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FINANCIAL ECONOMICS

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I - Financial System and Financial Markets

1 Financial System

The financial system is the set of markets and intermediaries used to carry out economic agents' (households, companies, governments) financial decisions.

- **Markets:** Stock markets, Fixed-Income markets, Currency (Foreign Exchange) markets, Options markets, Futures markets, etc.
- **Financial intermediaries**: Banks, insurance companies, mutual funds, pension funds, hedge funds, finance companies, etc...

In the financial system, funds flow from agents (typically households and foreign investors) with surpluses of funds to agents with deficits (typically corporations and governments). The latter sell ("issue") financial claims (shares of stock, bonds, etc...) which are purchased by the former.

1.1 Functions

The role of financial markets is to:
- Transfer resources across time.
- Transfer and manage risks (diversification, insurance, hedging).
- Pool resources to finance large-scale indivisible investments/Financing
- Provide price information.

2 Financial Instruments

2.1 Common Stock

Also known as "stock" or "equity", each share of common stock entitles its holder to:

- An equal share in the ownership of the firm.
- One vote used to elect the board of directors who in turn appoints senior managers.
- A dividend if the management chooses to pay one.

Shareholders are *residual claimants* (that is, they are the last in line to receive any payment in case of bankruptcy) and have a *limited liability* (that is, their potential losses are limited to their initial investment in the stock).

2.2 Debt

Debt instruments are issued by the agents who borrow money. They promise to pay the debtholder fixed sums of money at pre-determined points in time in the future. As such debt instruments are often called *fixed income securities.* They include:

- Corporate bonds and commercial paper.
- Government bills (short-term), notes (medium term) and bonds (long-term).
- Banks loans and Certificates of Deposits (CDs).

In contrast with common stocks, debt instruments do not grant a share in the ownership of a firm.

2.3 Derivative assets

A derivative asset's value depends on (i.e., derives from) the value of another asset. Two of the main classes of derivative products are:

- **Forward or futures contracts**: oblige the buyer (seller) of the contract to buy (sell) one asset at a pre-specified price at some future date.

- **Options.** The holder of an option has the *right* to buy or sell (depending on the option type) an asset at a pre-specified price up to or at a specific date in the future. The seller of the option has the *obligation* to buy or to deliver the asset if the option buyer exercises his/her option.

Equity and debt instruments are issued by corporations to obtain financing. As such they are claims to income generated by the real assets of the firms that sell these securities. In contrast, derivatives are mainly used to manage or/and to transfer risks. For instance, a forward contract allows the parties in the contract to eliminate the uncertainty on the future price of the asset which is underlying the contract. Common stocks, bonds and derivatives can in general be traded on the financial market. For a detailed description of how securities are traded in financial markets, see Chapter 1 in the textbook: Investments, by S. Bodie, A. Kane and A. Marcus. This book is available at the HEC library.

3 Different types of financial markets

It is common to categorize financial markets according to various criteria:

- **Primary markets vs. secondary markets.**
  - *Primary markets:* the markets where firms initially sell (issue) their securities for the first time (generating cash flow for the firm).
  - *Secondary markets:* the markets in which securities already issued by firms are traded (which generates no cash flow for the firm).

- **Capital markets vs. money markets.**
  - *Capital markets:* This market provides long term financing and involves financial instruments with maturities greater than one year.
  - *Money markets:* This market provides short term financing and involves financial instruments with maturities lower than one year (commercial papers, treasury bills, CDs,

etc)

4 Rates of return on financial assets

Suppose that you buy one share of a stock XYZ at price P_0. You resell it in one year at price P_1, just after receiving a dividend of D_1. The realized rate of return, r_{XYZ}, on this investment is:

$$r_{XYZ} = \frac{P_1 - P_0}{P_0} + \frac{D_1}{P_0}$$

This formula can be applied to calculate the rate of return on the investment in any financial assets, whatever the holding period for the asset. There are two components of this rate of return:

- The **capital gain or capital loss component** which comes from the possibility to resell the asset at a markup or a discount relative to the purchase price.

- The **yield component** which comes from the possibility to receive an income (e.g. the dividend in the case of a stock) from holding the asset.

For most of the assets, the capital gain or loss component is not known at the time of the purchase because the future resale value is not known with certainty. The yield component can be certain (e.g., in the case of government bonds, which do not have default risk) or uncertain (e.g., in the case of stocks). Thus, holding financial assets is risky because, in general, their rates of return are unknown at the time of the investment decision.

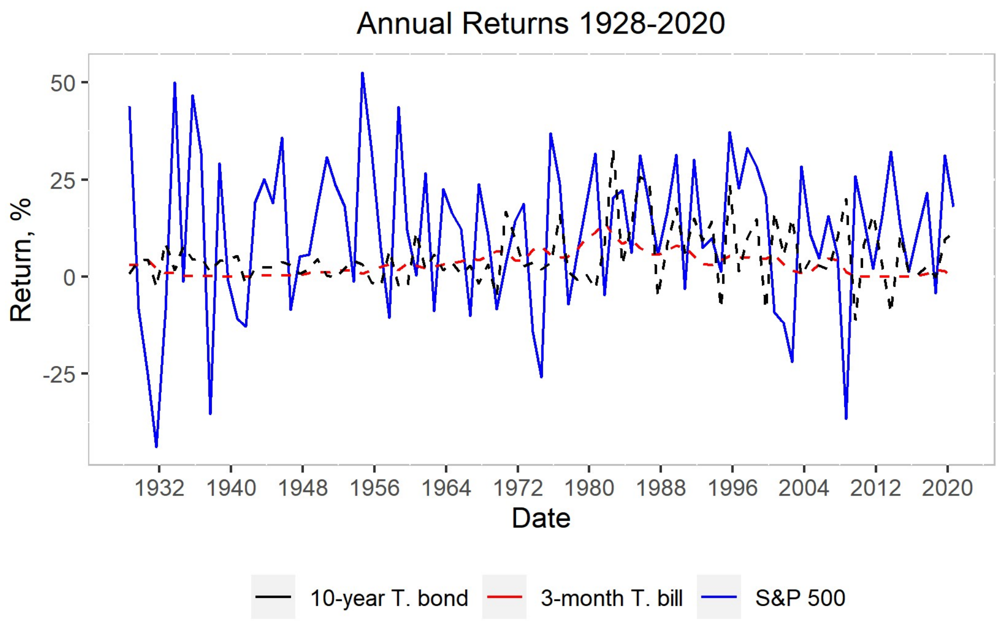
Table 1: Common stocks versus government bonds

Securities	Capital Gain/Loss	Income
Common Stocks	Uncertain	Dividends/Uncertain
Government bonds	Uncertain	Coupons/Certain

Graph 1 represents the evolution of the annual rates of returns on three types of assets for the period 1928-2020: (a) large American stocks (S&P 500), (b) 3-month American treasury bills and (c) 10-year American Treasury bonds.

It is clear that the rates of returns on stocks are much more volatile than the rates of returns on treasury bills. This reflects the greater uncertainty (risk) of investing in stocks. One of the objectives of *Financial Economics* course is to measure this risk quantitatively. It is also interesting to note that the average rate of return on stocks (11,6%) has been higher than the average rate of return on treasury bills (3,4%) and treasury bonds (5,2%) for the period 1928-2020. Thus, *there is a tradeoff between risk and return*. We will see in the next lectures that the variance of the returns of an asset is just one measure of its *risk*. Another important goal of *Financial Economics* is to understand the reasons and the implications of such a trade-off.

Graph 1: Annual returns 1928-2020



Graph 1 represents the evolution of the annual rates of returns on three types of assets for the period 1928-2020: (a) an index of large American stocks (S&P 500), (b) 3-month American treasury bills and (c) 10-year American Treasury bonds.

5 Some Important Financial Intermediaries

5.1 Banks

- **Commercial banks:** borrow through deposits and make loans to firms and individuals that cannot borrow from financial markets.
- **Investment banks:** advise firms and governments trying to raise funds in order to finance their activities. They also facilitate mergers and can act as brokers and dealers in financial markets (see again Chapter 1 in Bodie/Kane/Marcus).

In many countries, banks can be both commercial and investment banks. Such institutions are called universal banks and are more prevalent in Europe. In the U.S., the Glass Steagall Act of 1933 imposed a separation between investment and commercial banks. But the Act has been repealed during the 1990s. Since then, large American commercial banks have been acquiring investment banking firms to become universal banks. For example, in 1999 Hambrecht & Quist, an independent investment bank, was acquired by Chase Manhattan Corp. In 2000, Chase Manhattan Corp. acquired JP Morgan& Co. Inc. to give birth to JP Morgan Chase & Co. JP Morgan

Chase & Co. then acquired Bear Stearns in March 2008 and Washington Mutual in September 2008, thus becoming the second largest commercial bank in the U.S..

5.2 Mutual funds

A mutual fund is a portfolio of financial assets that is managed in the name of a group of investors. Each investor is entitled to a *pro rata* share of distributions by the fund and can redeem his/her share at the current market value calculated at the end of the trading day.

The determination of the market value depends on whether the fund is *open-ended* or *closed-ended*.

- **Open-end Funds:** the number of shares is not fixed. Shares are redeemed at their *net asset value*, i.e. the market value of all securities in the fund divided by the number of shares issued by the fund.
- **Closed-end Funds:** the number of shares is fixed. The shares of the fund are traded through brokers at a price that can differ from their net asset value.

Mutual funds can be *active* or *passive*.

- **Passive management** is an investment approach of holding highly diversified portfolios without spending effort on attempts to improve investment performance through security analysis.
- **Active management** is an attempt to improve investment performance either by identifying mispriced securities or by timing the performance of broad asset classes (e.g. by increasing exposure to the stock market when the market is expected to perform well).

Passive mutual funds track a stock market index (such as S&P 500, Russel 1000) and do *not* try to outperform the index by selecting stocks with superior expected return. To the contrary, active mutual funds engage in security analysis in attempt to identify which stocks will have higher future return.

5.3 Exchange-traded funds

Like mutual funds, exchange-traded funds (ETFs) offer investors an opportunity to obtain a share in a portfolio of financial assets. However, unlike mutual funds, ETFs are traded directly on stock exchanges as any typical stock. ETFs are often cheaper (have lower fees) than mutual funds. Whereas the majority of ETFs are passive and track a specific stock market index without an attempt to outperform it, a new type of actively managed ETFs has emerged recently.

5.4 Pension funds

A pension fund is a fund set up to pay the pension benefits after retirement. Such funds can be set up by governments and companies. In Defined Benefit Plans the retiree receives a payment (monthly or quarterly) based on a formula that takes into account years worked, last or average salary over time, among other ingredients. In Defined Contribution Plans the employee makes payments and manages his/her investments. In both cases, the employer requires that the employee makes a minimum payment and provides matching contributions.

Pension funds are the largest category in terms of assets holdings, totaling more than $20 trillion in January 2008 according to Morgan Stanley. Open pension funds support at least one pension plan but display no restriction on membership, while closed pension funds are restricted to certain employees. Not surprisingly, many public sector pension funds stand among the largest. In 2006, the government pension funds of Japan, Norway and Netherlands ranked first and the California Public Employees Retirement System (CalPers) was the largest American pension fund.

5.5 Hedge funds

A hedge fund is a fund that uses financial instruments generally beyond the reach of mutual funds, which are subjected to financial rules and regulations (as well as disclosure requirements) that largely prevent them from using short selling, leverage, concentrated investments, and derivatives. Namely a hedge fund can take both long and short positions, use arbitrage, buy and sell securities, trade options and other derivatives. Many hedge funds trade against downturns in the markets (hence their generic name), their behavior is associated with more volatile, possibly overheated, markets in the popular press.

However, their strategies vary enormously and need not entail contrarian positions. Event-driven funds draw on corporate events, either by taking positions in bankruptcies and reorganizations through bank debt or ‘high yield’ bonds (‘distressed securities’ funds), or by betting on the stock price reactions of corporate mergers (‘merger arbitrage’ funds). Macro funds rely on macroeconomic analysis to take positions in major risk factors such as currencies, interest rates, stock indices and commodities. Market neutral funds use a wide array of strategies aimed at expunging exposure to major risks and bet on relative price movements. Examples include ‘long-short equity’ funds, ‘stock index arbitrage’ funds or ‘fixed income arbitrage’ funds. Sector funds focus on specific sectors of the economy for which they have expertise. Quantitative hedge funds use statistics and machine learning to predict which stocks will perform well in the future.

5.6 Insurance companies

They sell insurance policies to households in exchange of a periodic payment called a *premium*. The insurance policy is a financial asset which allows the buyer to receive payments that are contingent to the occurrence of an event (e.g., a car accident, fire, or theft). The insurance companies reinvest the insurance premiums that they receive in financial instruments in fixed-income securities (bonds) and other financial contracts. For this reason, along with mutual funds, they are major players in financial markets.

5.7 Private equity and venture capital

Large firms can raise funds directly from the stock and bond markets. However, smaller and younger firms that have not issued securities to the public do not have such option. Instead, they rely on bank loans and investors who are willing to invest in them in return for an ownership stake in the firm. While *venture capital* (VC) firms specialize on such investments in the start-up companies, the *private equity* in general might include other strategies of investing in private companies (private as opposed to companies whose equity trades on public markets). For example, some private equity firms purchase distressed companies, “improve” them through direct intervention into the management, and aim to sell later these companies at a profit.

5.8 Financial intermediaries and FinTech

Financial technology (FinTech) is the technology and innovation that that aims to compete with traditional methods in the delivery of financial services. A prominent financial innovation in the space of financial intermediation is *peer-to-peer lending* (P2P lending), which is the practice of lending money to individuals or businesses through online services that match lenders with borrowers.

