Alexander Garcia June 28, 2017

Assignment-5

1.

$$I = \int_{-1}^{1} e^{-2x} dx$$

(a) Approximation using the composite trapezoidal rule.
The general formula for the composite trapezoidal rule is

$$I(f) = \frac{h}{2}[f(a) + f(h) + 2\sum_{j=1}^{k-1} f(a + jp)]$$

where h is the width of the total interval, p is the node width, and k is the total number of nodes. In this case, $h = p = 0.5$, $k = 4$, and $f = e^{-2x}$.
Then, according to the rule,

$$I(f) = \frac{0.5}{2}[f(-1) + f(1) + 2\sum_{j=1}^{3} f(a + jp)]$$

or

$$I(f) = \frac{1}{4}[e^{-1} + e + 2[e^{-1+0.5} + e^{-1+1} + e^{-1+1.5}]]$$

When the expression is evaluated, we get the composite trapezoidal approximation of the integral to be

$$I(f) \approx 2.399166\ldots$$

We know that the error for this formula can be expressed by $-\frac{h^2}{12}(b - a)f''(\eta)$, where η is an unknown location within the interval. Rather than use an unknown location, we can bound the error by the maximum of the second derivative over the interval.

$$f''(x) = 4e^{-2x} \leq 4e^2 \quad x \in [-1, 1]$$

Therefore, the maximum error of this approximation is

$$\frac{0.5^2}{12}(1 - (-1))(4e^2) \approx 1.231509\ldots$$

which makes our result not very significant.

(b) Approximation using composite Simpson's rule

We can define composite Simpson's rule in this case as the following

$$I(f) = \frac{h}{3}\sum_{j=2}^{8}(f(a+(j-2)h) + 4f(a+(j-1)h) + f(a+jh))$$

In our case, $k = 4$, and $p = \frac{b-a}{k}$, so $h = \frac{p}{2}$. We can write out the full summation then as

$$\frac{h}{3}\left[\begin{array}{ll} f(-1)+4f(-0.75)+f(-0.5) & + \\ f(-0.5)+4f(-0.25)+f(0) & + \\ f(0)+4f(0.25)+f(0.5) & + \\ f(0.5)+4f(0.75)+f(1) & \end{array}\right]$$

The overall error of Simpson's rule approximation can be written as

$$E \leq -\frac{b-a}{180}h^4 max(f''(x)) \quad x \in [-1,1]$$

We can easily see that the $max(f''(x)) = 16e^2$ for our interval, so the upper bound on the error is

We can then calculate the upper bound on the error.

$$E \leq -\frac{2}{180}(0.25^4)(16e^2)$$
$$E \leq -0.00513129\ldots$$

This makes composite Simpson's approximation a much better estimate of the integral than composite trapezoidal approximation.

(c) For the composite trapezoidal rule, we are given that the error is defined by

$$|E| \leq \frac{h^2}{12}(b-a)max(f''(x)) < 10^{-6} \quad x \in [a,b]$$

As we are simply solving for the node spacing h, we can rearrange this equation to gain an expression for its value.

$$h < \sqrt{\frac{12\delta}{(b-a)M}}$$

where

$M = max(f''(x))\ x \in [-1,1] = 4e^2$

$\delta = 10^{-6}$, and

$b - a = 2$.

When these values quantities are used, we get

$$h < (\sqrt{\frac{12 * 10^{-6}}{2 * 4e^2}} \approx 0.000450558\ldots)$$

(d) We have a similar expression for the error of Simpson's rule

$$|E| \leq \frac{b-a}{180} h^4 max(f^{(iv)}(x)) < 10^{-6} \quad x \in [a,b]$$

We can again rearrange this equation to solve for h, the minimum node spacing to ensure an error of $\leq 10^{-6}$

$$h < (\frac{180\delta}{(b-a)M})^{1/4}$$

where $M = max(f^{(iv)}(x)) \quad x \in [-1,1] = 16e^2$
$\delta = 10^{-6}$, and
$b - a = 2$.
This inequality gives us a minimum node spacing of

$$h < ((\frac{180 * 10^{-6}}{2 * 16e^2})^{1/4} \approx 0.0295381\ldots)$$

2. Holmes 6.18

(a) $I_S(n) = \frac{2}{3} I_T(n) + \frac{1}{3} I_M(n/2)$
The approximation of this integral must take place over an interval $[a,b]$, the size of which determines the node spacing h. For n nodes, $h = \frac{b-a}{n}$, and for $n/2$ nodes, $h = 2\frac{b-a}{n}$. Going forward, we use these values of h for $I_T(n)$ and $I_M(n)$ respectively.

