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1. Question 1

   (a) $f(x) = -3x^2 + 7$
   $f(-1) = 4$
   $f(1) = 4$

   Obviously, $1 \in \mathbb{R} \wedge -1 \in \mathbb{R}$, so both are within the domain of f. This goes against the definition of a bijection, since two different elements in the domain of f have the same image. In order to rectify this, the domain of f should be $\{x \in \mathbb{R} : x \geq 0\}$. The range would also have to be modified to be $\{y \in \mathbb{R} : y \geq 7\}$, since this function will never be less than 7.

   Inverse: $x = -3y^2 + 7$
   $\frac{7-x}{3} = y^2$
   $y = \sqrt{\frac{7-x}{3}}$
   $f^{-1}(x) = \sqrt{\frac{7-x}{3}}$
   This is, of course, given the modified domain and range.

   (b) $f(x) = \frac{x+2}{x+2}$
   $f(-2) = DNE$

   In order for f to be a bijection, every element in its domain must have an image in its range. Since $f(-2)$ is undefined, the conditions are not satisfied. Here, the domain of f could be modified to be $\{x \in \mathbb{R} : x \neq -2\}$. The range could be modified to be $\{y \in \mathbb{R} : y \neq 1\}$, since this function, by definition, can never be equal to 1.

   Inverse: $x = \frac{y+1}{y+2}$
   $x * (y + 2) = y + 1$
   $xy + 2x = y + 1$
   $2x - 1 = y - xy$
   $2x - 1 = y(1 - x)$
   $f^{-1}(x) = \frac{2x-1}{1-x}$

   (c) $f(x) = x^5 + 1$
   This function is a bijection, since every element in the domain has exactly one unique image.
   Inverse: $x = y^5 + 1$
   $x - 1 = y^5$
   $f^{-1}(x) = \sqrt[5]{x-1}\ \{y \in \mathbb{R}\}\ \{x \in \mathbb{R}\}$

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2. Question 2

   $f(x) = ax + b$

   $g(x) = cx + d$

   $\{a, b, c, d \in \mathbb{R}\}$

   $f \circ g = a(cx + d) + b$

   $g \circ f = c(ax + b) + d$

   $a(cx + d) + b = c(ax + b) + d$

   $acx + ad + b = cax + cb + d$

   $ad + b = cb + d$

   $(f \circ g = g \circ f) \leftrightarrow (ad + b = cb + d)$

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3. Question 3

Proof by Cases:

In each case, $x = n + q$

Case 1: $0 \leq q < \frac{1}{3}$

$$
\begin{array}{ll}
3x = 3n + 3q & \\
\lfloor 3x \rfloor = 3n & \text{Because } 0 \leq 3n < 1 \\
\left\lfloor x + \frac{1}{3} \right\rfloor = n & x + \frac{1}{3} = n + \frac{1}{3} + q \text{ and } 0 \leq \frac{1}{3} + q < 1 \\
\left\lfloor x + \frac{2}{3} \right\rfloor = n & x + \frac{2}{3} = n + \frac{2}{3} + q \text{ and } 0 \leq \frac{1}{3} + q < 1 \\
\lfloor x \rfloor = n &
\end{array}
$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + n = 3n = \lfloor 3x \rfloor$

Case 2: $\frac{1}{3} \leq q < \frac{2}{3}$

$$
\begin{array}{ll}
3x = 3n + 3q & 3x = (3n + 1) + (3q - 1) \\
\lfloor 3x \rfloor = 3n + 1 & 0 \leq 3q - 1 < 1 \\
\left\lfloor x + \frac{1}{3} \right\rfloor = n & x + \frac{1}{3} = n + \frac{1}{3} + q \text{ and } 0 \leq q - \frac{1}{3} < 1 \\
\left\lfloor x + \frac{2}{3} \right\rfloor = \left\lfloor n + \frac{2}{3} + q \right\rfloor & \\
\left\lfloor x + \frac{2}{3} \right\rfloor = n + 1 &
\end{array}
$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + n + 1 = 3n + 1 = \lfloor 3x \rfloor$

Case 3: $\frac{2}{3} \leq q < 1$

$$
\begin{array}{ll}
3x = 3n + 3q & 3x = (3n + 2) + (3q - 2) \\
\lfloor 3x \rfloor = 3n + 2 & 0 \leq 3q - 2 < 1 \\
\left\lfloor x + \frac{1}{3} \right\rfloor = n + 1 & x + \frac{1}{3} = n + 1 + (q - \frac{2}{3}) \text{ and } 0 \leq q - \frac{2}{3} < 1 \\
\left\lfloor x + \frac{2}{3} \right\rfloor = n + 2 & x + \frac{2}{3} = n + 1 + (q - \frac{1}{3}) \text{ and } 0 \leq q - \frac{2}{3} < 1
\end{array}
$$

$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + 1 + n + 1 = 3n + 2 = \lfloor 3x \rfloor$