Figure 1: The Financial System and its flows. Plain arrows denote demand and supply for cash flows; dotted arrows denote demand and supply for financial assets**.**

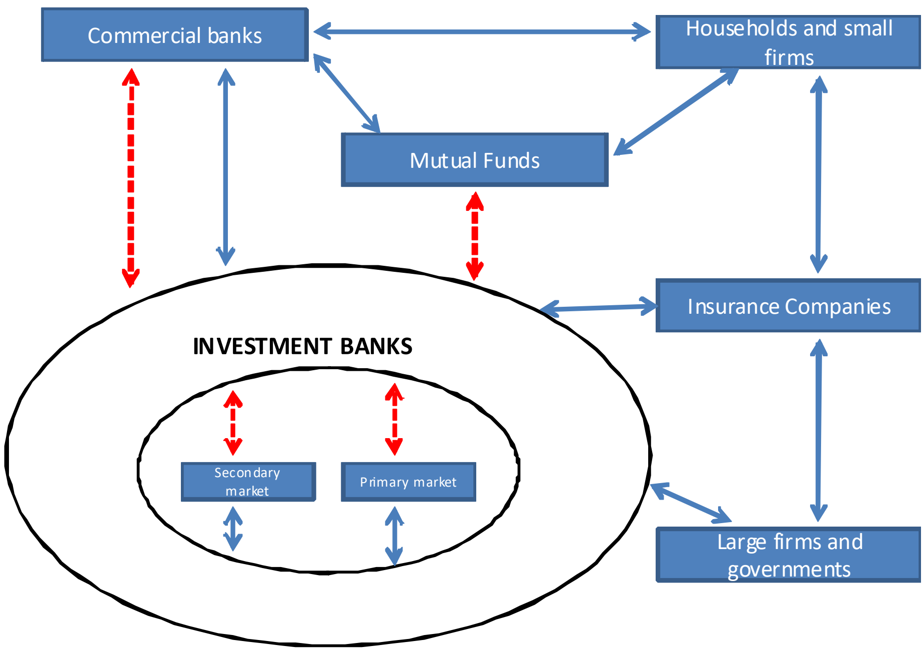
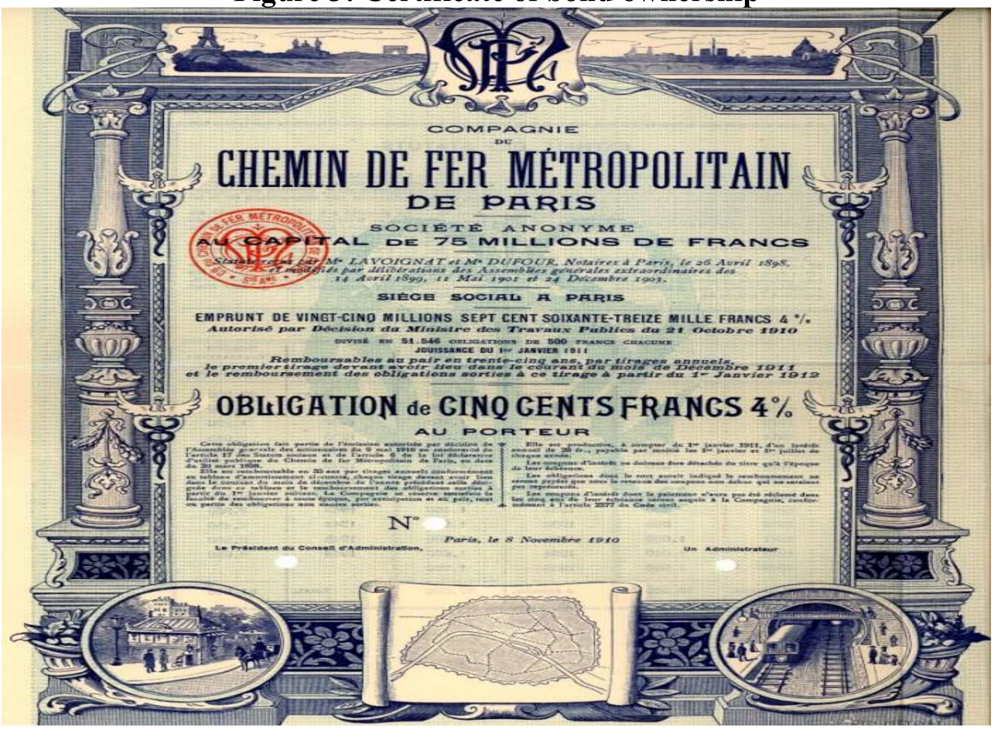


Figure 2: Certificates of stock share ownership (note that shares are now registered and that certificates are only available upon request)

Figure 3: Certificate of bond ownership



II - Capital Budgeting

1 What is capital budgeting?

Firms carry on business by investing in a variety of real assets (machines, factories, trademarks,...). Capital budgeting is the process by which the management:

- proposes new investment projects;
- evaluates projects;
- decides which projects to undertake and which projects to reject.

Here, we are mainly interested in the evaluation and the screening parts of the capital budgeting process. Note that the course does not address the issue of how to finance an investment project (issue debt or equity?). Investing and financing decisions can be considered separately because, under certain conditions, the value of investment projects is independent of the way the projects will be financed. The *Corporate Finance* course will address firm's financing issues in detail.

An investment project is characterized by its cash flows $\{CF_0, CF_1, \ldots, CF_t, \ldots, CF_T\}$ where CF_t is the cash flow of the project at date t. The cash flows can be positive (cash inflows, i.e., cash generated by the project) or negative (cash outflows, typically investments).

Example: Design of a new software to help medical doctors in their diagnoses.

Table 2: An example of cash-flows

Year	0	1	2	3	4	5
Cash Flow	– 2000	750	750	775	900	50

How should we evaluate this project? Can we simply sum the different cash-flows over time and accept the project if this sum is positive?

NO! Because this would neglect *the time value of money*: Intuitively, €1 "today" is worth more than €1 "tomorrow" for three reasons:

- *Opportunity cost:* One euro today can be invested and will then earn interest.
- *Inflation:* the purchasing power of one euro today is higher than the purchasing power of one euro tomorrow.
- *Uncertainty*: cash flows may not materialize as projected.

For this reason, comparing sums of money at different points in time is not straightforward. We first describe how to perform this comparison and we derive the Net Present Value (NPV) rule, a simple decision tool to accept or to reject an investment project. In a second step, we study some of the issues that arise in forecasting cash flows. Finally, we present other rules that are used to select investment projects and show why they can be wrong.

2 Time Value of Money

2.1 Future Value and Compounding

Compounding is the process of computing the future value of a given sum of money today at some fixed date in the future. Central to this computation is the *interest rate*. An interest rate over a given period of time is a promised rate of return.

Example: What is the future value of $100 in two years if the annual interest rate is 5%?

At the end of Year 1: $FV = 100 \times (1 + 0.05) = 105$

At the end of Year 2: $FV = 105 \times (1.05) = 100 \times (1.05)^2 = 110.25$

Note that, for the second period, the interest received at the end of the first period ($\$5$) earns interest as well. This way of computing interests is called *compounding*.

Definition 1*: If r is the interest rate per year, the future value of a sum S in n years is:*

$$FV(r, n) = S \times (1 + r)^n$$

Example: How many years are necessary to double a given initial stake, *S*? We are looking for *n* such that:

$$2S = S \times (1 + r)^n$$

i.e., *n=ln(2)/ln(1+r)* For instance with *r* = 5%, 14.2 years are necessary. With *r* = 10%, only 7.27 years are necessary.

2.1.1 Interest Rates

In practice interest rates vary according to:

- **The unit of account:** Interest rates on dollar deposits are different from interest rates on euro deposits.
- **The maturity:** In general, there is a relationship between the maturity of an investment or a loan and the interest rate. This relationship is depicted by the *yield curve* that we study in more details in the Financial Markets course.
- **Default risk:** Other things equal, interest rates are larger when default risk is higher as investors require a higher risk premium for a higher probability of default.

Table 3: Interest rates on U.S. Treasury and corporate bonds as of July 2023

	US treasury bonds	US AA corporate bonds
2-year maturity	4.74%	5.75%
5-year maturity	4.00%	4.90%
10-year maturity	3.80%	4.71%

Table 4: London Interbank Offered Rate (LIBOR) as of July 2023

	In US dollars	In Japanese Yens
1-month maturity	5.37%	-0.06%
3-month maturity	5.59%	-0.03%
1-year maturity	5.83%	0.07%

Source: Yahoo finance & bbalibor

Inflation is an important determinant of the level of interest rates. Why? Suppose that the basket of goods which is used to compute the rate of inflation is worth \$100 today. The rate of inflation per year is $\pi = 2\%$. This means that the basket of goods will be worth \$102 in one year. Thus, for investing \$100, you would require at least an interest rate of 2% in order to maintain your purchasing power unchanged. In general, the interest rate you will require would be larger than the inflation rate. Suppose that this interest rate is $r = 5\%$, what is the increase in your purchasing power? Given an investment of \$100, in one year you could purchase: $1.0294 = \frac{(1+r)*100}{(1+\pi)*100}$ times the basket of goods worth €100 today. Thus, your purchasing power has really increased by 2.94% (and not by 5%). This is called *the real rate of return* in contrast to r which is called *the nominal rate of return*. The real rate of return, r_e is given by:

$$1 + r_e = \frac{(1+r)}{(1+\pi)}$$

and a useful approximation is given by:

$$r \approx \pi + r_e$$

which shows that inflation is one of the main determinants of the nominal rate. The other determinant is the real interest rate which is determined by factors such as the productivity of capital goods, time preferences, and risk aversion.

We will be mostly considering nominal cash-flows. Hence, we will use only nominal interest rates.

3 Present Value and Discounting

What is the amount of money that we must invest today in order to obtain a given future value S in n years? If the interest rate is r, then we search for the value PV that solves:

$$S = (1+r)^n PV$$

$$\text{i.e. } PV(r,n) = \frac{S}{(1+r)^n}$$

.

Definition 2: *If r is the interest rate per year, the present value of an amount S received in n years is:*

$$PV(r,n) = \frac{S}{(1+r)^n}$$

Computing a present value is called *discounting* (the reverse of compounding) and r is often referred to as the *discount rate*. Note that when $r > 0$, the present value of one dollar in one year is less than one dollar, which reflects the time value of money.

3.1 Multiple Cash Flows

Result 1: If the interest rate is r, the present value of a stream of cash-flows $\{CF_0, ..., CF_T\}$ is:

$$PV = CF_0 + \frac{CF_1}{(1+r)} + ... + \frac{CF_t}{(1+r)^t} + ... + \frac{CF_T}{(1+r)^T} = \sum_{t=0}^{t=T} \frac{CF_t}{(1+r)^t}$$

This formula is often called *the Discounted Cash Flow* (DCF) formula. Note that the value of the stream of cash flows is just the sum of the present value of each cash flow. This is in fact the amount one would have to save at date 0 to secure the cash flow CF_0 at date 0, CF_1 at date 1,..., and CF_T at date T.

Example: Consider the investment project which is described in Table 2 and assume that the interest rate is $r = 11.5\%$. The present value is of this project is:

$$-2000 + \frac{750}{1+r} + \frac{750}{(1+r)^2} + \frac{775}{(1+r)^3} + \frac{900}{(1+r)^4} + \frac{50}{(1+r)^5} = 446.31$$

The interest rate can depend on the duration of the investment period. Let r_t be the annual interest rate for an investment that matures in t periods. In this case the present value of the stream of cash flows $\{CF_0, CF_1, ..., CF_t, ..., CF_T\}$ is:

$$PV = \sum_{t=0}^{t=T} \frac{CF_t}{(1+r_t)^t}$$

Note that in this case each cash flow CF_t is discounted using the corresponding interest rate r_t.

Important remark: The DCF formula can be used to compute the present value of any stream of future cash flows. This means that the formula can be applied to compute the present value of the cash flows associated with an investment project, as we do here but also to compute the present value (price) of **a financial asset** once the cash flows of the security have been forecasted. For instance, the price of a bond must be the present value of the stream of coupons and the reimbursement value that the bondholder can expect. This will be explained in more details in the *Financial Markets* course.

3.1.1 Annuities

Definition 3: *An ordinary annuity of length* n *is a sequence of* ***constant*** *cash-flows during* n *periods, where the cash flows are paid (received) at the end of each period.*

An ***immediate*** *annuity is an annuity in which the cash flows are paid at the beginning of each period. We only consider ordinary annuities below.*

Definition 4: *A perpetuity is an ordinary annuity with an infinite length.*

Result 2: The present value $P(C,r)$ of a perpetuity with a constant cash flow C when the interest rate is r is:

$$P(C,r) = \frac{C}{r}$$

Using this result we can now deduce the present value $A(C,r,n)$ of an ordinary annuity of length n with a cash flow C. Actually, a perpetuity with cash flow C has the same cash flows as a portfolio that is made of one ordinary annuity of length n with cash flow C and one perpetuity whose first cash flow is paid at the end of period $n+1$. It follows that:

$$P(C,r) = A(C,r,n) + \frac{P(C,r)}{(1+r)^n}$$

Result 3: The present value $A(C,r,n)$ of an ordinary annuity with a cash flow C when the interest is r is:

$$A(C,r,n) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Some applications:

Example 1: You win a prize. You are offered two ways to cash-in the prize. In the first case, you receive \$1,000,000 immediately. In the second case, you receive a perpetuity of \$100,000 per year with a first payment today. The interest rate is 9.5%. Which one would you choose?

Solution: In the second case, your prize is equivalent to 100,000+100,000/9.5%=1,152,631.58
Hence, you prefer the second case.

Example 2: You plan to borrow \$1,000,000, that you will reimburse in 25 years, with monthly payments (300 payments). If the interest rate is 16% per year, what is the monthly payment C?

Solution: Use the formula for ordinary annuity. r=16%/12, n=300, A(C, r, n)=1,000,000.
Then C=13588,89.

Result 4: Consider a stream of cash-flows which grow at rate g in perpetuity, that is $CF_{t+1} = (1+g)CF_t$. The first cash flow is received in one year and is equal to C. The present value of this stream of cash flows is:

$$PV = \frac{C}{r-g} \quad if \quad g < r$$

4 Net Present Value

4.1 The Net Present Value Rule

Consider an investment project for which the initial investment outlay is *I* and which yields a stream of cash-flows $\{CF_1, CF_2, ..., CF_T\}$, up to year *T*.

The Net Present Value of this project (NPV) is the difference between the present value of the stream of cash flows received between *t=1 and t=T that are* associated with the investment and the initial investment outlay I_0 at *t=0*:

$$NPV = \sum_{t=1}^{t=T} \frac{CF_t}{(1+r)^t} - I_0$$

The NPV is the difference between the positive cash flows generated by the investment (in present value) and the cost of investment. The *NPV rule* consists of accepting an investment project if and only if its NPV is positive. For instance, as shown previously, the NPV of the investment project described in Table 2 is positive, therefore the project must be accepted.

If we evaluate *several investment projects*, two cases must be distinguished:

1) **If the projects are not mutually exclusive**: all the projects with a positive NPV must be undertaken.

2) **If the projects are mutually exclusive**: choose the project with the greatest positive NPV.

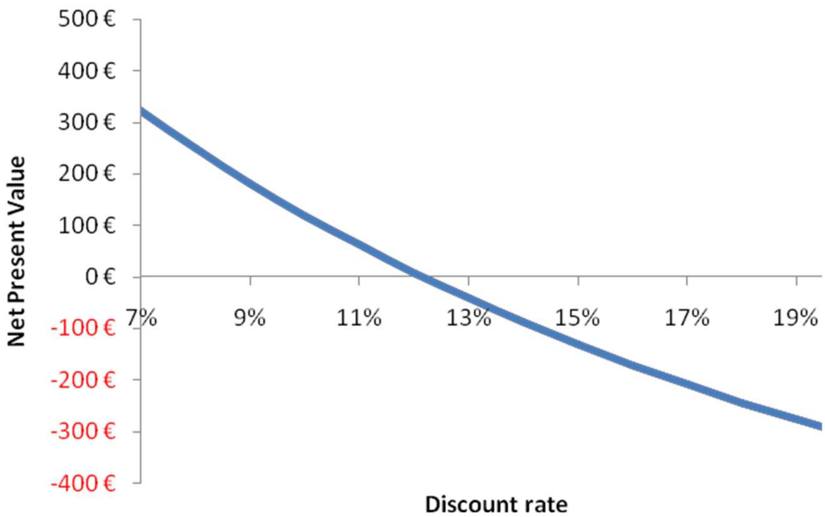
4.2 Opportunity Cost of Capital.

The NPV depends on the discount rate. For instance, consider the following investment project:

Year	0	1	2	3	4	5	6	7	8	9	10
Cash Flow	-1000	100	115	132	152	175	201	231	266	306	352

Graph 2 illustrates the evolution of the NPV for this project as the interest rate increases between 7% and 20%. As can be seen the NPV decreases and is positive if $r < 12.2\%$ and is negative otherwise. Thus, the discount rate is crucial for the investment decision. In general, the NPV will decrease with the interest rate but not always (more on this below).

Graph 2: NPV and discount rate



How should we choose the discount rate? We must choose a discount rate equal to the interest rate that we could earn elsewhere **in the same risk profile** if we were not to invest in the project under evaluation. This rate is called **the opportunity cost of capital**. Two problems arise. What does *elsewhere* mean? Investments in financial assets always provide an alternative investment. Thus, we can use rates of return offered on financial assets to determine the appropriate discount rate. This is one of the functions of financial markets. A more difficult problem is to evaluate the risk of the project and to find financial assets with a similar risk. The only case in which there is no difficulty is when the project is *riskless* (no uncertainty on the cash flows) in which case, the opportunity cost of capital is the rate of return on riskless financial asset (for instance, a Treasury bill). For the other cases, we first need to define what is the risk of an asset. This will be considered in the next Chapter.