We know the definition of $I_M(n)$ as

$$I_M(n) = h[f(a + h/2) + f(a + h) + f(a + 3h/2) + f(a + 2h) + \ldots + f(a + (n-1)h/2)]$$

Considering the fact that $h = 2\frac{b-a}{n}$ when we take $I_M(n/2)$, we can rewrite this equation as

$$I_M(n/2) = 2\frac{b-a}{n}[f(a + 2\frac{b-a}{n}) + f(a + 4\frac{b-a}{n}) + \ldots]$$

and

$$\frac{1}{3} I_M(n/2) = \frac{2(b-a)}{3n}[f(a + 2\frac{b-a}{n}) + \ldots]$$

We also know the definition of $I_T(n)$

$$I_T(n) = \frac{b-a}{2n}[f(a) + f(b) + 2f(a + \frac{b-a}{n}) + 2f(a + 2\frac{b-a}{n}) + 2f(a + 3\frac{b-a}{n}) + \ldots]$$

and

$$\frac{2}{3} I_T(n) = \frac{b-a}{3n}[f(a) + f(b) + 2f(a + \frac{b-a}{n}) + 2f(a + 2\frac{b-a}{n}) + \ldots]$$

These two sequences share terms that are of the form $f(a+2i\frac{b-a}{n})$, meaning the even terms are seen twice. In $I_T(n)$, there are also multiplied by a factor of 2, meaning that in total, each even term appears 4 times. The odd terms in $I_T(n)$, then, are only found in $I_T(n)$, but are still multiplied by 2, so each odd term appears 2 times. Each end point $f(a), f(b)$ only appears in $I_T(n)$, so each has a weight of 1. When the two sequences are combined, we get

$$\frac{(b-a)}{3n}[f(a)+f(b)+4f(x_{even})+2f(x_{odd})]$$

where $f(x_{odd}) = f(a+odd\frac{b-a}{n})$ and $f(x_{even}) = a+even\frac{b-a}{n}$. This is the definition of composite Simpson's rule.

(b) $I_S(n) = \frac{4}{3}I_T(n) - \frac{1}{3}I_M(n/2)$
A very similar idea is used to demonstrate this alternate definition of composite Simpson's rule.

$$\frac{1}{3}I_M(n/2) = \frac{2(b-a)}{3n}[f(a+2\frac{b-a}{n})+\dots]$$

and

$$\frac{4}{3}I_T(n) = \frac{2(b-a)}{3n}[f(a)+f(b)+2f(a+\frac{b-a}{n})+2f(a+2\frac{b-a}{n})+\dots]$$

Again, once the difference is taken, the definition of composite Simpson's rule appears.

$$\frac{4}{3}I_T(n) - \frac{1}{3}I_M(n/2) = \frac{b-a}{3n}[f(a)+f(b)+4f(x_{even})+2f(x_{odd})]$$

3. $Q_3(f) = \frac{4h}{3}[2f(h) - f(2h) + 2f(3h)] \quad 0 \le x \le 4h$

(a) Given that this approximation uses 3 nodes, and therefore uses three function evaluations to approximate the integral, we can say that $Q_3(f)$ is of precision 3. This means $Q_3(f)$ can approximate integrals exactly for polynomials up to and including degree 3.

(b) We know that in general, the formula for the error in an odd degree Quadrature rule is

$$E_n = Kh^{n+1}p^n(\eta)$$

where h is the node spacing, p^n is a polynomial of degree $n+1$, and η is an unknown location on the interval. In order to find the value of K, we chose the function $f(x) = x^4$, where $f^{(iv)}(x) = 24$. It is known that

$$\int x^4 = \frac{x^5}{5}$$

so

$$\int_0^{4h} x^4 = \frac{(4h)^5}{5}$$

We also know the approximation, as it was given to be

$$\frac{4h}{3}[2f(h) - f(2h) + 2f(3h)]$$

If we expand each side as a Taylor series, we get

$$F(x) = \int f(x) = F(a) + hF'(a) + \frac{h^2}{2}F''(a) + \frac{h^3}{3!}F'''(a) + \dots$$

where a is given as 0. We know that $F(0)$ is zero, since F is a one term polynomial. F can now be rewritten in terms of f, as the previously unknown term $F(a)$ is now gone.

$$F(x) = (x-a)f(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(iv)}(a)$$

What we are looking for is $F(4h)$. We use the Taylor expansion of F to find this value. As we know that $x - a = 4h$, we can make this substitution as well.

$$F(4h) = 4hf(a) + \frac{(4h)^2}{2!}f'(a) + \frac{(4h)^3}{3!}f''(a) + \frac{(4h)^4}{4!}f^{(iv)}(a)$$

Remembering that $F(a)$ was being approximated by $\frac{4h}{3}[2f(h) - f(2h) + 2f(3h)]$, we can expand each f term in the approximation to achieve

$$F^*(h) = \frac{4h}{3}[2f(a) +$$

$$[f(a) + 2hf'(a) + \frac{4h^2}{2!}f''(a) + \frac{8h^3}{3!}f'''(a) + \dots] -$$

$$2[f(a) + 3hf'(a) + \frac{9h^2}{2!}f''(a) + \frac{27h^3}{3!}f'''(a) + \dots]] + E$$

If we simply compare the $f^{(iv)}(x)$ terms in each of the Taylor expansions, we see that the term for F is $\frac{32h^5}{5!}f^{(iv)}(a)$, and the term for F^* is $\frac{4h}{3}[\frac{(2h)^5}{5!} - \frac{(3h)^5}{5!}]f^{(iv)}(a) = -\frac{211h^5}{5!}f^{(iv)}(a) + \dots + E$

When we take the difference between the actual and the approximate terms, we finally see E

$$E = [\frac{32h^5}{5!} + \frac{211h^5}{5!}]f^{(iv)}(a) = \frac{241h^5}{5!} + \dots$$

or, if we want to truncate the sequence by using an unknown a,

$$E = \frac{241h^5}{5!}f^{(iv)}(\eta)$$

4.