Alexander Garcia
6 April 2017

1. Question 1

   (a) Let $\mathbb{S}$ be the set of odd multiples of 3
   Basis step: $3 \in \mathbb{S}$
   Recursive step: If $x \in \mathbb{S} \wedge y \in \mathbb{S}$
   $x + 2y \in \mathbb{S}$
   $x - 2y \in \mathbb{S}$
   Because $x, y \in \mathbb{S}$, both are known to be odd. It is also known that $even * odd = even$, and that $even + odd = odd$. Each of the terms is also a multiple of 3, and multiplying that value by an integer does not change that fact. Therefore, an odd multiple of 3 $\pm$ an even multiple of 3 generates a new odd multiple of 3.

   (b) Let $\mathbb{S}$ be the set of bit strings with an even number of zeros
   Note: This question is done assuming that a string of zero length has an even number of zeros.
   Basis Step: $\emptyset \in \mathbb{S}$
   Recursive Step: $x \in \mathbb{S}$
   $1x \in \mathbb{S}$
   $x1 \in \mathbb{S}$
   $00x, 0x0, x00 \in \mathbb{S}$
   Adding a 1 to the string will not affect the number of zeros, so the string is still valid. Adding two zeros in any location will then maintain an even number of zeros.

   (c) Let $\mathbb{S}$ be the set of strings of even length from the alphabet $\Sigma = \{a, b\}$
   Basis Step: $\emptyset \in \mathbb{S}$
   Recursive Step: $X \in \mathbb{S}$
   $Xaa, Xab, Xba, Xbb \in \mathbb{S}$

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2. Question 2

   (a) The maximum number of members is the product of the possible choices for each character in the membership number.

   Total capital letters: 26

   Total number of length 5 bit strings: $2^5 = 32$

   Total number of 2 digit numers where the second digit is less than the first:

   $(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 45$

   $26 * 32 * 45 = 37440$ possible unique member numbers

   (b) In this case, the lowest possible number of shared first letters would be $\frac{1000}{26}$. This is because all numbers must start with a capital letter. Any one letter not being shared means the rest must be shared by more people. So, the lowest number for any given letter would be in an even distribution, which is represented by $\frac{1000}{26}$. This means that there are at least 38 people with the same first letter in a group of 1000.

   (c) The total number of possible numbers that don't start with "J" is

   $25 * 32 * 45 = 36000$

   This answer means that there are more than enough possible numbers for 2000 members that do not begin with "J". So, in a group of 2000, there are at least 0 membership numbers that begin wtih "J".

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3. Question 3

   Take an arbitrary vertex V_1

   Because the graph is complete, V_1 shares an edge (i.e. has some relationship with) all of the eight other vertices.

   It is given that in any complete, two color, 9 vertex graph, there is at least one node incident to 6 red edges, or 4 blue edges. We will then assume that this is the case for V_1

   We then consider the set of 6 friends of V_1, from the previous statement. This "sub graph" can be treated as its own 6 vertex party, which had been covered previously.

   For a 6 vertex graph, it is known that there exists a 3-clique of either friends or enemies.

   If there is a 3 clique of friends, and all members of the clique are also friends with V_1, then we have found a 4-clique of friends (red edges)

   If there is a 3 clique of enemies, then we have found the 3-clique of enemies we were looking for in the beginning.

   $\therefore$ Any complete 2-colored graph with 9 vertices contains either a red 4-clique or a blue 3-clique

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4. Question 4

   (a) A 10 bit string with fewer ones than zeros has either 6, 7, 8, 9, or 10 zeros.
   The number of bit strings with exactly 6 zeros is $C(10,6) = \frac{10!}{6!(10-6)!} = 210$

   With 7 zeros: $C(10,7) = \frac{10!}{7!(10-7)!} = 120$

   With 8 zeros: $C(10,8) = \frac{10!}{8!(10-8)!} = 45$

   Wtih 9 zeros: $C(10,9) = \frac{10!}{9!(10-9)!} = 10$

   Wtih 10 zeros: $C(10,10) = 1$

   The sum of these is the total number of bit strings of length 10 with fewer ones than zeros

   $210 + 120 + 45 + 10 + 1 = 386$

   (b) We can determine the number of length 10 bit strings with at least 5 consecutive ones or zeros by treating each group of 5 as a single item. This changes the number of positions from 10 to 6.

   When the 5 bit sequence starts at the beginning of the string, there are 2^5 possible strings. However, because the rest of the string could contain more zeros, we must avoid counting some combinations twice.

   This means that the sequence being in positions 2-6 only have 2^4 new possible strings associated with them.

   The same principle applies for a sequence of ones. The only remaining overlap is for the strings 0000011111 and 1111100000, as they are counted by both the sequences of 5 ones and 5 zeros.

   Therefore, the total number of length 10 bit strings is $2 * (2^5 + 5 * 2^4) - 2 = 222$

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5. Question 5

   (a) Treat the problem as $w + x + y + z + u = 100$

   The number of solutions to this problem is the same as the number of ways one can select n_1 of w, n_2 of x, n_3, of y, n_4 of z, and n_5 of u, where $n_1 + n_2 + n_3 + n_4 + n_5 = 100$

   This is equal to the number of 100 combinations from a set of 5 elements.

   $\binom{5+100-1}{100} = \binom{104}{100}$

   According to Theorem 2 from the text, $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$

   So, the number of solutions for this equation is $\binom{104}{4} = \frac{104!}{4!(100!)} = 4598126$

   (b) Again, treat the problem as $w + x + y + z + u = 100$

   This time, $x > 1$ or $y > 1$

   In order to solve, find the number of solutions that result from $x, y = 0$, and the number that result $x, y = 1$.

   Using the "stars and bars" method for finding these combinations, the number of remaining variables is 4 (w, y, z, u) or (w, x, z, u), depending on which is chosen.

   If x or y is zero, the remaining 4 variables must sum to the original 100. This makes the number of options $\binom{103}{3} = 176851$.

   If x or y is one, the remaining 4 variables must sum to 99. This makes the number of options $\binom{102}{3} = 171700$

   The original number of possible solutions to this problem was 4598126. This minus the sum of the two "illegal" combinations, is the number of solutions to the equation where $x > 1$ or $y > 1$.

   $4598126 - (176851 + 171700) = 4249575$ possible solutions.

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6. Question 6

   (a) The number of different words formed by the letters in "PEPPERONI" is the same as placing distinguishable objects in distinguishable boxes.
   The formula for this is that for objects in k boxes is $\frac{k!}{n_1! * n_2! * n_3! * \cdots * n_k!}$, where n_i is the number of occurrances of that object (in this case, how many times the letter is repeated).
   $\frac{9!}{3!*2!*1!*1!*1!*1!} = 30250$

   (b) When the word is required to begin and end with P, this leaves 7 boxes, and only 1 remaining use of P.
   $\frac{7!}{2!*1!*1!*1!*1!*1!} = 2520$

   (c) When all three P's are consecutive, we can treat the group as one object. This leaves us with 7 boxes, and only 6 objects, E being the only one repeated.
   $\frac{7!}{2!*1!*1!*1!*1!} = 2520$