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Assignment-1

1. Let $f(x) = e^x \cos 2x$.

   (a) Find $T_4(x)$, the degree-4 Taylor polynomial of $f(x)$ centered at $x = 0$.

   The definition of the Taylor polynomial is as follows:

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$$

   In this case, $n = 4$, and $x_0 = 0$.
   We start by finding $f^{(1)}(x), f^{(2)}(x), f^{(3)}(x)$, and $f^{(4)}(x)$

$$f^{(1)}(x) = -2e^x sin(2x) + e^x cos(2x) = e^x(cos(2x) - 2sin(2x))$$
$$f^{(2)}(x) = e^x(-2sin(2x) - 4cos(2x)) + e^x(cos(2x) - 2sin(2x)) = -e^x(4sin(2x) + 3cos(2x))$$
$$f^{(3)}(x) = -e^x(8cos(2x) - 6sin(2x)) - e^x(4sin(2x) + 3cos(2x)) = e^x(2sin(2x) - 11cos(2x))$$
$$f^{(4)}(x) = e^x(4cos(2x) + 2sin(22sin(2x)) + e^x(2sin(2x) - 11cos(2x)) = e^x(24sin(2x) - 7cos(2x))$$

   Evaluating these functions at the center point, we get:

$$f(0) = e^0 cos(0) = 1$$
$$f^{(1)}(0) = e^0(cos(0) - 2sin(0)) = 1$$
$$f^{(2)}(0) = -e^0(4sin(0) + 3cos(0)) = -3$$
$$f^{(3)}(0) = e^0(2sin(0) - 11cos(0)) = -11$$
$$f^{(4)}(0) = -e^0(24sin(0) - 7cos(0) = -7$$

   We now have all the pieces we need to determine the polynomial expansion.

$$P_n(x) = \frac{x^0}{0!}f(0) + \frac{x^1}{1!}f^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \frac{x^4}{4!}f^{(4)}(0)$$
$$= 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3 - \frac{7}{24}x^4$$

   (b) Find the derivative form of the remainder $R_4(x)$ in the expression

$$f(x) - T_4(x) = R_4(x).$$

   By definition, the derivitive form of the remainder is

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

   where c is close to x_0.
   In this case, we have $n = 4, x_0 = 0$ from the previous question.

$R_n(x) = \frac{x^5}{5!} f^{(5)}(c)$
$f^{(5)}(c) = e^x(48cos(2x) + 14sin(2x)) + e^x(24sin(2x) - 7cos(2x))$
$= e^x(41cos(2x) + 38sin(2x))\,]_{\,x=c}$

Therefore,

$$R_4(x) = \frac{x^5}{120} e^c(41cos(2c) + 38sin(2c))$$

(c) Suppose that $f(x)$ is approximated by $T_4(x)$ in the interval $x \in [-\pi/4, \pi/4]$. Find a bound on the absolute error of the approximation.

We know the remainder, $R_4(x)$ from part (c). Given a closed interval, we can find the maximum error by finding the maximum of $f^{(5)}(x)$ in this interval. To do this, we must find its derivative. Its zeroes will show all maxima and minima.
$f^{(6)}(x) = e^x(-82sin(2x) + 76cos(2x)) + e^x(41cos(2x) + 38sin(2x)) = e^x(117cos(2x) - 44sin(2x))$

Now find zeroes.
$e^x(117cos(2x) - 44sin(2x)) = 0$
$117cos(2x) - 44sin(2x) = 0$
$\frac{117}{44} = \frac{sin(2x)}{cos(2x)}$
$x = \frac{1}{2} tan^{-1}(\frac{117}{44}) \approx 0.606$

Our candidate points are then $\{-\frac{\pi}{4}, 0.606, \frac{\pi}{4}\}$

$f(-\frac{\pi}{4}) = e^{\frac{-\pi}{4}} cos(-\frac{\pi}{2}) = 0 = f(\frac{\pi}{4})$
$f(0.606) = e^{0.606} cos(2 * 0.606) \approx 0.645$

As $f(0.606)$ is clearly the max, this is what we use for determining the error bound.

$f^{(5)}(0.606) \approx$

(d) Using Matlab, plot $f(x)$ and $T_4(x)$ on the same graph for $x \in [-\pi/4, \pi/4]$. On a second graph plot the absolute error $|f(x) - T_4(x)|$, again for $x \in [-\pi/4, \pi/4]$. Check if the error does indeed obey the bound you found in part (c) above.

2. (Pencil-and-paper) Convert the binary number $10.\overline{110}$ to base 10. Give the answer both as a decimal and a fraction if you can.

3. (Pencil-and-paper) Express $x = 6.7$ as an IEEE single-precision float $\text{fl}(x)$ using the round-to-nearest rule. Compute the relative error $d = |x - fl(x)|/|x|$ exactly as a base-10 number, and show that d satisfies $d \le (1/2)\epsilon_{mach}$.

4. (Pencil-and-paper and MATLAB) Holmes 1.5(a).

5. (Pencil-and-paper and MATLAB) Holmes 1.12.

6. (Pencil-and-paper, adapted from Holmes Problem 1.16.) Assume single-precision IEEE arithmetic. Assume that the round-to-nearest rule is used with one modification: if there is a tie then the smaller value is picked (this rule for ties is used to make the problem easier).

   (a) For what real numbers x will the computer claim that the inequalities $1 < x < 2$ hold?

   (b) For what real numbers x will the computer claim $x = 4$?

(c) Suppose it is stated that there is a floating point number x^* that is the exact solution of $x^2 - 2 = 0$. Why is this not possible? Also, suppose x_L^* and x_R^* are the floats to the left and right of $\sqrt{2}$ respectively. What is the value of $x_R^* - x_L^*$?

7. (a) A problem is ill-conditioned if its solution is highly sensitive to small changes in the input data. True or False?

   (b) Using higher-precision arithmetic will make an ill-conditioned problem better conditioned. True or False?

   (c) If two real numbers are exactly representable as floating-point numbers on a finite-precision machine, then so is their product. True or False?

   (d) Consider the sum

$$S = \frac{1}{x+1} + \frac{1}{x-1}, \quad x \neq 1.$$

   For what range of values is it difficult to compute S accurately in a finite-precision system? How will you rearrange the terms in S so that the difficulty disappears?

   (e) In a finite-precision system with UFL $= 10^{-40}$, which of the following operations will incur an underflow?

   i. $\sqrt{a^2 + b^2}$, with $a = 1$, $b = 10^{-25}$.
   ii. $\sqrt{a^2 + b^2}$, with $a = b = 10^{-25}$.
   iii. $(a \times b)/(c \times d)$, with $a = 10^{-20}$, $b = 10^{-25}$, $c = 10^{-10}$, $d = 10^{-35}$.