Descomposición en Valores Singulares: Ejercicios a mano

Sergio García Prado

sergio@garciparedes.me

28 de abril de 2018

$$A_{mxn} = U_{mxm} \cdot \Sigma_{mxn} \cdot V^T_{nxn} \tag{1}$$

1. Ejercicio 1

La matriz a la cual se aplica la *descomposición en valores singulares* es:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{2}$$

Para ello necesitamos calcular la multiplicación de A por su transpuesta en los dos sentidos (mantienen el rango):

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 11 \\ 11 & 21 \end{bmatrix} \tag{3}$$

$$A A^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 10 \\ 6 & 10 & 20 \end{bmatrix} \tag{4}$$

$$\tag{5}$$

Seguidamente, buscamos los autovalores en la de menor dimensión:

$$det(A^T A - \lambda I) = \begin{vmatrix} 6 - \lambda & 11 \\ 11 & 21 - \lambda \end{vmatrix} = \lambda^2 - 27\lambda + 5 \tag{6}$$

$$det(A^T A - \lambda I) = 0 \Rightarrow \lambda = \begin{cases} \frac{27+\sqrt{709}}{2} \\ \frac{27-\sqrt{709}}{2} \end{cases} \tag{7}$$

A partir de los autovalores obtenemos los valores singulares:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{27+\sqrt{709}}{2}} & 0 \\ 0 & \sqrt{\frac{27-\sqrt{709}}{2}} \\ 0 & 0 \end{bmatrix} \tag{8}$$

Ahora calculamos los autovectores asociados a cada autovalor:

$$\lambda_1 = \frac{27+\sqrt{709}}{2} \Rightarrow \begin{bmatrix} 6 - \frac{27+\sqrt{709}}{2} & 11 \\ 11 & 21 - \frac{27+\sqrt{709}}{2} \end{bmatrix} \cdot \begin{bmatrix} v_{11}^* \\ v_{12}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1^* \begin{bmatrix} 1 \\ \frac{15+\sqrt{709}}{22} \end{bmatrix} \tag{9}$$

$$\lambda_2 = \frac{27-\sqrt{709}}{2} \Rightarrow \begin{bmatrix} 6 - \frac{27-\sqrt{709}}{2} & 11 \\ 11 & 21 - \frac{27-\sqrt{709}}{2} \end{bmatrix} \cdot \begin{bmatrix} v_{21}^* \\ v_{22}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2^* \begin{bmatrix} 1 \\ \frac{15-\sqrt{709}}{22} \end{bmatrix} \tag{10}$$

$$V^* = \begin{bmatrix} v_1^* & v_2^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{15+\sqrt{709}}{22} & \frac{15-\sqrt{709}}{22} \end{bmatrix} \tag{11}$$

El siguiente paso es normalizar la matriz de autovectores obtenida para llegar a V:

$$v_1 = \frac{v_1^*}{\|v_1^*\|} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{12}$$

$$v_2 = \frac{v_2^* - \langle v_2^*, v_1 \rangle v_1}{\|v_2^* - \langle v_2^*, v_1 \rangle v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{13}$$

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{14}$$

Por último, obtenemos los vectores de U a partir de las propiedades que se indican:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{\frac{27+\sqrt{709}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} \\ \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 2\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} \\ \frac{2\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 4\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} \end{bmatrix} \tag{15}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{\frac{27-\sqrt{709}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} \\ \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 2\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} \\ \frac{2\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 4\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} \end{bmatrix} \tag{16}$$

$$AA^T u_3 = 0 \Rightarrow \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 10 \\ 6 & 10 & 20 \end{bmatrix} \cdot \begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_3^* = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \tag{17}$$

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & 0 \\ \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 2\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 2\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & \frac{-2}{\sqrt{5}} \\ \frac{2\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 4\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{2\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 4\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & \frac{1}{\sqrt{5}} \end{bmatrix} \tag{18}$$

Las matrices resultantes se muestran a continuación:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{19}$$

$$U = \begin{bmatrix} \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & 0 \\ \frac{\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 2\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 2\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & \frac{-2}{\sqrt{5}} \\ \frac{2\frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} + 4\frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27+\sqrt{709}}{2}}} & \frac{2\frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} + 4\frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}}}{\sqrt{\frac{27-\sqrt{709}}{2}}} & \frac{1}{\sqrt{5}} \end{bmatrix} \tag{20}$$

$$\Sigma = \begin{bmatrix} \sqrt{\frac{27+\sqrt{709}}{2}} & 0 \\ 0 & \sqrt{\frac{27-\sqrt{709}}{2}} \\ 0 & 0 \end{bmatrix} \tag{21}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} & \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{22}$$

2. Ejercicio 2

La matriz a la cual se aplica la *descomposición en valores singulares* es:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \tag{23}$$

Para ello necesitamos calcular la multiplicación de A por su transpuesta en los dos sentidos (mantienen el rango):

$$A^T A = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix} \tag{24}$$

$$AA^T = \begin{bmatrix} 6 & -3 & 3 \\ -3 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix} \tag{25}$$

Seguidamente, buscamos los autovalores en la de menor dimensión:

$$det(A^T A - \lambda I) = \begin{vmatrix} 6-\lambda & 3 & 3 \\ 3 & 6-\lambda & -3 \\ 3 & -3 & 6-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 - 81\lambda \tag{26}$$

$$det(A^T A - \lambda I) = 0 \Rightarrow \lambda = \begin{cases} 9 & \text{(multiplicidad doble)} \\ 0 & \text{(multiplicidad simple)} \end{cases} \tag{27}$$

A partir de los autovalores obtenemos los valores singulares:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_1} & 0 \\ 0 & 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Ahora calculamos los autovectores asociados a cada autovalor:

$$\lambda_1 = 3 \Rightarrow \begin{bmatrix} 6-3 & 3 & 3 \\ 3 & 6-3 & -3 \\ 3 & -3 & 6-3 \end{bmatrix} \cdot \begin{bmatrix} v_{11}^* & v_{21}^* \\ v_{12}^* & v_{23}^* \\ v_{13}^* & v_{23}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2^* = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{29}$$

$$\lambda_1 = 3 \Rightarrow \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} v_{31}^* \\ v_{32}^* \\ v_{33}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{30}$$

$$V^* = \begin{bmatrix} v_1^* & v_2^* & v_3^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{31}$$

El siguiente paso es normalizar la matriz de autovectores obtenida para llegar a V:

$$v_1 = \frac{v_1^*}{\|v_1^*\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{32}$$

$$v_2 = \frac{v_2^* - \langle v_2^*, v_1 \rangle v_1}{\|v_2^* - \langle v_2^*, v_1 \rangle v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix} \tag{33}$$

$$v_3 = \frac{v_3^* - \langle v_3^*, v_1 \rangle v_1 - \langle v_3^*, v_2 \rangle v_2}{\|v_3^* - \langle v_3^*, v_1 \rangle v_1 - \langle v_3^*, v_2 \rangle v_2\|} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \tag{34}$$

$$\tag{35}$$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{36}$$

Por último, obtenemos los vectores de U a partir de las propiedades que se indican:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{37}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{3} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \tag{38}$$

$$AA^T u_3 = 0 \Rightarrow \begin{bmatrix} 6 & -3 & 3 \\ -3 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} u_{31}^* \\ u_{32}^* \\ u_{33}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_3^* = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \tag{39}$$

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{40}$$

Las matrices resultantes se muestran a continuación:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \tag{41}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{42}$$

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{44}$$