GSERM - Oslo 2019

Hierarchical / Multilevel Models

January 8, 2019 (morning session)

“Robust” Variance-Covariance Estimators

Linear Model: $\text{Var}(\hat{\boldsymbol{\beta}})$ with $\mathbf{uu}' = \sigma^2\Omega$:

$$
\begin{aligned}
\text{Var}(\boldsymbol{\beta}_{\text{Het.}}) &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{Q}\,(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
$$

where $\mathbf{Q} = (\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})$ and $\mathbf{W} = \sigma^2\boldsymbol{\Omega}$.

Rewrite:

$$
\begin{aligned}
\mathbf{Q} &= \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}) \\
&= \sum_{i=1}^{N}\sigma_i^2\mathbf{X}_i\mathbf{X}_i'
\end{aligned}
$$

"Robust" Variance-Covariance Estimators

White's Insight:

$$\widehat{\mathbf{Q}} = \sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'$$

$$\begin{aligned}
\widehat{\text{Var}(\beta)}_{\text{Robust}} &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\widehat{\mathbf{Q}}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\left[\mathbf{X}'\left(\sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'\right)^{-1}\mathbf{X}\right](\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

What about MLE?

Recall:

$$\begin{aligned}\mathrm{Var}(\hat{\theta}) &= \mathrm{E}[(\hat{\theta}-\theta)(\hat{\theta}-\theta)'] \\ &= \mathrm{E}\left[\left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right)^{-1} \frac{\partial \ln L}{\partial \theta} \frac{\partial \ln L'}{\partial \theta} \left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right)^{-1}\right]\end{aligned}$$

We assumed:

$$\mathrm{E}\left[\frac{\partial \ln L}{\partial \theta} \frac{\partial \ln L'}{\partial \theta}\right] = \mathrm{E}\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]$$

So,

$$\begin{aligned}\mathrm{Var}(\hat{\theta}) &= \left[-\mathrm{E}\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right)\right]^{-1} \\ &= [\mathbf{I}(\theta)]^{-1}\end{aligned}$$

Alternatively:

$$\mathrm{Var}(\hat{\theta})_{\text{Robust}} = [\mathbf{I}(\theta)]^{-1} \left(\frac{\partial \ln L}{\partial \hat{\theta}} \frac{\partial \ln L'}{\partial \hat{\theta}}\right) [\mathbf{I}(\theta)]^{-1}$$

“Clustering”

Suppose N “clusters” $i = \{1, 2, ...N\}$, each with n_i observations $j = \{1, 2, ...n_i\}$.

Model:

$$Y_{ij} = \mathbf{X}_{ij}\beta + u_{ij}$$

Then:

$$\widehat{\text{Var}(\beta)}_{\text{Clustered}} = (\mathbf{X}'\mathbf{X})^{-1}\left\{\mathbf{X}'\left[\sum_{i=1}^{N}\left(\sum_{j=1}^{n_j}\hat{u}_{ij}^2\mathbf{X}_{ij}\mathbf{X}_{ij}'\right)\right]^{-1}\mathbf{X}\right\}(\mathbf{X}'\mathbf{X})^{-1}$$

An Illustration

"Regular" OLS:

```
> id<-seq(1,100,1) # 100 observations
> set.seed(7222009)
> x<-rnorm(100) # N(0,1) noise
> y<-1+1*x+rnorm(100)*abs(x)
> library(rms)
> fit<-ols(y~x,x=TRUE,y=TRUE)
> fit

Linear Regression Model

ols(formula = y ~ x, x = TRUE, y = TRUE)
                Model Likelihood     Discrimination
                   Ratio Test           Indexes
Obs      100    LR chi2     61.54    R2       0.460
sigma 0.9538    d.f.            1    R2 adj   0.454
d.f.      98    Pr(> chi2) 0.0000    g        1.002

Residuals

     Min       1Q   Median       3Q      Max
-3.27767 -0.54898  0.09069  0.35771  2.95014

          Coef   S.E.   t    Pr(>|t|)
Intercept 0.8867 0.0954 9.30 <0.0001
x         0.8822 0.0966 9.13 <0.0001
```