Example: You can invest in a riskless project which will yield a cash flow equal to $10,500 in one year and which requires an investment outlay equal to $10,000. The rate of return on a one-year treasury bill is 6%. Should you invest in the project? Investing $10,000 in the treasury bill will yield a future value of FV=$10,600 whereas the project has a future value of $10,500. Thus, the project should be rejected. In fact, its NPV is:

$$NPV = \frac{10{,}500}{1.06} - 10{,}000 = -94.34 < 0$$

4.3 What is the NPV?

The NPV of an investment project is the increase in the wealth of the existing shareholders when the project is undertaken. Note that this vindicates the NPV rule: the shareholders of the company do not want an investment project with a negative NPV to be chosen since this would decrease the value of their shares.

Example 1. Consider again the project described in Table 2. The firm with this project has one shareholder-manager who does not have the cash ($2,000.000) which is necessary to undertake the project. The shareholder contacts one financier who will finance the investment project. The financier pays $2,000.000 and receives interests at 11.5%. The initial shareholder-manager invests the amount provided by the financier, pays back the financier, and retains the NPV. He is richer by $446.310.
Why is it so?
The entrepreneur borrows 2000 at 11.5% for 5 years (maturity of the project). It must pay back the financier : 2000 x 1.115^5 = 3446.7 at the end of the 5th year.
Now, evaluate the project at maturity date, T=5, instead of at t=0, i.e. reinvest all cash flows at the opportunity cost of capital 11.5%.

Date	Cash flow (at t)	Cash flow valued at T=5
0	-2000	
1	750	1159.2
2	750	1039.6
3	775	963.5
4	900	1003.5
5	50	50
	TOTAL	4215.8

Out of the sum of 4215.8, the entrepreneur pays back 3446.7 to the financier. It remains for him : 4215.8 – 3446.7 = 769.1 that is the NPV of the project but evaluated at date 5. To get the NPV at date 0, as we usually do, 769.1/1.115^5 = 446.31.

So the NPV of 446.1 is the net return to the entrepreneur.

Example 2. Now consider an entrepreneur with the following one-period project: a cash flow of €240,000 in one year for an initial investment equal to €200,000. The opportunity cost of capital is 10%. Thus the *NPV*=€18,181.82. Without undertaking the project, the entrepreneur has an income equal to €100,000 in Year 0 and in Year 1. The consumption needs of the entrepreneur are at least equal to this amount in each year. Does it mean that the entrepreneur should not undertake the project?
NO! If there is a financial market, the entrepreneur can borrow money on the financial market and undertake the project without sacrifices in term of consumption, as long as the NPV is positive. He can even increase his consumption in Year 0 and Year 1 of an amount (in present value) equal to the NPV of the project.

Fisher's Separation Theorem. The previous examples show that all shareholders should agree on undertaking positive NPV projects, independently of their consumption plans. This result is the cornerstone of the *separation between ownership and management* since this implies that all managers have to do is to select investments with positive NPV (they do not need to know shareholders' desired consumption plans for instance). This result is known as *Fisher's Separation Theorem*. The second example also illustrates another function of financial markets: they allow consumption plans to be different from the cash flows associated with an investment project.

Problem: Moral-hazard: do managers have sufficient incentives to act in shareholders' best interests? Not necessarily. Thus, there is a need to align shareholders and managers' incentives (incentive contracts, stock options, etc...).

4.4 The Internal Rate of Return Criterion

Definition 5: *The internal rate of return (IRR) of an investment project is the discount rate, y, such that the NPV of the project is zero. It is the solution of the following equation:*

$$\sum_{t=1}^{t=T} \frac{CF_t}{(1+y)^t} - I_0 = 0$$

Consider the investment project whose NPV is described in Graph 2. It can be seen on the graph that the IRR for this project is approximately 12.2%.

The IRR is often used as a criterion to select investment projects. The rule is to undertake investment projects for which the IRR is larger than the opportunity cost of capital. When there are several mutually exclusive investment projects, the rule is to select the project with the highest IRR.

Note that for the investment project described in Graph 2, the IRR criterion is equivalent to the NPV criterion. However, this is not the case in general. The IRR rule is always equivalent to the NPV rule in the case of a "standard" investment project. By “standard,” we mean that the first cash flow is negative (investment cost) and the second is positive. Actually, in this case, if the NPV is positive this means that:

$$NPV = -I + \frac{CF_1}{1+r} > 0$$

Now, by definition, the IRR satisfies the following equation: $-I + \frac{CF_1}{1+y} = 0$. It follows that if the NPV is positive and if $CF_1 > 0$ then $r < y$. In the other cases, the decision rule based on IRR has several caveats that we expose below.

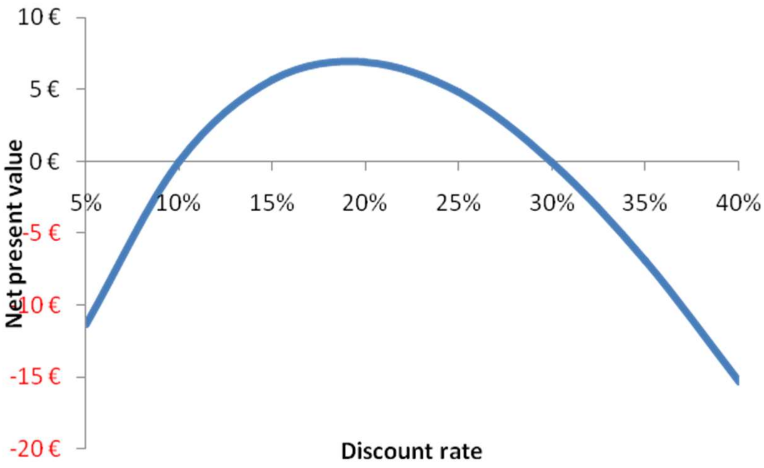
Caveat 1: There can be several IRR for projects that last more than one period.

Example: Consider the following investment project.

Year	0	1	2
Cash Flow	-1000	2400	-1430

As shown in Graph 3, there are two IRR for this project that are respectively 10% and 30%. In this case, there are 2 problems. First, a decision rule based on the IRR can suggest that the project be accepted even though its NPV is negative. This is the case if the opportunity cost of capital is lower than 10%. Second, there is no basis on which to choose one of the two IRR.

Graph 3: 2 IRR case



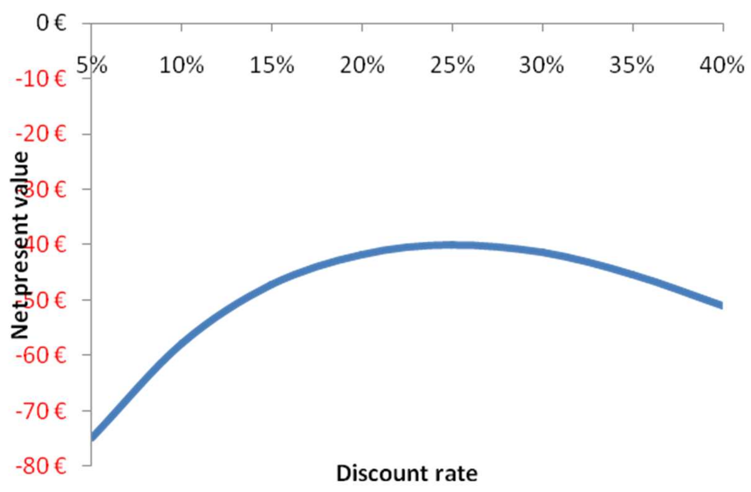
Caveat 2: There can be no IRR for an investment project.

Example: Consider the following investment project.

Year	0	1	2
Cash Flow	-1000	2400	-1500

Graph 4 shows that the NPV is always negative for this project.

Graph 4: No IRR case

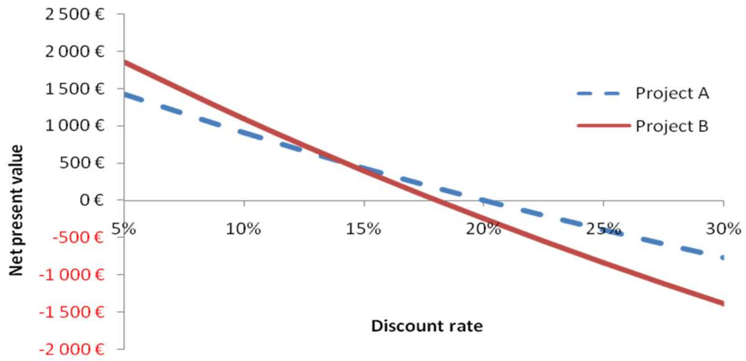


Caveat 3: The IRR criterion and the NPV criterion do not necessarily lead to the same decision in the case of mutually exclusive projects.
Example: Consider the two following investment projects *A* and *B*:

Year	0	1
Cash Flow A	-10000	12000
Cash Flow B	-15000	17700

Graph 5 represents the NPVs of these two projects as a function of the discount rate. It can be seen that the IRR of project *A* is 20% whereas the IRR of project *B* is 18%. Thus, according to the decision rule based on the IRR, investment A should be chosen. However, for all discount rates lower than 14%, Project B has a higher NPV.

Graph 5: IRR criterion and NPV criterion contradict



To sum up, only the criterion based on NPV is universally applicable.

4.5 Other criteria

Definition 6: *A project* ***payback*** *is the time required to recover the initial investment.*

Example: Consider the following investment projects.

Project A

Year	0	1
Cash Flow	-1000	1000

Project B

Year	0	1	2	3
Cash Flows	-1000	200	1600	3000

Project C

Year	0	1	2	3
Cash Flows	-1000	0	1000	300

Project D

Year	0	1	2	3
Cash Flows	-1000	500	500	300

Project A's payback is one year as the initial investment is recovered in exactly on year time. Project B's payback is computed as follows: After 1 year 200 have been recovered, then remaining 800 will be recovered in 800/1600=0.5 years. Thus, payback for project B is 1.5 years. The payback for project C and D is 2 years.

The payback criterion states that one should undertake the project with the shorter payback. However, this decision rule has at least two caveats. First, the payback criterion ignores all cash flows received after the payback period. For example, according to this criterion, Project A should be preferred to project B even if project A's NPV is clearly negative whereas Project B's NPV can be positive. Second, the payback criterion ignores the timing of cash flows within the payback period. For example, Project C and D have the same payback but clearly the NPV of project D is larger than the NPV of project C. Why? (hint: timing of cash flows matter. Compare cash flows at t=1 and t=2 of the project C and project D).

Definition 7: *The normalized NPV of project is the NPV of the project divided by the initial investment.*

Example: Consider the following two mutually exclusive investment projects.

Project E

Year	0	1	2
Cash Flows	-200	200	300

Project F

Year	0	1	2
Cash Flows	-1000	800	600

If the discount rate is 5%, then the NPV of project E and F are

$$NPV_E = -200 + \frac{200}{1.05} + \frac{300}{1.05^2} = 262.59$$

$$NPV_F = -1000 + \frac{800}{1.05} + \frac{600}{1.05^2} = 306.12$$

The normalized NPV are
$NNPV_E = 262.59 / 200 = 1.313$
and $NNPV_F = 306.12 / 1000 = 0.30612$.

The rule of the normalized NPV is to undertake investment projects with the largest NNPV. Still, from the previous example it is straightforward that if project E cannot be replicated, then project F should be preferred to project E as it provides a larger increase in the current wealth.

III - Portfolio Management

1 What is portfolio management?

A portfolio is a combination of assets.

For instance, an allocation of one's wealth into 10 shares of IBM, 5 shares of ITT and one share of Natixis is a portfolio. Portfolio management consists of choosing a particular combination of assets with the goal of achieving specific targets in terms of portfolio risk and return. The purpose of this Chapter is to provide an overview of the basic principles and techniques of portfolio management, which are also known as *the Modern Portfolio Theory* (MPT).

The rate of return, r_p, on a portfolio can be calculated as the rate of return on any other asset.

$$r_p = \frac{V_1 - V_0}{V_0} + \frac{D_1}{V_0}$$

where, V_0 is the value of the portfolio at date 0, V_1 is the value of the portfolio at date 1 and D_1 is the income generated by the portfolio during the holding period. Assume that today and in one year, the prices and the dividends of the stocks in the portfolio considered previously are as follows (keep in mind that this example is fictitious):

Stocks	Price at $t=0$	Price at $t=1$	Dividend	Rate of Return
IBM	200	215	10	12.5%
ITT	350	320	5	-7.14%
Natixis	250	310	0	24%

Then the rate of return on the portfolio is:

$$r_p = \frac{4060 - 4000}{4000} + \frac{125}{4000} = 4.63\%$$

Very often portfolios are described by the proportion of the total wealth which is invested in each asset – which we call “weights”. In the previous example, the portfolio can be described by the following weights:

x_{IBM} = 2000/4000=50.00%, x_{ITT}=1750/4000 =43.75%, and x_{NAT}=250/4000= 6.25%

Result 1: Consider a portfolio of N assets, with weights $\{x_i\}_{i=1}^{i=N}$ where x_i is the proportion invested in asset i. Let r_i be the rate of return on asset i over the portfolio's holding period. Then the rate of return on the portfolio is given by:

$$r_p = x_1 r_1 + x_2 r_2 + ... + x_i r_i + ... + x_N r_N$$

This formula relates the rate of return on a portfolio, the composition of the portfolio and the rates of return on each individual asset in the portfolio. In the example above, we can check that:

$$50\% \times 12.5\% - 43.75\% \times 7.14\% + 6.25\% \times 24\% = 4.63\%$$

Short-sales: A short sale allows an investor to profit from a decline in the price of a security. There are two steps in a short sale. In the first step, the investor borrows the security (from her broker) and sells it, say at price P_0.. In a second step, later on in time, the investor *covers the short position*: sshe purchases the security, say at price P_1, in order to return the share she initially borrowed. She also pays the lender any income (e.g., dividends) paid by the security during the period she borrowed the security. The total gain (loss) for the short seller is $P_0 - P_1 - D_1$. This is positive if P_1 is sufficiently low relative to the initial price, i.e., if the price of the security declines. The following tables represent the flows of cash and assets generated by a short sale:

Period 0	**Action**	**Asset flows**	**Cash flows**
	Borrow one asset from your broker	$+1$	
	Sell the asset	-1	P_0
Net flows		**0**	$\mathbf{P_0}$

Period 1	**Action**	**Asset flows**	**Cash flows**
	Buy one asset	$+1$	$-P_1$
	Return the asset to your broker	-1	
	Pay the dividend to your broker		$-D_1$
Net flows		**0**	$\mathbf{-P_1-D_1}$

In fact the rate of return on the short-sale is equal *in absolute value* to the rate of return on the purchase of the security but has the opposite sign. A short-seller gets a positive rate of return if the price decreases and a negative rate of return otherwise. Note that a short-sale creates a liability for the investor: in exchange of cash today (the price of the security that is sold short), the investor has to return (that is, to purchase) the security at some point in the future. Thus, short selling \$1,000 of ITT shares is equivalent to borrowing \$1,000 at an interest rate equal to the return rate of ITT shares.

How to describe a portfolio that contains short sales? For instance, consider an investor Adam Smith who considers buying IBM and short selling ITT. His initial wealth is \$1,000 and he plans to short-sell 2 shares of ITT. The investor has eventually \$1,700 to invest in IBM and he buys 8.5 shares of this stock. He has invested x_{IBM} = 170% of his initial wealth in IBM and has short sold ITT for a proportion representing x_{ITT} = -70% of his initial wealth. His rate of return will be:

$$r_p = 170\% \times r_{IBM} - 70\% \times r_{ITT} = 26.25\%$$

or

$$r_p = x_{IBM} \times r_{IBM} - x_{ITT} \times r_{ITT}$$

This shows that one way to track the impact of short sales on the rate of return of a portfolio is to give *negative weights* to the securities that are sold short. For instance, the previous portfolio is

$$x_{IBM} = 170\% \text{ and } x_{ITT} = -70\%.$$

In this way, the formula announced in Result 1 is still valid. Note that in all the cases:

$$\sum_i x_i = 1$$

When short sales are allowed, x_i can take any values (positive or negative). When short sales are not allowed, it must be the case that $x_i \in [0,1]$.

