Descomposición en Valores Singulares: Ejercicios a mano

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27 de abril de 2018

$$A_{mxn} = U_{mxm}\Sigma_{mxn}V_{nxn}^T \tag{1}$$

1. Ejercicio 1

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{2}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 11 \\ 11 & 21 \end{bmatrix} \tag{3}$$

$$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 10 \\ 6 & 10 & 20 \end{bmatrix} \tag{4}$$

$$\tag{5}$$

$$det(A^TA - \lambda I) = \begin{vmatrix} 6-\lambda & 11 \\ 11 & 21-\lambda \end{vmatrix} = (6-\lambda)(21-lambda) - 121 = \lambda^2 - 27\lambda + 5 \tag{6}$$

$$det(A^TA - \lambda I) = 0 \Rightarrow \lambda = \begin{cases} \frac{27-\sqrt{709}}{2} \\ \frac{27+\sqrt{709}}{2} \end{cases} \tag{7}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{27-\sqrt{709}}{2}} & 0 \\ 0 & \sqrt{\frac{27+\sqrt{709}}{2}} \\ 0 & 0 \end{bmatrix} \tag{8}$$

$$\lambda_1 = \frac{27-\sqrt{709}}{2} \Rightarrow \begin{bmatrix} 6-\frac{27-\sqrt{709}}{2} & 11 \\ 11 & 21-\frac{27-\sqrt{709}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} q \\ \frac{15+\sqrt{709}}{22} q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ \frac{15+\sqrt{709}}{22} \end{bmatrix} \tag{9}$$

$$\lambda_2 = \frac{27+\sqrt{709}}{2} \Rightarrow \begin{bmatrix} 6-\frac{27+\sqrt{709}}{2} & 11 \\ 11 & 21-\frac{27+\sqrt{709}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} q \\ \frac{15-\sqrt{709}}{22} q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ \frac{15-\sqrt{709}}{22} \end{bmatrix} \tag{10}$$

$$V^* = \begin{bmatrix} v_1^* & v_2* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{15+\sqrt{709}}{22} & \frac{15-\sqrt{709}}{22} \end{bmatrix} \tag{11}$$

$$v_1 = \frac{v_1^*}{\|v_1^*\|} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{12}$$

$$v_2 = \frac{v_2^* - \langle v_2^*, v_1 \rangle v_1}{\|v_2^* - \langle v_2^*, v_1 \rangle v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{13}$$

$$\tag{14}$$

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{15}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{\frac{27-\sqrt{709}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \end{bmatrix} = \begin{bmatrix} \frac{-23-\sqrt{709}}{12} \\ \frac{1}{2} \\ 1 \end{bmatrix} \tag{16}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{\frac{27+\sqrt{709}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \\ \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} = \begin{bmatrix} \frac{-23+\sqrt{709}}{12} \\ \frac{1}{2} \\ 1 \end{bmatrix} \tag{17}$$

$$A^T A u_3 = 0 \Rightarrow \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 10 \\ 6 & 10 & 20 \end{bmatrix} \cdot \begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_3 = \begin{bmatrix} 0 \\ -2 \\ q \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \tag{18}$$

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{-23-\sqrt{709}}{12} & \frac{-23+\sqrt{709}}{12} & 0 \\ \frac{1}{2} & \frac{1}{2} & -2 \\ 1 & 1 & 1 \end{bmatrix} \tag{19}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{20}$$

$$U = \begin{bmatrix} \frac{-23-\sqrt{709}}{12} & \frac{-23+\sqrt{709}}{12} & 0 \\ \frac{1}{2} & \frac{1}{2} & -2 \\ 1 & 1 & 1 \end{bmatrix} \tag{21}$$

$$\Sigma = \begin{bmatrix} \sqrt{\frac{27-\sqrt{709}}{2}} & 0 \\ 0 & \sqrt{\frac{27+\sqrt{709}}{2}} \\ 0 & 0 \end{bmatrix} \tag{22}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} & \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-23-\sqrt{709}}{12} & \frac{-23+\sqrt{709}}{12} & 0 \\ \frac{1}{2} & \frac{1}{2} & -2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\frac{27-\sqrt{709}}{2}} & 0 \\ 0 & \sqrt{\frac{27+\sqrt{709}}{2}} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} & \frac{15+\sqrt{709}}{22\sqrt{1+\frac{(15+\sqrt{709})^2}{484}}} \\ \frac{1}{\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} & \frac{15-\sqrt{709}}{22\sqrt{1+\frac{(-15+\sqrt{709})^2}{484}}} \end{bmatrix} \tag{24}$$

2. Ejercicio 2