Further Illustration: "Robust" $\hat{V}$

```
> RVCV<-robcov(fit)
> RVCV

Linear Regression Model

ols(formula = y ~ x, x = TRUE, y = TRUE)
                Model Likelihood     Discrimination
                   Ratio Test           Indexes
Obs       100    LR chi2     61.54    R2       0.460
sigma 0.9538     d.f.            1    R2 adj   0.454
d.f.       98    Pr(> chi2) 0.0000    g        1.002

Residuals

     Min       1Q   Median       3Q      Max
-3.27767 -0.54898  0.09069  0.35771  2.95014

          Coef   S.E.   t    Pr(>|t|)
Intercept 0.8867 0.0943 9.41 <0.0001
x         0.8822 0.1352 6.52 <0.0001
```

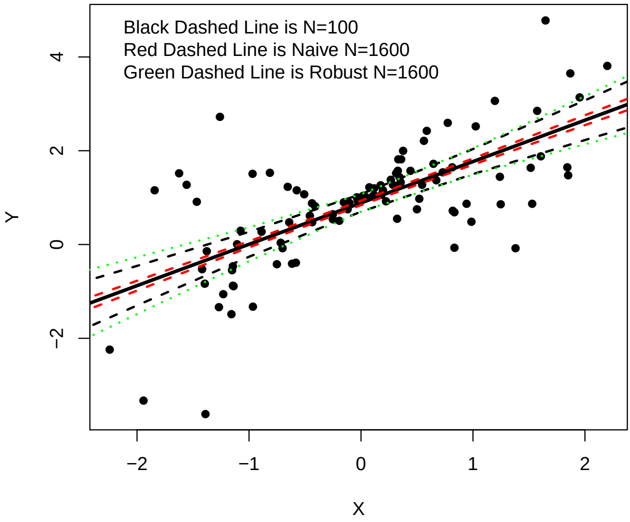
Attack of the Clones

```
> bigID<-rep(id,16)
> bigX<-rep(x,16)
> bigY<-rep(y,16)
> bigdata<-as.data.frame(cbind(bigID,bigY,bigX))
> bigOLS<-ols(bigY~bigX,data=bigdata,x=TRUE,y=TRUE)
> bigOLS

Linear Regression Model

ols(formula = bigY ~ bigX, data = bigdata, x = TRUE, y = TRUE)
                Model Likelihood     Discrimination
                   Ratio Test           Indexes
Obs      1600    LR chi2    984.69    R2         0.460
sigma 0.9448     d.f.            1    R2 adj     0.459
d.f.     1598    Pr(> chi2) 0.0000    g          0.993

Residuals

     Min       1Q   Median       3Q      Max
-3.27767 -0.54898  0.09069  0.35771  2.95014

          Coef   S.E.   t     Pr(>|t|)
Intercept 0.8867 0.0236 37.54 <0.0001
bigX      0.8822 0.0239 36.86 <0.0001
```

Peter and Hal To The Rescue

```
> BigRVCV<-robcov(bigOLS,bigdata$bigID)
> BigRVCV

Linear Regression Model

ols(formula = bigY ~ bigX, data = bigdata, x = TRUE, y = TRUE)
                            Model Likelihood     Discrimination
                               Ratio Test           Indexes
Obs                  1600    LR chi2    984.69    R2        0.460
sigma              0.9448    d.f.            1    R2 adj    0.459
d.f.                 1598    Pr(> chi2) 0.0000    g         0.993
Cluster on bigdata$bigID
Clusters              100

Residuals

     Min       1Q   Median       3Q      Max
-3.27767 -0.54898  0.09069  0.35771  2.95014

          Coef   S.E.   t    Pr(>|t|)
Intercept 0.8867 0.0943 9.41 <0.0001
bigX      0.8822 0.1352 6.52 <0.0001
```

Illustrated...



‘Robust” Variance Estimators: Cautions

- Are *only* consistent (Chesher and Jewitt 1987)
- Efficiency loss if homoscedastic (Kauermann and Carroll 2001)
- “Even if the key assumption holds, bias should be of greater interest than variance, especially when the sample is large and causal inferences are based on a model that is incorrectly specifed. Variances will be small, and bias may be large.” (Freedman 2006)

Things you should read...

Freedman, D. A. 2006. “On the So-Called ‘Huber Sandwich Estimator’ and ‘Robust’ Standard Errors.” *The American Statistician* 60:299-302.

Huber, P. J. 1967. “The Behavior of Maximum Likelihood Estimates under Nonstandard Conditions.” *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability* I:221-33.

White, H. 1994. *Estimation, Inference, and Specifcation Analysis*. New York: Cambridge University Press.

