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Assignment-5

1.

$$I = \int_{-1}^{1} e^{-2x} dx$$

(a) Approximation using the composite trapezoidal rule.
The general formula for the composite trapezoidal rule is

$$I(f) = \frac{h}{2}[f(a) + f(h) + 2\sum_{j=1}^{k-1} f(a + jp)]$$

where h is the width of the total interval, p is the node width, and k is the total number of nodes. In this case, $h = p = 0.5$, $k = 4$, and $f = e^{-2x}$.
Then, according to the rule,

$$I(f) = \frac{0.5}{2}[f(-1) + f(1) + 2\sum_{j=1}^{3} f(a + jp)]$$

or

$$I(f) = \frac{1}{4}[e^{-1} + e + 2[e^{-1+0.5} + e^{-1+1} + e^{-1+1.5}]]$$

When the expression is evaluated, we get the composite trapezoidal approximation of the integral to be

$$I(f) \approx 2.399166\ldots$$

We know that the error for this formula can be expressed by $-\frac{h^2}{12}(b-a)f''(\eta)$, where η is an unknown location within the interval. Rather than use an unknown location, we can bound the error by the maximum of the second derivative over the interval.

$$f''(x) = 4e^{-2x} \leq 4e^2 \quad x \in [-1, 1]$$

Therefore, the maximum error of this approximation is

$$\frac{0.5^2}{12}(1 - (-1))(4e^2) \approx 1.231509\ldots$$

which makes our result not very significant.

(b) Approximation using composite Simpson's rule

We can define composite Simpson's rule in this case as the following

$$I(f) = \frac{h}{3}\sum_{j=2}^{8}(f(a+(j-2)h) + 4f(a+(j-1)h) + f(a+jh))$$

In our case, $k = 4$, and $p = \frac{b-a}{k}$, so $h = \frac{p}{2}$. We can write out the full summation then as

$$\frac{h}{3}[\begin{array}{ll} f(-1)+4f(-0.75)+f(-0.5) & + \\ f(-0.5)+4f(-0.25)+f(0) & + \\ f(0)+4f(0.25)+f(0.5) & + \\ f(0.5)+4f(0.75)+f(1) & \end{array}]$$

The overall error of Simpson's rule approximation can be written as

$$E \leq -\frac{b-a}{180}h^4 max(f''(x)) \quad x \in [-1,1]$$

We can easily see that the $max(f''(x)) = 16e^2$ for our interval, so the upper bound on the error is

We can then calculate the upper bound on the error.

$$E \leq -\frac{2}{180}(0.25^4)(16e^2)$$
$$E \leq -0.00513129\ldots$$

This makes composite Simpson's approximation a much better estimate of the integral than composite trapezoidal approximation.

(c) For the composite trapezoidal rule, we are given that the error is defined by

$$|E| \leq \frac{h^2}{12}(b-a)max(f''(x)) < 10^{-6} \quad x \in [a,b]$$

As we are simply solving for the node spacing h, we can rearrange this equation to gain an expression for its value.

$$h < \sqrt{\frac{12\delta}{(b-a)M}}$$

where

$M = max(f''(x))\ x \in [-1,1] = 4e^2$

$\delta = 10^{-6}$, and

$b - a = 2$.

When these values quantities are used, we get

$$h < (\sqrt{\frac{12 * 10^{-6}}{2 * 4e^2}} \approx 0.000450558\ldots)$$

(d) We have a similar expression for the error of Simpson's rule

$$|E| \leq \frac{b-a}{180} h^4 max(f^{(iv)}(x)) < 10^{-6} \quad x \in [a,b]$$

We can again rearrange this equation to solve for h, the minimum node spacing to ensure an error of $\leq 10^{-6}$

$$h < (\frac{180\delta}{(b-a)M})^{1/4}$$

where $M = max(f^{(iv)}(x)) \quad x \in [-1,1] = 16e^2$
$\delta = 10^{-6}$, and
$b - a = 2$.
This inequality gives us a minimum node spacing of

$$h < ((\frac{180 * 10^{-6}}{2 * 16e^2})^{1/4} \approx 0.0295381\ldots)$$

2. Holmes 6.18

(a) $I_S(n) = \frac{2}{3} I_T(n) + \frac{1}{3} I_M(n/2)$
(b) $I_S(n) = \frac{4}{3} I_T(n) - \frac{1}{3} I_T(n/2)$