2 Uncertainty and rates of return

In the previous example, the rates of return have been computed *ex-post*, i.e. after observing the rates of return on the different securities in the portfolio. Of course, portfolio managers must form their portfolios *ex-ante*, i.e., before knowing the actual rates of return on the securities they can include in their portfolios. How can we describe or quantify the uncertainty on the rates of return of individual securities? How is the uncertainty on the rate of return of a portfolio related to the uncertainty on the rates of return on the securities in the portfolio? To answer these questions, we will use basic concepts from the Probability Theory that we describe below.

2.1 Probability Distribution

Consider the following table which gives the rate of return on one share of IBM, ITT and a treasury bill according to the state of the economy in one year.

Stocks	IBM	ITT	Treasury Bill	Probability
State 1: Economy is booming	20%	5%	3%	$p_1 = 0.2$
State 2: Economy is normal	11%	2%	3%	$p_2 = 0.5$
State 3: Economy is in recession	-15%	2%	3%	$p_3 = 0.3$

Note that the rate of return on the treasury bill does not depend on the state of the economy. Its rate of return is known with certainty ex-ante. Such a security is called *a riskless asset*. This is not the case for the other two securities, however. The uncertainty on their rate of return depends on the likelihood of each state of the economy. This likelihood is described by a *probability distribution* $\{p_i\}_{i=1}^{i=3}$ which gives the probability of each possible state for the economy. There are two requirements on a probability distribution:

$$p_i \in [0,1]$$

$$\sum_i p_i = 1$$

Given the probability distribution on the states of the economy and the table above, we can derive the probability distribution of the rates of return of the different securities, that is, the probability of occurrence of each possible rate of return for a given security. This probability distribution is often given in the form of a table which associates each rate of return with its probability of occurrence. For instance, the probability distribution of r_{ITT}, the rate of return on ITT, is:

Rate of return	5%	2%
Probability p_i	0.2	0.8

Note that the probability distribution for the risk-free treasury bill is "degenerate": the rate of return is 3% with probability $p = 1$.

2.2 Expectation

Now consider a security S that can have n different possible rates of return with the following probability distribution:

Rate of return	r_{S1}	r_{S2}	...	r_{Sk}	...	r_{Sn}
Probability p_i	p_1	p_2	...	p_i	...	p_n

The rate of return on security S is unknown ex-ante. It is *a random variable*. We denote this random variable $\tilde{r}_S$ The tilde " ~ " is used to distinguish the random variable from the possible occurrences (realizations) for the rate of return (r_{S1}, r_{S2}, r_{S3},…).

Definition 1: *The expected (probability weighted-average) rate of return on security S is*

$$E(\tilde{r}_S) = p_1 r_{S1} + p_2 r_{S2} ... + p_n r_{Sn} = \sum_k p_k r_{Sk}$$

The expected rate of return on a security can be considered as a "forecast" of the rate of return of this security. In the example above, it can be checked that: $E(r_{IBM})$ = 5%, $E(r_{ITT})$ = 2.6%, and $E(r_{TB})$ = 3%.

Consider two securities *S* and *S'*. Two useful properties of expectations are as follows:

- **Property 1**: $E(\tilde{r}_S + \tilde{r}_{S'}) = E(\tilde{r}_S) + E(\tilde{r}_{S'})$
- **Property 2**: Let a be a constant, $E(a\tilde{r}_S) = aE(\tilde{r}_S)$

2.3 Risk: Variance and Standard Deviation

To which extent is the expected rate of return a good forecast of the rate of return that will eventually be observed? Intuitively the lower are the deviations (in absolute value) between the actual rate of return and the forecast, the better is the forecast. This suggests using the following measure to quantify the uncertainty on possible rates of return of an investment in a security *S*:

$$Var(\tilde{r}_S) = p_1(r_{S1} - E(\tilde{r}_S))^2 + ... + p_k(r_{Sk} - E(\tilde{r}_S))^2 + ... + p_n(r_{Sn} - E(\tilde{r}_S))^2 = \sum_k p_k(r_{Sk} - E(\tilde{r}_S))^2$$

This measure is called *the variance of the rate of return*. The square-root of the variance, denoted σ, is called the standard deviation. For security *S* the standard deviation is:

$$\sigma_S = \sqrt{Var(r_S)} = \sqrt{\sum_k p_k(r_{Sk} - E(r_S))^2}$$

The standard deviation is a measure of the dispersion of the possible future rates of return for security *S* around its expected value. The larger is this dispersion, the less reliable is the expected rate of return as a forecast. Thus, the standard deviation (or the variance) is a measure of the degree of uncertainty regarding the future rate of return. In this sense the standard deviation is one measure of the risk of a security.

Consider the previous example. Intuitively, investing in IBM seems riskier than investing in ITT in the sense that the range of possible rates of return is wider for IBM. This is indeed confirmed by computing the standard deviation of these two securities: σ_{IBM} = 13.53% > σ_{ITT} = 1.2%.

The variance has some important properties that can sometimes be used to simplify computations:

- **Property 1:** $Var(\tilde{r}_S) = E(\tilde{r}_S^2) - [E(\tilde{r}_S)]^2$.
- **Property 2:** If *k* is a constant then $Var(k) = 0$.
- **Property 3:** Let *k* be constant then $Var(k\tilde{r}_S) = k^2 Var(\tilde{r}_S)$.

For a riskless asset, there is no risk since the rate of return is known with certainty. Indeed, for a riskless asset, Property 2 of the variance implies that the standard deviation is zero.

2.4 Covariance and Correlation

The degree to which the rates of return of two different securities move together is an important information for portfolio management. This linkage is measured by the *covariance* and the *correlation* between the rates of return of the securities. These measures are defined below.

Consider two securities *S* and *S'*. The rate of return on security *S* over the next period of time can take *n* different values whereas the rate of return on security *S'* can take *n'* different values. The next table gives the *joint probability distribution* of the rate of return for security *S* and *S'*:

Rate of return	r_{S1}	r_{S2}	...	r_{Sk}	...	r_{Sn}
$r_{S'1}$	p_{11}	p_{21}	...	p_{k1}	...	p_{n1}
$r_{S'2}$	p_{12}	p_{22}	...	p_{k2}	...	p_{n2}
...	...	...	...	...	...	...
$r_{S'j}$	p_{1j}	p_{2j}	...	p_{kj}	...	p_{nj}
...	...	....	...	...	...	...
$r_{S'n'}$	$p_{1n'}$	$p_{2n'}$	...	$p_{kn'}$	...	$p_{nn'}$

In this table, p_{kj} gives the probability of obtaining simultaneously a rate of return equal to r_{Sk} for security S and a rate of return r_{Sj} for security S'.

Example: The table below presents the joint probability distribution of the rates of return of IBM and ITT:

Rate of return	5%	2%
20%	0.2	0
11%	0	0.5
−15%	0	0.3

The joint probability distribution is used to compute the covariance.

Definition 2: *The covariance between the rate of return on security S and security S' is:*

$$\text{cov}(\tilde{r}_S, \tilde{r}_{S'}) = E\left[(\tilde{r}_S - E[\tilde{r}_S])\,(\tilde{r}_{S'} - E[\tilde{r}_{S'}])\right]$$

The sign of the covariance indicates the tendency of the rates of return on two securities to move in the *same* direction (covariance is positive) or in opposite directions (covariance is negative).

Example: Do the rates of return on IBM and ITT tend to move in the same direction or in opposite directions?
Solution: Compute covariance between rates of return on IBM and ITT and verify that it is a positive number (cov=0.0009). As a result, the rates of return tend to move in the same direction.

There are several important properties for the covariance:

- **Property 1**: $\text{cov}(\tilde{r}_S, \tilde{r}_{S'}) = E(\tilde{r}_S \tilde{r}_{S'}) - E(\tilde{r}_S)E(\tilde{r}_{S'})$.
- **Property 2**: $Cov(\tilde{r}_S, \tilde{r}_S) = Var(\tilde{r}_S)$.
- **Property 3**: Let k be a constant: $\text{cov}(k\tilde{r}_S, \tilde{r}_{S'}) = k\,\text{cov}(\tilde{r}_S, \tilde{r}_{S'})$
- **Property 4**: $\text{cov}(\tilde{r}_S, \tilde{r}_{S'} + \tilde{r}_{S''}) = \text{cov}(\tilde{r}_S, \tilde{r}_{S'}) + \text{cov}(\tilde{r}_S, \tilde{r}_{S''})$
- **Property 5**: $\text{cov}(\tilde{r}_S, \tilde{r}_{S'}) = \text{cov}(\tilde{r}_{S'}, \tilde{r}_S)$

Property 3 shows that the size of the covariance is sensitive to scale. In order to obtain a measure of the degree to which two rates of return move together, which is not sensitive to scale, we use the

correlation coefficient.
Definition 3: *The correlation coefficient between the rates of return of security S and security S' is:*

$$\rho_{SS'} = \frac{\operatorname{cov}(\tilde{r}_S, \tilde{r}_{S'})}{\sigma_S \sigma_{S'}}$$

An important property of the correlation coefficient is that its value is always in between -1 and +1. When $\rho = +1$ ($\rho = -1$), there is perfect positive (negative) correlation and when $\rho = 0$, there is no correlation.

Example: The correlation between the rates of return on IBM and ITT is: $\rho = 0.55$.

2.5 Expected rate of return and risk of a portfolio

Consider a portfolio p of N securities with weights $x_1, x_2, \ldots, x_N$,. How is the expected rate of return on the portfolio related to the expected rates of return on the securities in the portfolio? How is the risk of the portfolio related to the risk of the securities in the portfolio?[1]

Result 2: The expected rate of return on the portfolio is:

$$E(r_p) = x_1 E(r_1) + x_2 E(r_2) + \ldots x_i E(r_i) + \ldots + x_N E(r_N) = \sum_{i=1}^{N} x_i E(r_i)$$

Result 3: The variance of the rate of return on the portfolio is:

$$Var(r_p) = \sum_i x_i^2 Var(r_i) + 2\sum_{i=1}^{i=N}\sum_{j>i}^{j=N} x_i x_j \operatorname{cov}(r_i, r_j)$$

In the case of a portfolio with two securities, the previous formula simplifies as:

$$Var(r_p) = x_1^2 Var(r_1) + x_2^2 Var(r_2) + 2x_1 x_2 \operatorname{cov}(r_1, r_2)$$

Example: You build a portfolio with ITT and IBM in which you invest €10,000 of which €6,000 is invested in IBM. What are the expected rate of return and the standard deviation of the rate of return on this portfolio? What is the expected future value of your portfolio? What is the standard deviation of this future value? How do the responses change if you invest €100,000?
Solution: The weight of your investment in IBM is 0.6; the weight of your portfolio in ITT is 0.4. As a result, the expected rate of return is 0.6*5+0.4*2.6=4.04%. Expected future value of your portfolio is 10000*1.0404=10404. The standard deviation of the return on portfolio is

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[1]From now on, to simplify notations, we drop the tilde "$\sim$" on rates of return when it is clear that we are talking about the random variable and not its possible realizations.

(0.6^2*0.1353^2+0.4^2*0.012^2 +2*0.6*0.4*0.0009)^0.5=0.0839. As a result, the standard deviation of the value of the portfolio is 10000*0.0839=839.

2.6 How to estimate expected rates of return, standard deviations and correlations?

The material in section 2.6 is for your information.

Problem: In the real world, we do not know what the probability distributions of the rates of return are.

Idea: We could use historical rates of return in order to infer (estimate) the characteristics (mean, standard deviation, etc.) of the probability distributions of the rates of return on securities and portfolio of securities.
This would make sense if the probability distribution of the rates of return for a security does not change over time. Suppose that you observe T occurrences of the rate of return on security S: r_{S1}, $r_{S2}, \ldots, r_{ST}$. The rates of return are computed with the same periodicity (e.g., every year). These observations form *a sample*.

Definition 4: *The sample mean of historical returns,* $\hat{r}_S$*, is:*

$$\hat{r}_S = \frac{1}{T}\sum_{t=1}^{t=T} r_{St}$$

The sample mean is just the arithmetic average of all the observations. Under some conditions, it provides a good estimate of the expected rate of return on security S (which again, unfortunately, cannot be directly observed).

Definition 5: *The sample variance of historical returns,* $\hat{\sigma}_S^2$*, is:*

$$\hat{\sigma}_S^2 = \frac{1}{T-1}\sum_{t=1}^{t=T}(r_{tS} - \hat{r}_S)^2$$

The sample variance provides an estimate of the variance of the rate of return on security S. The sample standard deviation is just the square root of the sample variance. Now assume that you also observe T realizations of past rates of return on another security, say S'.

Definition 6: *The sample covariance,* $\hat{\text{c}}\text{ov}_{SS'}$*, between security S and S' is:*

$$\hat{\text{c}}\text{ov}_{SS'} = \frac{1}{T-1}\sum_{t=1}^{t=T}(r_{tS} - \hat{r}_S)(r_{tS'} - \hat{r}_{S'})$$

The sample covariance provides an estimator of the covariance between the rates of return on security S and S'. The sample correlation is:

$$\hat{\rho}_{SS'} = \frac{\hat{\text{c}}\text{ov}_{SS'}}{\hat{\sigma}_S \hat{\sigma}_{S'}}$$

Example. Consider Table 5. Using the previous definitions and the per-year rates of return on a portfolio of large firms stocks (CAC40) over the period 1995-2021, we can deduce an estimate of the expected rate of return and the standard deviation on this portfolio. These estimates are respectively $\hat{r}_p = 9.88\%$, $\hat{\sigma}_p = 21.01\%$.

Table 5: CAC40 annual returns

Year	Return
1995	2.83%
1996	27.59%
1997	32.95%
1998	34.06%
1999	54.14%
2000	0.95%
2001	-20.33%
2002	-31.92%
2003	19.87%
2004	11.40%
2005	26.60%
2006	20.87%
2007	4.16%
2008	-40.33%
2009	27.58%
2010	0.55%
2011	-13.39%
2012	20.37%
2013	22.22%
2014	2.71%
2015	11.94%
2016	8.88%
2017	12.73%
2018	-8.00%
2019	30.46%
2020	-4.96%
2021	12.90%

2.7 Risk Premium

Consider the following table:

Table 6: Return and Risk, USA 1928-2022

Portfolio	Average Rate of Return	Standard Deviation	Risk Premium
Large firms stocks	11.1%	20.4%	7.3%
Long term Bonds	5.5%	7.6%	1.7%
Treasury Bills	3.8%	3.0%	0.0%

As it can be seen, the larger is the standard deviation of a portfolio, the larger is the average return. This shows that investors require a compensation in order to invest in more risky assets. This compensation is called the *risk premium*. The risk premium of a security is the difference between the expected rate of return offered by the security and the rate of return on the riskless asset. An important question that the future lectures will address is: how are risk premia related to risk?

3 Diversification or why you should not put all your eggs in the same basket

Consider Figure 4. Diversification is a basic principle of portfolio management.

