**Alexander Garcia**
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1. Question 1

    (a) $\sum_{i=1}^{n}(2i-1)$
    $n=1 \quad \sum_{i=1}^{n}(2i-1)=1$
    $n=2 \quad \sum_{i=1}^{n}(2i-1)=1+3=4$
    $n=3 \quad \sum_{i=1}^{n}(2i-1)=4+5=9$
    $n=4 \quad \sum_{i=1}^{n}(2i-1)=9+7=16$
    A possible formula for this summation would be $f(n)=n^2, n \geq 1$

    (b) Proof by mathematical induction
    $n^2=\sum_{i=1}^{n}(2i-1)$
    Base case: $1^2=\sum_{i=1}^{1}(2i-1)$
    $1=2-1$

    Assume:
    $n^2=\sum_{i=1}^{n}(2i-1)$
    $(n+1)^2=\sum_{i=1}^{n+1}(2i-1)$
    $\sum_{i=1}^{n+1}(2i-1)=\sum_{i=1}^{n}(2i-1)+2(n+1)-1$
    $(n+1)^2=n^2+2n+1$
    $n^2+2n+1=\sum_{i=1}^{n}(2i-1)+2n+1$

    Subtract $(2n+1)$ from each side
    $n^2=\sum_{i=1}^{n}(2i-1)$

    It is assumed that $n^2=\sum_{i=1}^{n}(2i-1)$ from the inductive step. Therefore, the formula is correct.

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2. Question 2

Proof by mathematical induction

Base Case: $P(3) := \frac{3(2)(1)}{6} = 1$

This is true, since a set of size 3 can only have one subset of size 3.

Inductive Step: Assume a set $S$ of size $n$ has $\frac{n(n-1)(n-2)}{6}$ subsets of size 3. Also it is given that $S$ has $\frac{n(n-1)}{2}$ subsets of size 2.

$R = S + \{x_{n+1}\}$

$|R| = n + 1$

$S \subset R$

By adding the new element $\{x_{n+1}\}$, $R$ now has a certain number of subsets of size 3 that contain $\{x_{n+1}\}$, and a certain number that do not.

The number of subsets of size 3 without the new element is simply the number of size 3 subsets in $S$.

$S$ has $\frac{n(n-1)}{2}$ subsets of size 2, as given by the problem.

The number of subsets of size 3 that include the new element is also $\frac{n(n-1)}{2}$. This is due to the fact that $S$ and $R$ are identical, save the new element. By adding the new element to every size 2 subset in $S$, one can generate every size 3 subset containing $\{x_{n+1}\}$ in $R$.

Thus, the number of size 3 subsets in $R$ is $\frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} = \frac{n(n+1)(n-1)}{6}$

$P(n+1) := \frac{n(n+1)(n-1)}{6}$

$\therefore P(n) \rightarrow P(n+1)$

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3. Question 3

Proof by Strong Induction

Base Cases:

$n = 1 : 1 \leq 2$

$n = 2 : 2 \leq 4$

$n = 3 : 3 \leq 8$

Inductive Hypothesis:

$P(n) := a_{n-1} + a_{n-2} + a_{n-3} \leq 2^n$

$a_{n+1} = a_n + a_{n-1} + a_{n-2}$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Substitute in $a_n$

$a_{n+1} = 2a_{n-1} + 2a_{n-2} + a_{n-3}$

$2a_{n-1} + 2a_{n-2} + a_{n-3} < 2(a_{n-1} + a_{n-2} + a_{n-3})$

$2^{n+1} = 2^n * 2$

We just showed that $a_{n+1} < 2 * a_n$

$2 * a_n \leq 2 * 2^n$, since by the inductive hypothesis, $a_n \leq 2^n$

So

$a_{n+1} < 2 * a_n \leq 2 * 2^n$

Simplifying this down,

$a_{n+1} \leq 2^{n+1} = P(n+1)$

$\therefore P(n) \to P(n+1)$

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4. Question 4

   Proof by Strong Induction

   Base Case:

   $20 = 5r + 6s$

   $r = 4, s = 0$

   Inductive Hypothesis:

   $\forall n \in \mathbb{Z}_{>20}\ n = 5r + 6s$

   $21 = 5(3) + 6(1)$

   $22 = 5(2) + 6(2)$

   $23 = 5(1) + 6(3)$

   $24 = 5(0) + 6(4)$

   We now let $r_n$ represent the $r$ coefficient used in the construction of $n$, and $b_n$ represent the $b$ coefficient of $n$

   $25 = 5(5) + 6(0)$

   $25 = 5(r_{20} + 1) + 6(b_{20})$

   $n = 5(r_n) + 6(b_n)$

   $n + 5 = 5(r_n + 1) + 6(b_n)$

   By increasing a number by 5, the only change that needs to be made to its $r$ value is to increase it by 1. The $b$ value can remain the same. Therefore, $P(n) \to P(n + 5)$

   By proving the first 4 base cases, we then have proven $P(n)\forall n \geq 20$.

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5. Question 5

   (a) Basis Step:
   For $T$ of size 1, $H(T) = 0$
   Recursive Step:
   For $T = T_1 \bullet T_2 \bullet T_3$,
   $H(T) = max(H(T_1), H(T_2), H(T_3)) + 1$

   (b) Basis Step:
   For $T$ that is a single vertex, $N(T) = 1$
   Recursive Step:
   For $T = T_1 \bullet T_2 \bullet T_3$,
   $N(T) = 1 + N(T_1) + N(T_2) + N(T_3)$

   (c) Base Case:
   For $T$ of one vertex $r$
   $N(T) = 1$
   $H(T) = 0$
   $3^{H(T)+1} - 1 = 2$
   $1 < 2$

   Inductive Hypothesis:
   For a full ternary tree $T$,
   $N(T) \leq 3^{H(T)+1} - 1$

   Recursive Step:
   $T = T_1 \bullet T_2 \bullet T_3$
   $N(T) = 1 + N(T_1) + N(T_2) + N(T_3)$
   $N(T) \leq 1 + (3^{H(T_1)+1} - 1) + (3^{H(T_2)+1} - 1) + (3^{H(T_3)+1} - 1)$, by the inducive hypothesis
   $N(T) \leq 3 * max(3^{H(T_1)+1}, 3^{H(T_2)+1}, 3^{H(T_3)+1}) - 2$

   This is true since the sum of these three terms cannot be more than $3*$ the largest term.

   $max(3^{H(T_1)+1}, 3^{H(T_2)+1}, 3^{H(T_3)+1}) = 3^{max(H(T_1), H(T_2), H(T_3))+1}$
   $3^{max(H(T_1), H(T_2), H(T_3))+1} = H(T)$
   Substituting this into the earlier equation gives us
   $N(T) \leq 3 * 3^{H(T)} - 2$
   $N(T) \leq 3^{H(T)+1} - 2 < 3^{H(T)+1} - 1$