: $2^3 + 2 * 2 = 12$

   $3|12$

   Inductive Hypothesis: $P(k) := 3|k^3 + 2k$, when k is a positive integer.

   Prove $P(k) \to P(k+2)$, since k must be a positive integer

   $(k+2)^3 + 2(k+2) = (k^3 + 6k^2 + 12k + 8) + (2k + 4)$

   $= (k^3 + 2k) + (6k^2 + 12k + 12)$

   $= (k^3 + 2k) + 3(2k^2 + 4k + 4)$

   $3|k^3 + 2k$ by the inductive hypothesis

   $3|3(2k^2 + 4k + 4)$ because by definition, $n|an$

   $\therefore 3|(k+1)^3 + 2(k+1)$, proving the initial statement.