Figure 4: The benefits of diversification

Table 7: Number of stocks and portfolio risk

Number of Stocks in Portfolio	Standard Deviation of portfolio returns
1	49,24%
2	37,36%
4	29,69%
6	26,64%
8	24,98%
10	23,93%
20	21,68%
30	20,87%
40	20,46%
50	20,20%
100	19,69%
200	19,42%
300	19,34%
400	19,29%
500	19,27%
1000	19,21%

(Source: Meir Statman, "How Many Stocks Make a Diversified Portfolio?", Journal of Financial and Quantitative Analysis, 22 (September 1987), pp.353-64)

3.1 Diversification and risk: Why is diversification attractive?

Consider the following naive strategy. You randomly choose N stocks listed on the New York Stock Exchange and you allocate your wealth equally across these N stocks, i.e., you build an *equally weighted portfolio* ($x_1 = x_2 = ... = x_N = \frac{1}{N}$). You estimate the standard deviation of this portfolio using historical data. Then you repeat the experience with $N+1$ stocks, etc... What will be the evolution of the risk (standard deviation) of the portfolio as it gets bigger and bigger? Table 7 shows that this risk will decrease with the number of stocks in the portfolio. This illustrates the following principle:

Principle 1: Diversification is attractive because it reduces risk.

Why is this the case? Consider the following example: two securities A and B that have the same variances for their rate of return: $\sigma_A^2 = \sigma_B^2 = 20\%$. We build an equally weighted portfolio that combines securities A and B. Using Result 3 in the case of two stocks, we obtain

$$Var(r_p) = (50\%)^2 \sigma_A^2 + (50\%)^2 \sigma_B^2 + 2(50\%)^2 \sigma_A \sigma_B \rho_{AB}$$

As $\sigma_A^2 = \sigma_B^2$

$$Var(r_p) = 2\sigma_A^2 (50\%)^2 (1+\rho_{AB}) = \frac{\sigma_A^2}{2}(1+\rho_{AB})$$

It is clear that as long as ρ_{AB} is lower than +1 the variance of the portfolio is strictly lower than the variance of asset *A*. This shows that the risk of the portfolio is lower than the risk each individual asset in the portfolio. In this case, diversification has also reduced the risk despite the fact that each asset in the portfolio has the same risk. Furthermore, it is clear that the lower is the correlation ρ_{AB} between the rates of return on securities *A* and *B*, the lower is the risk of the portfolio compared to the risk of each security.

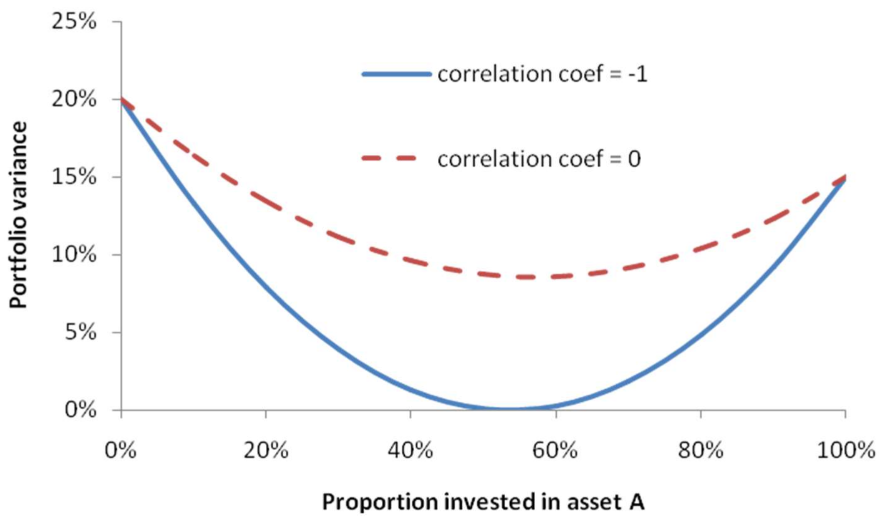
Principle 2: Diversification reduces risk because the correlation between the rates of return of the securities in the portfolio is less than perfectly positive (equal to +1).

3.2 Minimum variance portfolio

We modify the previous example in such a way that securities A and B do not have the same risk:

$$\sigma_A = 15\% \quad and \quad \sigma_B = 20\%$$

Graph 6: Minimum variance portfolio



Graph 6 represents the evolution of the standard deviation of a portfolio that combines securities *A* and *B* when the weight of asset *A* in the portfolio varies in between 0% and 100%, for two different levels of the correlation: $\rho_{AB}=0$ and $\rho_{AB}=-1$. For most weights, the risk of the portfolio is lower than the risk of each asset in the portfolio. As explained previously, this is due to the effect of diversification. More importantly, for each correlation level, there is one combination of securities *A* and *B* that minimizes the risk of the portfolio. For instance when $\rho_{AB}=0$, this combination is

$x_A^{\min} = 64\%$ and $x_B^{\min} = 36\%$. The portfolio with the lowest possible risk among all feasible portfolios is called *the minimum variance portfolio*. How can we compute the minimum variance portfolio for arbitrary standard deviations and correlations for the rates of return?

Result 4: The minimum variance portfolio with two securities *A* and *B* has the following weights:

$$x_A^{\min} = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}$$

$$x_B^{\min} = 1 - x_A^{\min}$$

Remarks:
1) Note that $x_A^{\min}$ or $x_B^{\min}$ can be negative. This means that in order to build the minimum variance portfolio, it may be necessary to short-sell one of the securities.
2) Suppose that $\sigma_B < \sigma_A$. It is easy to check that the minimum variance portfolio never consists in investing only in the asset with the lowest risk (*B*) except in the particular case in which $\sigma_B = \rho_{AB}\sigma_A$. This vindicates the claim that diversification is a way to reduce risk.

Result 5: If the correlation between the rates of return on two assets *A* and *B* is equal to -1, the minimum variance portfolio built with *A* and *B* is riskless. If the correlation between the rates of return of two assets *A* and *B* is equal to $+1$, it is possible to build a riskless portfolio if short sales are allowed.

An important remark: Is standard deviation a good measure of risk? Suppose that the correlation between securities *A* and *B* is equal to -1 and that the standard deviation of security *B* is very large. Is *B* a very risky investment or not? If we only invest in security *B*, the answer is yes. But if we already hold asset *A*, then investing in asset *B* becomes attractive since it can be used to cancel (hedge) the risk of asset *A*. This suggests that when we can invest in a portfolio of securities, the standard deviation of an individual security by itself is not a good measure of its risk. Rather, it is the contribution of the asset to the risk of the portfolio that should be taken into account when evaluating its riskiness. We'll say more on this in the sections below.

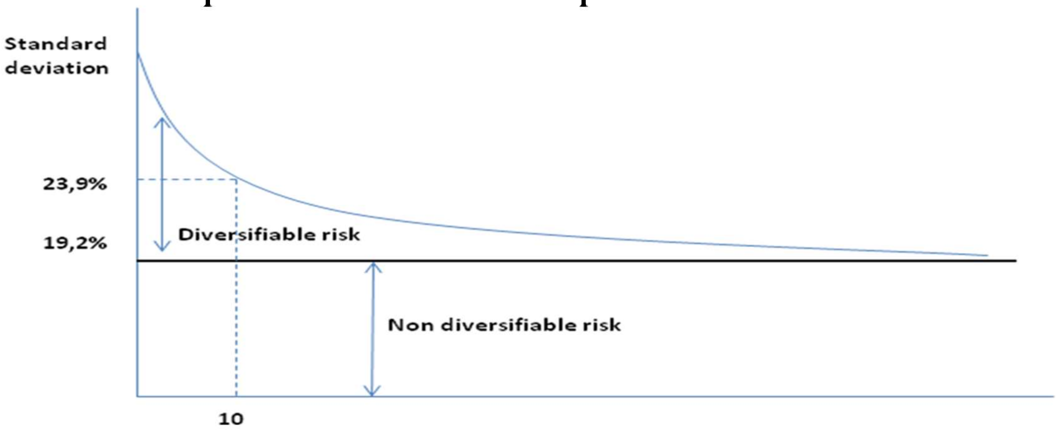
3.3 Why is there a limit to risk reduction with diversification?

Consider again Table 7. When we initially increase the number of stocks in the portfolio from $N = 1$ to $N = 10$, the risk of the portfolio is reduced by about 25 percentage points. However, when we subsequently increase the number of stocks from $N = 10$ to $N = 20$, the risk is reduced by only about 2 percentage points. For portfolios that contain more than 20 stocks, the addition of new stocks to the portfolio has almost no impact on portfolio's risk. In any case, it seems impossible to get a risk lower than 19.2%, whatever the size of the portfolio. This shows that there is a limit to the gains in risk reduction that can be obtained by diversification.

Definition 7: *The risk that can be reduced by adding securities in a portfolio is called* ***diversifiable***

(or non-systematic, or firm-specific, or idiosyncratic) ***risk,*** *whereas the risk that cannot be diversified is called* ***non-diversifiable*** *(or systematic, or market)* ***risk****.*

Graph 7: Number of stocks and portfolio diversification



Why is part of the risk non-diversifiable? We can distinguish two types of factors that affect the rate of return on a security issued by a firm:

1) *Firm-specific factors*: strikes, changes in the management, innovations, etc. These factors are not a source of correlation between the rates of return of securities issued by different firms. As such this source of risk can be diversified and is called *firm specific risk*.

2) *Common factors*: wars, political changes, country-wide economic downturns, etc. These factors affect all the firms in the economy. Thus, they are a source of positive correlation between the rates of return of securities issued by firms. This source of risk cannot be diversified and is called *market risk*.

4 Optimal portfolios

Mutual funds are often classified according to their risk level. Investors can then choose among more or less risky mutual funds. For instance, investors will be offered a choice between a portfolio with a low risk level that contains mainly bonds, and a riskier portfolio that contains mainly stocks. Investors ultimately choose their portfolio according to their willingness to bear risk. But for a given level of risk, they prefer the portfolio that offers the best return prospect. For this reason, portfolio managers strive to build portfolios that offer the greatest possible expected rate of return for a given level of risk. In this section, we explain how to build such portfolios that are called *efficient portfolios*. We also present a simple theory of how investors choose among the different efficient portfolios according to their willingness to bear risk.

4.1 Efficient Portfolios

In this subsection, we consider the problem of building a portfolio that achieves the greatest possible expected rate of return for a given level of risk. We consider different cases. In the first case, we assume that the portfolio manager can only invest in one riskless asset and one risky asset. In the second case, we suppose that the portfolio manager can invest in two risky assets. In the third case, the portfolio manager is allowed to invest in two risky assets and one riskless asset. Finally, we consider the general case in which the portfolio manager can invest in *N* risky assets and one riskless asset. In each of these cases, we proceed in two steps:

- In a first step, we identify the set of all the risk-expected return combinations $(\sigma_p, E(r_p))$ that can be obtained by building portfolios with the available assets. This is the set of all *feasible portfolios*.
- In a second step, among the feasible portfolios, we identify the portfolios that offer the greatest expected rate of return for each level of risk, that is, the set of *efficient portfolios.*

Case 1: One riskless asset and one risky asset

Let r_f be the rate of return on the riskless asset and let $(\sigma_A, E(r_A))$ be the standard deviation and the expected rate of return on the only risky asset *A*. We assume that $E(r_A) > r_f$. Let $E(r_p)$ and σ_p be the expected rate of return and the standard deviation of a portfolio that combines asset A and the riskless asset. What are the possible risk/return combinations, $(\sigma_p, E(r_p))$, that can be obtained?

The variance of a portfolio with weights x_A for security *A* and $(1-x_A)$ for the riskless asset is:

$$Var(r_p) = x_A^2 Var(r_A)$$

It follows that there are two ways to build a portfolio with a given risk level σ_p. Either we invest a proportion $x_A = \frac{\sigma_p}{\sigma_A}$ in the risky asset or we short-sell the risky asset, in which case $x_A = -\frac{\sigma_p}{\sigma_A}$. In the first case, we obtain an expected rate of return which is:

$$E(r_p) = \frac{\sigma_p}{\sigma_A} E(r_A) + (1 - \frac{\sigma_p}{\sigma_A}) r_f = r_f + \lambda \sigma_p$$

with $\lambda = \frac{E(r_A) - r_f}{\sigma_A}$. In the second case, we obtain an expected rate of return which is:

$$E(r_p) = -\frac{\sigma_p}{\sigma_A} E(r_A) + (1 + \frac{\sigma_p}{\sigma_A}) r_f = r_f - \lambda \sigma_p$$

Thus, we have proved the following result.

Result 6: The risk/return combinations, $(\sigma_p, E(r_p))$, that can be obtained with a portfolio that combines a risky asset and the riskless asset must satisfy one of the two following relationships:

$$E(r_p) = r_f + \lambda \sigma_p$$

Or

$$E(r_p) = r_f - \lambda\sigma_p$$

with $\lambda = \frac{E(r_A) - r_f}{\sigma_A}$.

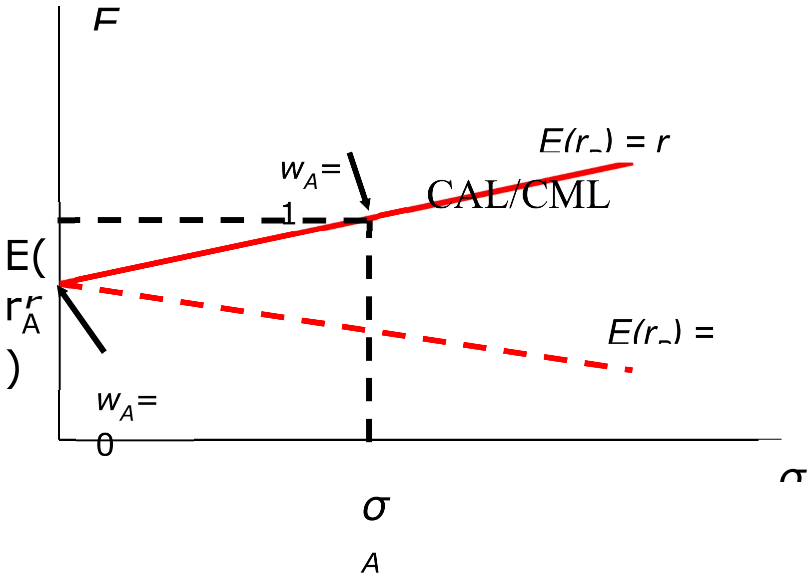
Graphically the set of all risk/return combinations that can be obtained with the riskless asset and the risky asset are two lines, as shown in Graph 8. Note that in order to obtain some combinations, it is necessary to short sell the riskless asset ($x_A > 1$) or the risky asset ($x_A < 0$).

We say that a portfolio is efficient if for a given level of risk, it offers the largest expected rate of return. Inspection of Graph 8 reveals that the set of efficient portfolio is the line with equation

$$E(r_p) = r_f + \lambda\sigma_p .$$

This line is called *the Capital Allocation Line* (CAL), or *Capital Market Line (CML).* This yields the following result. In Graph 8, the CAL is the upper branch (plain line) of the graph.

Graph 8: Set of feasible portfolios with one riskless and one risky asset



Result 7: A risk/return combination *(σ_p, $E(r_p)$)* corresponds to an efficient portfolio if and only if

$$E(r_p) = r_f + \lambda\sigma_p .$$

Thus, we have established a way to check if a portfolio is efficient. It is efficient if and only if its risk/return profile belongs to the CAL. Note that the slope of the CAL (λ) gives the extra expected rate of return that an investor can obtain by increasing by one unit the amount of risk he is willing to bear and by choosing an efficient portfolio. Thus, the slope quantifies the trade-off between risk and return.

Example: Suppose that $E(r_A)$= 11% and that σ_A=15%. The rate of return on the riskless asset is 10%. Build an efficient portfolio with a risk σ_p=18%. What is the expected rate of return of this portfolio?

Solution: The efficient portfolio consists of the asset A and riskless asset. Find the weight of asset A, $\boldsymbol{x_A}$, in the efficient portfolio: 15%^2*$\boldsymbol{x_A}^{\mathbf{2}}$=18^2. Thus, the weight of asset A is 18/15=1.2, and therefore the weight on the riskless asset is -0.2 (the two weights should sum up to 1). As a result, the expected rate of return on the efficient portfolio is 1.2*11%-0.2*10%=11.2%.

Case 2: Two risky assets.

Now, we consider the case in which the portfolio manager can invest in two risky assets A and B . We assume that $E(r_A) < E(r_B)$ and $\sigma_A < \sigma_B$. We start by considering the following example:

$$E(r_A) = 11\% \ \ \textit{and} \ \ \sigma_A = 15\%$$

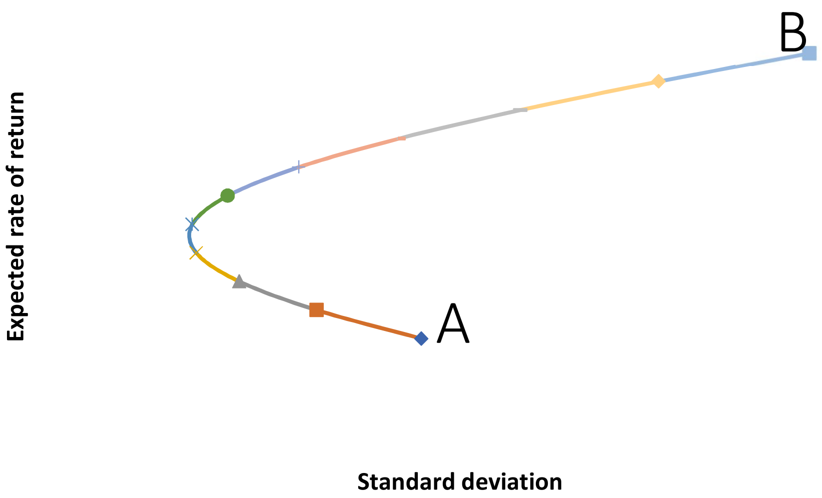
$$E(r_B) = 18\% \ \ \textit{and} \ \ \sigma_B = 20\%$$

Finally we assume that the correlation between the rates of return of the two securities is zero. What are all the possible risk/return combinations that we can obtain by building portfolios combining these two risky securities? In order to study this question, we let the proportion of security *A* in the portfolio vary between 0 and 100%, with increments of 10%. For each proportion, we compute the expected rate of return and the standard deviation of the corresponding portfolio. The outcome of this exercise is reported in Table 8. Using this table, we plot in Graph 9 the risk/return combinations associated with each of the 10 portfolios considered in the table:

Table 8: Portfolios with two risky assets A & B

Proportion of asset A	Expected rate of return	Standard deviation
100%	11,0%	0,150
90%	11,7%	0,136
80%	12,4%	0,126
70%	13,1%	0,121
60%	13,8%	0,120
50%	14,5%	0,125
40%	15,2%	0,134
30%	15,9%	0,147
20%	16,6%	0,163
10%	17,3%	0,181
0%	18,0%	0,200

Graph 9: Set of portfolios with two risky assets A & B



Several remarks are in order:

- The set of all feasible risk/return combinations, *(σ_p, $E(r_p)$)*, is not characterized by two lines but by an hyperbola. This is a general result when portfolio managers can only invest in risky assets.

- Adding security *B* in the portfolio always increases the expected rate of return on the portfolio. More surprisingly, adding asset *B* in the portfolio can reduce the risk of the portfolio. This is the case for all the portfolios with a proportion of asset *A* larger than 64%. This is due to the diversification effect that we discussed in the previous section.

- However, for all the portfolios with a proportion of security *A* less than 64%, it is not possible to increase the expected rate of return without bearing more risk. Note that the portfolio x_A = 64%, x_B = 36% , is the *minimum-variance portfolio*.

What is the set of efficient portfolios in this case? Clearly the portfolios with a proportion in security *A* strictly larger than 64% are not efficient. For each of these portfolios, we can build another portfolio at the same risk level but with a greater expected rate of return. For instance, using Table 7, it can be seen that a portfolio with a proportion x_A = 30% has approximately the same risk as security *A* but a much larger expected rate of return (15.9% instead of 11%). It follows that the portfolios corresponding to the risk-return combinations that belong to the lower branch of the hyperbola cannot be efficient. The efficient portfolios have risk-return combinations that belong to the upper branch of the hyperbola. This branch is called the *efficient portfolio frontier*. Note that the efficient portfolio with the lowest risk is the minimum variance portfolio. Note also that individual risky assets do not necessarily belong to the efficient frontier (it is the case for asset *A* in this example).

Although we have considered an example, the conclusions that we have derived in this section can be generalized. The equations that characterize the set of feasible portfolios and the efficient frontier are more complex than in the previous case. We will not attempt to derive these equations in the general case since they are not necessary for the rest of the course. There is however a case in which it is easy to characterize the efficient frontier, which is the case in which $\rho_{AB} = -1$.

Question: What is the efficient frontier when $\rho_{AB} = -1$?

Case 3: Two risky assets and one riskless asset.

A portfolio that contains risky assets and a riskless asset can be broken in two parts: (i) one portfolio that is only composed of risky assets and (ii) the riskless asset. Consider a portfolio *P* with weights x_A, x_B and x_f in securities *A*, *B* and the riskless asset *f*, respectively. This portfolio can always be seen as made of an investment in a portfolio *R* (that contains only the two risky assets) and the riskless asset. The proportion invested in portfolio *R* is $(x_A + x_B)$. In portfolio *R*, the proportions of securities *A* and *B* are respectively $x_A^R = \frac{x_A}{x_A + x_B}$ and $x_B^R = \frac{x_B}{x_A + x_B}$.

Example: Suppose that r_f = 10% and the risk/return characteristics of securities *A* and *B* are as given in the previous section. Consider the following portfolio: x_A = 12.8%, x_B = 7.2%, and x_f = 80%. The expected rate of return of this portfolio is $E(r_p)$=10.70% and its risk is 2.4%. There is another way to obtain this portfolio which is to invest 20% of one's wealth in a portfolio *R* with weights: $x_A^R = 64\%$ and $x_B^R = 36\%$ and 80% in the riskless asset.

How to construct efficient portfolios when there are N risky assets and one riskless asset? Clearly, the portfolio which is only made of risky securities must belong to the efficient frontier that would be obtained if the portfolio manager were restricted to invest in risky securities only. This remark suggests that the following method can be used to identify the set of efficient portfolios:

- **Step 1:** Identify the set of efficient portfolios when the portfolio manager can only invest in risky assets.

- **Step 2:** In this set, choose the portfolio *T* that is such that when it is combined with the riskless asset, the resulting portfolio offers the greatest expected rate of return, for all possible risk levels.

The remaining issue is to identify *T*. Consider again the example above. We know from the analysis of Case 1 that if we combine the minimum variance portfolio and the riskless asset, the optimal risk/return combinations belong to a line such that:

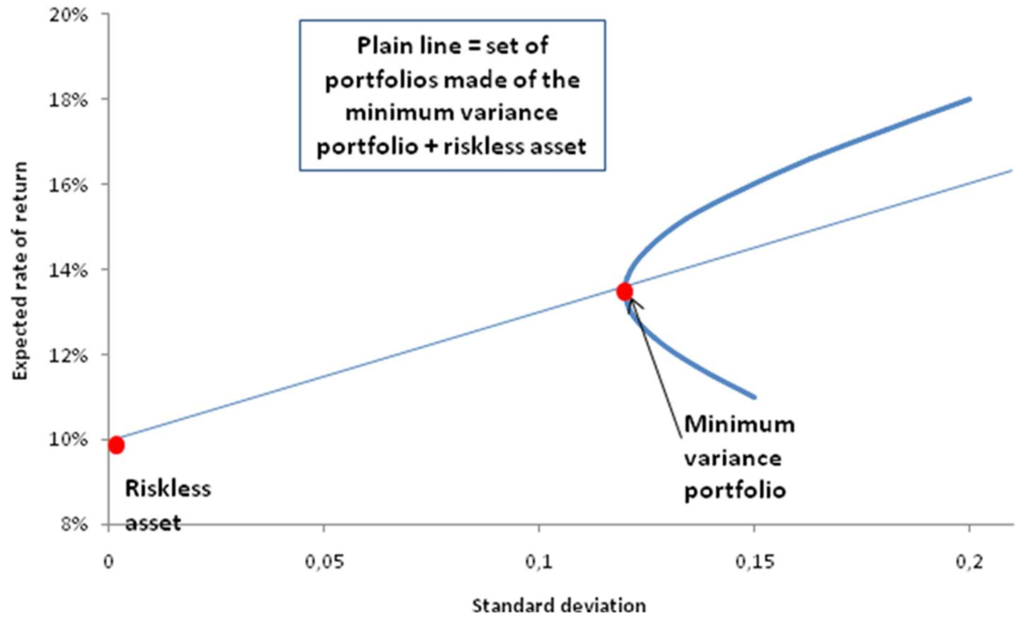
$$E(r_p) = r_f + \lambda\sigma_p$$

with $\lambda = \frac{E(r^{\min}) - r_f}{\sigma^{\min}}$ where $E(r^{\min})$ and $\sigma^{\min}$ are the expected rate of return and the standard deviation of the rate of return of the minimum variance portfolio. In the example above, we obtain $E(r_{min})$ = 13.52% and σ^{min} = 11.58%. It follows that λ = 0.30. The risk/return combinations that can be obtained by mixing the riskless asset and the minimum variance portfolio are plotted on Graph 10.

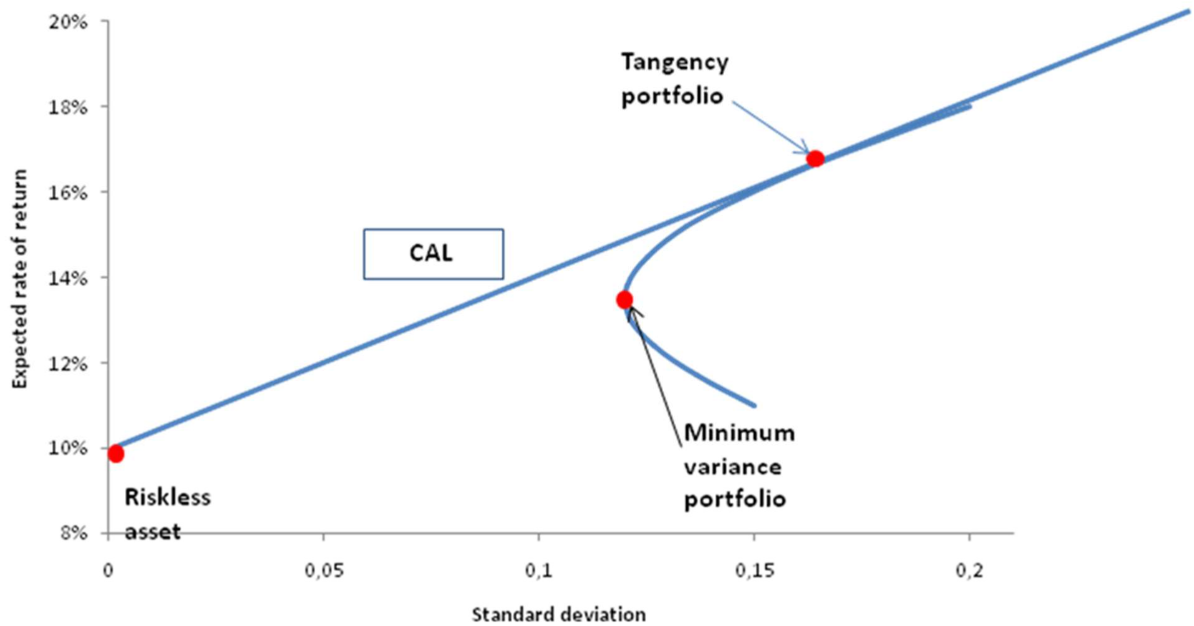
Clearly the choice of the variance minimum portfolio is not the best possible one. For instance, for a risk level equal to σ_p = 13.4%, this strategy yields an expected rate of return equal to 14.02%. But Table 8 (and Graph 10) reveals that for the same risk level, there exists a portfolio that combines only the two risky assets and which achieves an expected rate of return equal to 15.2%. Thus, the minimum variance portfolio is not the portfolio *T* that we are looking for.

The previous reasoning shows that the portfolio *T* should be chosen in such a way that the slope of the line depicting all the risk/return combinations obtained by mixing portfolio *T* and the riskless asset is at its maximum. It follows that *T* is at the point of tangency between the efficient frontier obtained by investing only in risky assets (the upper branch of the hyperbola) and a line whose intercept is the rate of return on the riskless asset. This is illustrated on Graph 11. *T* is called *the tangency portfolio*.

Graph 10: Minimum variance portfolio and the riskless asset



Graph 10: Capital allocation line with two risky assets and one riskless asset



In this example, the tangency portfolio is characterized by:

$$E(r_T) = 16.68\% \text{ and } \sigma_T = 16.5\%$$

A portfolio that is a combination of the riskless asset and the tangency portfolio, *T*, is efficient: it is not possible to find a portfolio with a larger expected rate of return for the same risk level. Furthermore, a risk/return combination belongs to the efficient frontier if and only if:

$$E(r_p) = r_f + \lambda_T \sigma_p$$

with $\lambda_T = \frac{E(r_T) - r_f}{\sigma_T}$. Note that the efficient frontier is a line, which is again called the *Capital Allocation Line*. In the example above the slope of this line is $\lambda_T = 0.4$

To sum up:

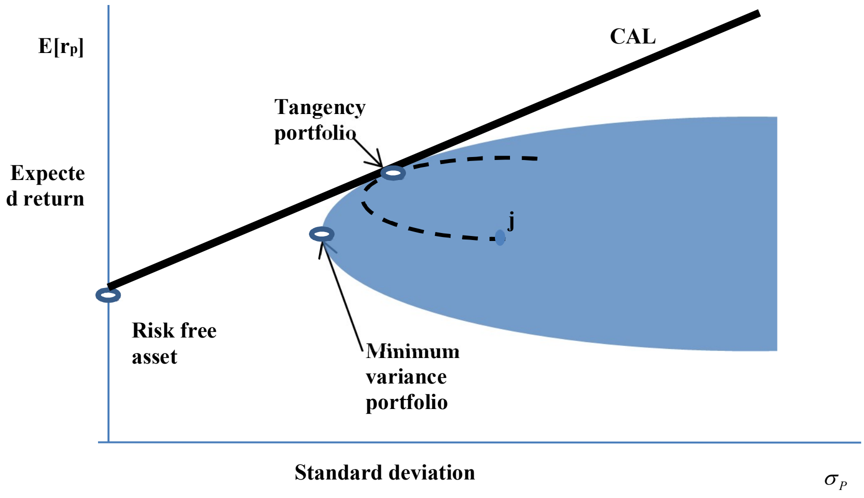
1) When portfolio managers can invest in a riskless asset the efficient frontier is a line. The intercept of this line is the rate of return on the riskless asset and the slope of this line is $\lambda_T = \frac{E(r_T) - r_f}{\sigma_T}$, where *T* designates the tangency portfolio.

2) A portfolio is efficient if and only if it is a combination of the riskless asset and the tangency portfolio T.

Case 4: N risky assets and a riskless asset.

With *N* risky assets and one riskless asset, the results and the conclusions that have been obtained previously are still valid. In particular the efficient frontier is a line which is tangent to the efficient frontier that would be obtained, were the portfolio manager restricted to invest in the *N* risky assets only. The set of all return-risk combinations that can be obtained by restricting to all possible portfolios that do not contain the risk-free asset is represented by a hyperbola and its interior (that is the shaded area in Graph 12). As in the previous case, the tangency portfolio is the point of tangency between the efficient frontier obtained by investing only in risky assets and a line whose intercept is the rate of return on the riskless asset. Any efficient portfolio will be composed of this tangency portfolio (with non-negative weights) and the risk-free asset.

Graph 12: CAL with N risky assets and one riskless asset



We close this subsection by deriving an important property of the tangency portfolio. This property will play a crucial role when we will derive the *Capital Asset Pricing Model* in the next chapter.

Result 8: **A property of the tangency portfolio.** The expected rates of return on a security, j, and on the tangency portfolio satisfy the following relationship:

$$E(r_j) = r_f + \beta_j (E(r_T) - r_f) \qquad (*)$$

$$\text{with } \beta_j = \frac{\text{cov}(r_j, r_T)}{Var(r_T)} \ .$$

Proof (*advanced material*): Consider a risky asset j and the portfolio obtained by combining the tangency portfolio with asset j. Let α be the weight of the tangency portfolio and (1-α) be the weight of asset j. The expected return of this portfolio and its standard deviation will be

$$E[r_P] = \alpha E[r_T] + (1-\alpha) E[r_j]$$
$$\sigma_P = \sqrt{\alpha^2 \sigma_T^2 + (1-\alpha)^2 \sigma_j^2 + 2\alpha(1-\alpha)\,\text{cov}(r_j, r_T)}$$

Portfolios built in this way do not contain the riskless asset and the risk/return combination that can be achieved by varying α is a hyperbola that passes through asset j (for $\alpha=0$) and the tangency portfolio T (for $\alpha=1$). This hyperbola is represented in Graph 12. Moreover, this hyperbola must be tangent to the Efficient Portfolio Frontier at point T. If not, one could build a portfolio that does not contain the riskless asset and that is above the efficient portfolio frontier, clearly a contradiction. In other words, the following relation will be true:

Slope of the Capital Allocation Line = Slope of the hyperbola at T

Considering that

$$\text{Slope of the CAL} = \frac{E[r_T]-r_f}{\sigma_T}$$

$$\text{Slope of the hyperbola in } T = \left.\frac{\partial E[r_P]/\partial\alpha}{\partial\sigma_P/\partial\alpha}\right|_{\alpha=1}$$

and that

$$\partial E[r_P]/\partial\alpha = E[r_T]-E[r_i]$$

$$\partial\sigma_P/\partial\alpha = \frac{\alpha\sigma_T^2+(1-\alpha)\sigma_j^2+(1-2\alpha)\operatorname{cov}(r_j,r_T)}{\sqrt{\alpha^2\sigma_T^2+(1-\alpha)^2\sigma_j^2+2\alpha(1-\alpha)\operatorname{cov}(r_j,r_T)}},$$

the fact that the hyperbola must be tangent to the Efficient Portfolio Frontier at point T implies the following equality:

$$\frac{E[r_T]-r_f}{\sigma_T}=\frac{\sigma_T(E[r_T]-E[r_i])}{\sigma_T^2-\operatorname{cov}(r_j,r_T)}$$

Multiplying both sides by $\sigma_T(\sigma_T^2-\operatorname{cov}(r_j,r_j))$ and rearranging, we obtain equation (*).

4.2 Mean-Variance Investors

Assume that:

- **H.1:** Investors care only about the expected rate of return of their portfolio and the risk of this portfolio.

- **H2:** For a given level of risk, they prefer the portfolio with the largest expected rate of return.

- **H3:** For a given expected return, they prefer the portfolio with the lowest risk.

Investors that behave in this way are said to have *mean-variance preferences*.

The question is then: in the set of efficient portfolios, which is the portfolio that will be chosen by a given mean-variance investor? It is likely that the answer will depend on the willingness of the investor to bear risk. As this willingness is different from investors to investors, they will choose different portfolios.

In order to investigate portfolio choice by investors further, we assume that investors seek to maximize the following objective (utility) function in selecting their portfolio:

$$U(\sigma_p,E(r_p))=E(r_p)-\frac{A}{2}\sigma_p^2$$

It can be checked that an investor who chooses his portfolio in order to maximize this utility function behaves as a mean-variance investor. Now suppose that the investor already holds a portfolio *P* that offers a risk/return combination $(\sigma_p, E(r_p))$. We offer the investor the possibility to choose between this portfolio and a portfolio *P'* that has a risk that is larger by one unit. What is the extra expected return ΔE that must be offered to the investor to make him indifferent? We can show that:

$$\Delta E = A\sigma + \frac{A}{2}$$

Thus the larger is *A*, the larger is the compensation that must be offered to the investor to bear one extra unit of risk. In this sense *A* measures the investor's willingness to bear risk. This coefficient is called *the risk aversion coefficient*. The risk aversion coefficient represents an individual's preference in risk taking, and it varies across individuals and may change during an individual's life.

Result 9: An investor with a mean-variance objective function and a risk aversion *A* will choose an efficient portfolio such that the proportions of his wealth invested in the tangency portfolio and the riskless asset are respectively:

$$x_T(A) = \frac{E(r_T) - r_f}{A\sigma_T^2}$$

$$x_f(A) = 1 - x_T(A)$$

Note that the larger is the risk aversion, the larger is the proportion that will be invested by the investor in the riskless asset. Note also that the larger is the risk premium on the tangency portfolio, the lower is the proportion invested in the riskless asset by the investor. This confirms the intuition that higher risk premia induce investors to bear more risk.

Question: Consider two investors 1 and 2 with risk aversions $A_1 > A_2$. Position the portfolios chosen by these investors on the capital market line in Graph 11.

According to the previous result, investors will choose different portfolios. However the fact that all mean-variance investors choose *efficient* portfolios has an important implication that we derive now. Suppose that all investors have the same forecasts on the expected rates of returns, correlations and standard deviations of the risky assets in the economy. In this case, the tangency portfolio and the Capital Allocation line are *identical* for all investors. Consequently all investors choose portfolios that are combinations of the riskless asset and the *same* tangency portfolio. It follows that the composition of the portfolio that is only made of risky assets in an investor's portfolio is identical for all investors: this is the tangency portfolio. This is one of the results of *the Capital Asset Pricing Model* (CAPM) that we study in the next chapter.

IV - The Capital Asset Pricing Model

1 Computing the price of a financial asset.

You consider buying one share of IBM that you will keep for one year. You expect the price of IBM in one year to be $E(P_{IBM})$=\$200 and you expect a dividend of $E(D)$=\$10 at that point in time. The price, P_0, you are willing to pay today for this share is the present value of the cash flow you expect to receive in one year. Let k be the discount rate:

$$P_0 = \frac{E(P_{IBM}) + E(D)}{1+k}$$

The discount rate must reflect the risk of investing in IBM. Note that the discount rate is just the expected rate of return. Thus, in order to value IBM stock, we need to be able to (a) quantify the risk of this security and (b) deduce from this risk, the expected rate of return on IBM for this stock to be fairly valued. This is exactly what is achieved by the *Capital Asset Pricing Model*. The CAPM is a model that provides the equilibrium expected rate of return on each security (or equivalently, as shown by the previous equation, its equilibrium price) as a function of the risk of the security. In fact, the CAPM shows that under certain assumptions, the equilibrium expected rate of return on (say) security j satisfies the following relationship:

$$E(r_j) = r_f + \beta_j (E(r_m) - r_f) \qquad (**)$$

where $\beta_j = \frac{\text{cov}(r_j, r_m)}{Var(r_m)}$ and r_m is the rate of return on a specific portfolio called *the market portfolio* that we will describe later on.

The CAPM is an equilibrium model, i.e., the prices of securities (or their expected rates of return) adjust so that the demand for financial assets is equal to the supply. Note that (**) is similar to Equation (*) in Chapter III on portfolio management. There is a very important difference, however. In (*), the portfolio that appears on the right hand side of the equation is the tangency portfolio and the relationship just derives from the definition of the efficient frontier. In contrast, in (**), the portfolio on the right hand side is the market portfolio. It turns out that *in equilibrium*, the market portfolio and the tangency portfolio are identical. Proving this result is in fact the gist of the derivation of the CAPM relationship, which has been developed by Sharpe and Lintner in the 1960s.

1.1 Assumptions

The CAPM rests on several assumptions that we describe below:

- **H.1** Investors behave competitively, that is, no investor is sufficiently large to influence security prices (expected returns) by his/her own demand for securities.

- **H.2** Investors have mean-variance preferences.

- **H.3** Investors have homogeneous expectations, i.e., their forecasts regarding expected rates of return, standard deviations of and correlations between rates of return are identical.

- **H.4** There are no transaction costs (e.g., no fees are paid to brokers or dealers in order to execute orders on the stock exchange).

- **H.5** There is a riskless asset and it is in zero net supply.

- **H.6** Short-sales are allowed

- **H.7** The model has only two dates: $t = 0$ (portfolios are formed and asset prices adjust to balance supply and demand) and $t = 1$ (assets returns are realized).

Note that in order to compute the expected rates of return that are such that demand is equal to supply, we need to explain how investors determine their holdings (demand) in the different securities. The model of portfolio management that we have analyzed in the previous lecture proposes such an explanation. The assumptions of the CAPM guarantee that we can use the results from portfolio theory to derive investors' holdings in financial securities. Note that the demand of a given security *j* by investor *i* can be described by the number of shares of security *j* the investor wants to acquire *or* by the proportion of the wealth that the investor wishes to invest in the security. For a mean-variance investor, the Portfolio Theory tells us the proportion of his/her wealth each investor should invest in security *j*, if he/she were to invest optimally, which he/she would under CAPM assumptions. We will explain the meaning of *zero-net supply* in the next subsection.

1.2 Market Capitalization.

Definition 1: *Let S_j be the number of shares of security j that have been issued in the financial market and let P_j be the price of security j. The market capitalization K_j of security j is:*

$$K_j = S_j * P_j$$

Example:

Stocks	IBM	ITT
Price	105	90
Number of outstanding shares (million)	1	2
Market Capitalization (USD millions)	105	180

Suppose that there are *N* risky securities listed on the financial market. The total market capitalization is just the sum of all individual security capitalizations. We can describe the supply of a given security *j* by the number of outstanding shares for this security or by the proportion w_j^m of the total market capitalization that the total value of these shares represents, that is:

$$w_j^m = \frac{K_j}{\sum_j K_j}$$

Definition 2: *The market portfolio is the portfolio with weights* $\{w_j^m\}_{j=1}^{j=N}$.

Example: If IBM and ITT are the only two securities traded in the financial market, the composition of the market portfolio is $w_{IBM}^m = 36.84\%$ and $w_{ITT}^m = 63.16\%$.

A security is in **zero net supply** when its proportion in the total market capitalization is zero. It means that each share of this security that is held by an investor has been sold short by another investor (and not issued by a firm). Note that CAPM assumes that the riskless asset is in zero net supply. This means that the riskless asset can be interpreted as the vehicle that is used by investors to lend/borrow money. Lending money is equivalent to purchasing the riskless asset whereas borrowing money is equivalent to short selling the riskless asset (why?).

Figure 5: Top 10 market capitalizations in Jan. 2000 and in July 2022

Rank	Company	Market Cap (Jan 1, 2000)
#1	Microsoft	\$606 billion
#2	General Electric	\$508 billion
#3	NTT Docomo	\$367 billion
#4	Cisco	\$352 billion
#5	Walmart	\$302 billion
#6	Intel	\$280 billion
#7	Nippon Telegraph	\$271 billion
#8	Nokia	\$219 billion
#9	Pfizer	\$206 billion
#10	Deutsche Telekom	\$197 billion

Rank	Company	Market Cap (Jul 1, 2022)
#1	Saudi Aramco	\$2.27 trillion
#2	Apple	\$2.25 trillion
#3	Microsoft	\$1.94 trillion
#4	Alphabet	\$1.43 trillion
#5	Amazon	\$1.11 trillion
#6	Tesla	\$707 billion
#7	Berkshire Hathaway	\$612 billion
#8	United Health Group	\$485 billion
#9	Johnson & Johnson	\$472 billion
#10	Tencent	\$435 billion

Source: Visualcapitalist.com

1.3 The CAPM

In this section, we show that if the assumptions described in Section 1.1 of this chapter are satisfied, then the equilibrium expected rate of return on each security satisfies the relationship described by (**). To this end, we prove that when the expected rates of return adjust so that the supply of each security is equal to its demand, then the market portfolio **is** the tangency portfolio: that is, the market portfolio is the only portfolio that is made exclusively of risky securities and that belongs to the Capital Allocation Line.[2]

Result 1: When the financial market is in equilibrium (supply is equal to demand for each security), then the market portfolio and the tangency portfolio are identical, that is:

$$w_j^T = w_j^m \quad \forall j \in \{1,..,N\}$$

We prove this result in two steps. First, if all investors are holding the same optimal risky portfolio, then that portfolio should equal the market portfolio. This is due to the fact that the aggregate demand for stocks should equal the aggregate supply of stock – in other words, the market is nothing more than the aggregate of all individual portfolios. Second, under CAPM assumptions, the optimal risky portfolio that investors should pick is the tangency portfolio. Hence, the market portfolio is nothing but the tangency portfolio.

Result 2: In equilibrium, the expected rates of return on a security j and on the market portfolio satisfy the following relationship:

$$E(r_j) = r_f + \beta_j (E(r_m) - r_f)$$

$$\text{with} \quad \beta_j = \frac{\mathrm{cov}(r_j, r_m)}{Var(r_m)} \quad .$$

Note: The coefficient β_j is called *the Beta of security j.*

Interpretation:
Note that the CAPM equation can be written:

$$E(r_j) - r_f = \beta_j (E(r_m) - r_f)$$

Thus, the expected risk premium on a security is proportional to its *Beta*. This makes clear that according to the CAPM, cross sectional variations in the average risk premia offered by financial assets are entirely due to variations in their Betas. This means that the *Beta* of a security is a measure of its risk. In equilibrium, investors require a lower risk premium to hold assets with low Betas than assets with high Betas. In fact, a security with a Beta equal to zero must offer the same expected rate of return as the riskless asset, even though the rate of return on this security might be

[2]Note the Capital Allocation Line is also often called the Capital Market Line (CML).

uncertain.

This last point shows again that the standard deviation of the rate of return of a security is not a good measure of the risk of this security. The reason is that part of the risk measured by the standard deviation can be diversified by holding the market portfolio. The agents do not need to be compensated for this diversifiable risk. In contrast, they require a compensation for holding the market portfolio, whose risk is not diversifiable. The *Beta* of a security measures the contribution of this security to the risk of the market portfolio. For this reason, *Beta* is said to be a measure of the **systematic risk** or **market risk** (i.e., the non-diversifiable risk) of a security.

Question: What is the *Beta* of a security with a rate of return that has a negative correlation with the rate of return on the market portfolio? What is the sign of the risk premium on such a security? Why?

Note that the CAPM applies not only to individual securities but also to portfolios of securities. There is a simple relationship between the *Beta* of a portfolio and the *Betas* of the securities in that portfolio:

Result 3: Consider a portfolio with weights $\{x_j\}_{j=1}^{j=N}$. The beta, β_p, of this portfolio is:

$$\beta_p = x_1\beta_1 + ... + x_j\beta_j + ... + x_N\beta_N$$

In equilibrium, the tangency portfolio and the market portfolio are identical. This means that the equation of the Capital Market Line (CML) is:

$$E(r_p) = r_f + \lambda_m \sigma_p$$

where $\lambda_m = \frac{E(r_m) - r_f}{\sigma_m}$. One must be cautious not to mix the CAPM relationship which is given in Result 2 and the previous equation. The CAPM relationship applies to any security, in particular to any portfolio whether the portfolio is efficient or not. In contrast, only the risk/return combinations of efficient portfolios satisfy the equation of the Capital Market Line.

Example: Consider again the example with securities *A* and *B* that we used previously (see pages 43--48). Suppose that the expected rates of return of securities *A* and *B* are such that the financial market is in equilibrium. It follows that the tangency portfolio is the market portfolio. We obtain that:

$$\beta_A = \frac{\text{cov}(r_A, r_m)}{Var(r_m)} = \frac{\text{cov}(r_A, r_T)}{Var(r_T)} = \frac{x_A^T \sigma_A^2}{Var(r_T)}$$

The last equality follows from the fact that $\rho_{AB} = 0$. It follows that $\beta_A = 0.16$ and it can be checked that the CAPM relationship is satisfied, that is:

$$E(r_A) = 11\% = 10\% + 0.16 \times (16.68\% - 10\%)$$

Furthermore, the equation of the Capital Market Line is:

$$E(r_p) = 10\% + 0.4\sigma_p$$

It can be checked that for security *A*, this equation is not satisfied since holding security *A* alone is not efficient. Similar computations can be performed for security *B*.

1.4 The Security Market Line

Definition 1: *The Security Market Line (SML) is the line with equation:*

$$E(r_j) = r_f + a\beta_j$$

with $a = (E(r_m) - r_f)$.

This equation expresses the fact that the expected rate of return on a security and its risk are linearly related according to the CAPM. In the previous example, $a = (E(r_m) - r_f) = 6.68\%$ and therefore the equation of the security market line is:

$$E(r_j) = 10\% + 6.68\%\beta_j$$

Suppose that you have data on the rates of return of individual securities over several months or years. Could you try to use this dataset to confirm or invalidate the CAPM? There are (at least) three important predictions of the CAPM:

1) There is a linear relationship between the average rate of return on a security and its risk, as measured by its beta.

2) The slope of the security market line is the average risk premium on the market portfolio.

3) The intercept of the security market line is the rate of return on the riskless asset.

These predictions are interesting because they offer a way to test the validity of the CAPM using data on rates of return. If one of the predictions is not confirmed empirically, then one of the assumptions of the CAPM must be wrong (e.g., investors do not have the same forecasts) and the CAPM is not likely to be a good asset pricing model.

Empirical tests of these predictions have been performed by many studies (e.g. ,Black, Jensen and Scholes (1972)) using regression analysis. In general, in these tests the following regression equation is estimated:

$$R_j = \gamma_0 + \gamma_1\hat{\beta}_j + \varepsilon_j$$

where $\hat{\beta}_j$ is an estimation of the Beta of security *j* (the Beta of a security cannot be directly observed).

The CAPM predicts that:

$$H_0 \quad : \quad \gamma_0 = r_f$$

$$H_0^{'} \quad : \quad \gamma_1 = \bar{r}_m - r_f$$

where r_f is the rate of return on the riskless asset (this can be observed) and $\bar{r}_m$ is the average rate of return on the market portfolio during the period of observation.

A main difficulty for testing the predictions associated with the CAPM is that the market portfolio cannot be observed directly. In practice, researchers have used an "approximation" of the market portfolio such as an overall stock market index. Stock indexes measure the overall level of prices in the stock market. Indexes are themselves portfolios that are made of stocks representative of an industry, a type of stocks (large caps, small caps) or a national economy. For the purpose of proxying the market portfolio, the important point is that these indexes are well diversified portfolios of stocks. Table 9 provides the names of some famous stock indexes in the

Table 9: Major stock indices by country

Country	Stock index
USA	Dow Jones, S&P500, Nasdaq Comp
Japan	Nikkei
UK	FTSE30, FTSE100
France	CAC40

Since the 1970s a lot of research has focused on the ability of the CAPM to predict prices and returns. It has been shown that the market risk is not the only risk factor. Other risk factors related to firm size and book-to-market should be used to price stocks more accurately. See for instance *"Common risk factors in the returns on stocks and bonds"*, Fama E., French K., Journal of Financial Economics, 1993.

2 Applications of the CAPM

The purpose of this section is to describe briefly some of the practical applications of the CAPM.

2.1 Discount rates in valuation problems

Let us come back to the initial example in this chapter and assume that you have the following additional information: β_{IBM} =1.3, r_f = 10% and the slope of the Security Market Line has been estimated to be 1.08% With this additional information, we can now determine the current price of one share of IBM. Actually, according to the CAPM, the expected rate of return on IBM must be:

$$k = 10\% + 1.08\% * 1.3 = 11.4\% \ .$$

It follows that the equilibrium price of IBM must be:

$$P_0 = \frac{210}{1.114} = \$188.5$$

If the price were to be lower, then the market for IBM shares would not be in equilibrium (or the CAPM assumptions are wrong). Demand would be larger than supply and as a result we would expect the price to adjust upward. Conversely, if the price were to be higher, supply would be larger than demand and we would expect the price to decrease.

Thus, a main interest of the CAPM is to provide a way to choose discount rates given the risk (measured by *Beta*) of the investment. Note that this applies to capital budgeting problems as well. We illustrate this point with the next example.

2.2 Portfolio management

Recall that a portfolio is efficient if and only if it is a combination of the tangency portfolio and the riskless asset. Since the market portfolio and the tangency portfolio are identical in equilibrium, the CAPM has an important implication for portfolio management. Namely, by investing parts of his wealth in the market portfolio on the one hand and in the riskless asset, on the other hand, an investor is sure to obtain a portfolio that is efficient. The exact combination that the investor will choose depends on his/her desired risk level.

Example: Consider again the example with securities *A* and *B*. Now consider an investor who wants to build an efficient portfolio with a risk σ_p = 10%. As explained above, he should allocate a proportion x_m of his wealth in the market portfolio and the remaining in the riskless asset, in such a way that:

$$Var(r_p) = x_m^2 Var(r_m) = (10\%)^2$$

.

It follows that x_m = 10%/16.5% = 0.6, that is by investing 60% of his wealth in the market portfolio, the investor will build a portfolio which has the desired risk level.

Thus, the CAPM calls for a *passive investment strategy*. Actually, combining the market portfolio and the riskless asset is particularly easy and it does not require active management of the portfolio. In fact, the previous reasoning vindicates the use of indexing strategies by portfolio managers. An *indexing strategy* consists of building a portfolio that has the same weights as a specific stock index (e.g., Dow Jones index, Nasdaq composite index, Nikkei index, CAC40, etc.. If the chosen index is a good proxy of the market portfolio, investors can then just combine the index fund and the riskless asset in order to achieve an efficient portfolio. As an empirical matter, indexing has historically performed better than most actively managed portfolios. Moreover, it requires less transactions than an actively managed fund and consequently involves smaller transaction costs. Finally, it does not require looking for underpriced securities and this reduces portfolio management costs.

Another benefit of the CAPM is that it provides useful benchmarks to measure the performance of portfolio managers. The issue is the following: let us assume that you observe the annual rates of return of two portfolio managers M_1 and M_2 over, say, the last 6 years. You estimate the average rate of return on their portfolios and you obtain $\bar{r}_1 > \bar{r}_2$. Which of the two managers is the better performer?

According to portfolio theory, this question cannot be answered without controlling for the risk of the portfolio. Actually, we know that there is a trade-off between risk and return: greater expected rates of return can be obtained at the price of investing in riskier portfolios. This means that in assessing portfolio managers we must first control for the risk of their portfolios. Now using past data, you estimate the risk of the portfolio chosen by M_1 to be σ_1 and the risk of the portfolio chosen by M_2 to be σ_2 with $\sigma_1 > \sigma_2$. We cannot tell whether M_1 is better at managing portfolios than M_2 since the former has obtained a larger average rate of return than the latter only at the cost of taking more risk.

The CAPM provides, however a simple benchmark against which we can compare the performance of each portfolio manager. This benchmark is the portfolio that is obtained by adopting a passive indexing strategy and that has the same risk as the portfolio of the manager we want to evaluate. This benchmark portfolio is efficient and can be obtained without any special skills in portfolio management. In order to justify his management fees, a portfolio manager must **outperform** the benchmark portfolio, that is he/she must provide a portfolio that has an expected rate of return above the CML for a given level of risk. In this case the portfolio manager is said to "beat the market". Benchmarking is a widespread technique to evaluate portfolio managers. Note that the only way for a portfolio manager to beat the market is to develop superior skills in predicting future returns (maybe in developing close relationships with firm management, etc.).

Example: Consider the environment with securities A and B and suppose that in this environment, you observe the average rate of return for M to be 12% for a level of risk equal to 10%. Has the portfolio manager outperformed or underperformed the market?
Often institutional investors such as pension funds use several portfolio managers, each of whom is specialized only in part of the whole portfolio. The performances of these specialized managers should be compared to another benchmark: the SML. According to the CAPM, every portfolio has a risk premium that is equal to its beta times the risk premium on the market portfolio (Result 2). The difference between the average return on a portfolio and its theoretical level provided by the SML is called the Alpha. Specialized managers that are able to consistently produce portfolios with positive alphas are judged superior independently from the fact that their portfolios beat the market when compared to the CML. In other words, a portfolio that has a positive Alpha can lie below the CML. Still it can be combined with the market portfolio and the risk free asset to build a total portfolio that outperforms the CML and thus beats the market.

Glossary

Accounts payables	Dettes fournisseur	Account receivables	Créances clients
Acquisition	Acquisition	Counterparty	Contrepartie
Actuaries	Actuaires	Coupon bond	Obligation à coupons
Adverse selection	Sélection adverse	Credit risk	Risque de crédit
Agency theory	Théorie de l'agence	Currency	Devise
Alpha	Alpha	Currency market	Marché des devises
Amortization	Amortissement des actifs immatériels	Current yield	Rendement actuel
Annuity	Annuité	Default risk	Risque de défaut
Asset	Actif	Depreciation	Amortissement des actifs matériels
Asset allocation	Allocation des actifs	Derivatives	Produits dérivés
At par	A parité	Discount	Décote
Beta	Beta	Discount rate	Taux d'escompte
Bond market	Marché obligataire	Discounted cash-flows	Cash-flows actualisés
Book value	Valeur comptable	Discounted dividends	Actualisation des dividendes
Break-even point	Point mort	Diversifiable risk	Risque diversifiable
Broker	Courtage	Diversification	Diversification
Call	Option d'achat	Diversification principle	Principe de diversification
Cap	Plafond	Dividend policy	Politique de dividende
Cap ; Interest rate cap	Taux plafond	Dividend yield	Rendement d'une action
Capital allocation line	Droite de marché	Duration	Duration
Capital asset pricing model (CAPM)	Modèle d'évaluation des actifs financiers	Earnings per share (EPS)	Bénéfice par action (BPA)
Capital gain	Plus-value	Effective annual rate	Taux d'intérêt effectif
Capital loss	Moins-value	Efficient frontier	Frontière efficiente
Capital market	Marché de capitaux	Efficient market hypothesis	Hyp. d'efficience des marchés
Capital market line	Droite de marché	Efficient portfolio	Portefeuille efficient
CAPM	MEDAF	Efficient portfolio frontier	Frontière efficiente
Cash dividend	Dividende en numéraire	Equity	Fonds propres
Cash settlement	Règlement en espèces	Exchange rate	Taux de change
Cash-flow	Flux de trésorerie	Exercise Price ; Strike Price	Prix d'exercice
Certficate of Deposit	Certificat de dépôt	Expected return	Espérance de rentabilité
Clearing ; Settlement	Compensation	Expiration date	Date d'échéance
Collateral	Collatéral	External financing	Financement externe
Commercial bank	Banque commerciale	Face value	Valeur nominale
Compounded interest	Intérêt composé	Financial guarantee	Garantie financière
Compounding	Capitalisation	Financial system	Système financier
Confidence interval	Intervalle de confiance	Fixed income instruments	Titres à revenu fixe
Constant annuity	Annuité constante (pour un emprunt)	Floor ; Interest rate floor	Taux plancher
Correlation	Corrélation	Forward contract	Contrat à terme
Cost of capital	Coût du capital	Forward price	Cours à terme
Cost of equity	Coût des capitaux propres	Frequency (of a bond)	Fréquence de versement d'intérêts

Fundamental value	Valeur fondamentale	Nominal	Nominal (adjectif)
Future contract	Contrat de future	Nominal future value	Valeur future nominale
Future value	Valeur future	Nominal interest rate	Taux d'intérêt nominal
Futures markets	Marché à terme	Nominal price	Prix nominal
Growth annuity	Rente croissante	Non diversifiable risk	Risque non diversifiable
Growth stocks	Valeurs de croissance	Normal distribution	Loi normale
Hedge (to) a risk	Couvrir (se) contre un risque	NPV	VAN
Hedge Fund	Fonds d'arbitrage	Opportunity cost of capital	Coût d'opportunité du capital
Hedging	Couverture	Optimal combination of risky assets	Combinaison optimale d'actifs risqués
Holding Period Return	Rendement sur la période de détention	Option	Option
Human capital	Capital humain	Options markets	Marché d'options
Immediate annuity	Annuité immédiate	Ordinary annuity	Annuité ordinaire
Intangible assets	Actifs immatériels	Organized market	Marché organisé
Interest rate arbitrage	Arbitrage de taux d'intérêt	Payback period	Période de retour sur inv.
Internal financing	Financement interne	Perpetuity	Perpetuité
Internal rate of return (IRR)	Taux interne de rentabilité (TIR)	Portfolio	Portefeuille
Inventories	Stocks	Portfolio management	Gestion de portefeuille
Investment bank	Banque d'affaire	Portfolio selection	Choix de portefeuille
Investment fund	Fonds d'investissement	Premium	Prime
IRR	TRI	Present value	Valeur actuelle
Issue price	Prix d'émission	Price-earnings ratio	Taux de bénéfice par action
Liabilities + Equity	Passif	Primary market	Marché primaire
Life annuities	Rente viagère	Principal-agent problem	Conflit principal-agent
Limited liability	Responsabilité limitée	Probability distribution	Distribution de probabilité
Liquidity	Liquidité	Rate of return	Taux de rentabilité
Long (to be)	Long (être)	Real future value	Valeur future réelle
Long position	Position longue	Real interest rate	Taux d'intérêt réel
Margin call	Appel de marge	Real price	Prix réel
Market capitalization rate	Exigence de rentabilité	Redemption value	Valeur de remboursement
Market portfolio	Portefeuille de marché	Reinvestment rate	Taux de réinvestissement
Market-weighted stock index	Indice pondéré	Residual claim	Créance de dernier rang
Maturity	Echéance	Retained earnings	Réserves
Mean	Espérance	Risk aversion	Aversion au risque
Merger	Fusion	Risk exposure	Exposition au risque
Minimum-variance portfolio	Portefeuille à variance minimale	Risk management	Gestion du risque
Money market	Marché monétaire	Risk premium	Prime de marché
Moral hazard	Aléa moral	Risk-adjusted discount rate	Taux de rentabilité exigé
Mutual fund	Fonds commun de placement	Secondary market	Marché secondaire
Net present value (NPV)	Valeur actuelle nette (VAN)	Security market line	Droite du MEDAF
Net worth	Actif net	Sensitivity	Sensibilité
Shareholder	Actionnaire	Share repurchase	Rachat d'actions

Short (to be)	Court (être)	Term structure of interest rates	Structure par termes des taux d'intérêt
Short position	Position courte	Time line	Droite de temps
Simple interest	Intérêt simple	Time value	Valeur temps
Specific risk	Risque spécifique	Time value of money	Valeur de l'argent dans le temps
Speculators	Spéculateurs	Transaction costs	Coûts de transaction
Spot	Comptant	Unit of account	Unité de compte
Spot price	Cours au comptant	Volatility	Volatilité
Stakeholder	Partie prenante	Working capital	Fonds de roulement
Standard deviation	Ecart-type	Yield	Taux de rendement
Stock market	Marché boursier	Yield curve	Courbe des taux
Sunk cost	Fonds perdus	Yield to maturity	Taux de rendement actuariel
Systematic risk	Risque systématique	Zero coupon bond	Obligation à coupon zéro
Takeover	Offre publique d'achat (OPA)	10 year treasury bonds	Obligations du trésor à 10 ans
Tangency portfolio	Portefeuille tangent	3 months treasury bills	Bons du trésor à 3 mois