Hierarchical Linear Models

HLM Starting Points

Begin by considering a two-level "nested" data structure, with:

$$\begin{aligned} i &\in \{1, 2, ...N\} \text{ indexing first-level units, and} \\ j &\in \{1, 2, ...J\} \text{ indexing second-level groups.} \end{aligned}$$

A general two-level HLM is an equation of the form:

$$Y_{ij} = \beta_{0j} + \mathbf{X}_{ij}\boldsymbol{\beta}_j + u_{ij} \tag{1}$$

where β_{0j} is a "constant" term, $\mathbf{X}_{ij}$ is a $NJ \times K$ matrix of K covariates, $\boldsymbol{\beta}_j$ is a $K \times 1$ vector of parameters, and $u_{ij} \sim$ i.i.d. $N(0, \sigma_u^2)$ is the usual random-disturbance assumption.

HLMs, continued

Each of these $K+1$ "level-one" parameters is then allowed to vary across Q "level-two" variables $\mathbf{Z}_j$, so that:

$$\beta_{0j} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \varepsilon_{0j} \tag{2}$$

for the "intercept" and

$$\beta_{kj} = \gamma_{k0} + \mathbf{Z}_j\gamma_k + \varepsilon_{kj} \tag{3}$$

for the "slopes" of $\mathbf{X}$. The εs are typically assumed to be distributed multivariate Normal, with parameters for the variances and covariances selected by the analyst. Substitution of (3) and (2) into (1) yields:

$$Y_{ij} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \mathbf{X}_{ij}\gamma_{k0} + \mathbf{X}_{ij}\mathbf{Z}_j\gamma_k + \mathbf{X}_{ij}\varepsilon_{kj} + \varepsilon_{0j} + u_{ij} \tag{4}$$

The form is essentially one of a model with "saturated" interaction effects across the various levels, as well as "errors" which are multivariate Normal.

Model Assumptions

- Linearity / Additivity
- Normality of us
- Homoscedasticity
- Residual Independence:
  - $\text{Cov}(\varepsilon_{\cdot j}, u_{ij}) = 0$
  - $\text{Cov}(u_{ij}, u_{i\ell}) = 0$

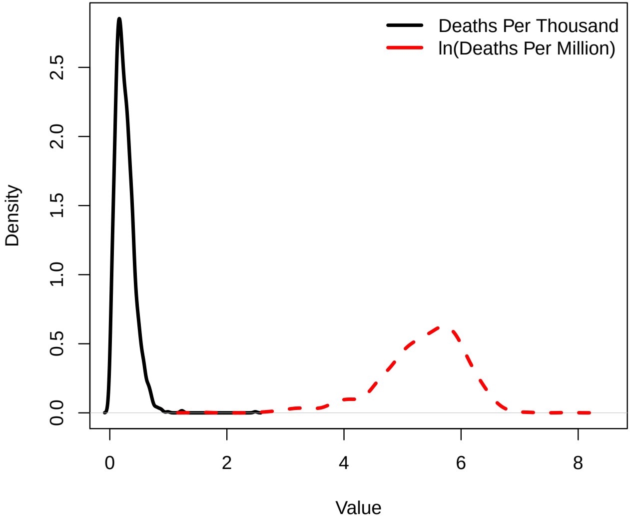
Two main alternatives:

- MLE
- "Restricted" MLE ("RMLE")
- Choosing:
  - MLE is biased in small samples, especially for estimating variance effects.
  - RMLE is not, but prevents use of LR tests when the models do not have identical fixed effects.
  - In general: RMLE is better with small sample sizes, but MLE is fine in larger ones.

An Example: HIV Death Rates, 1990-2007

```
> temp<-getURL("https://raw.githubusercontent.com/PrisonRodeo/GSERM-Oslo-2019-git/master/Data/HIVDeaths.csv")
> HIV<-read.csv(text=temp, header=TRUE)
> HIV<-HIV[ which(is.na(HIV$HIVDeathRate)==FALSE), ]
> HIV$LnDeathPM <- log(HIV$HIVDeathRate*1000)
> summary(HIV)
      country            ISO3           year       HIVDeathRate
 Angola    :  18   AGO    :  18   Min.   :1990   Min.   :0.00478
 Argentina:  18   ARG    :  18   1st Qu.:1995   1st Qu.:0.14429
 Australia:  18   AUS    :  18   Median :2000   Median :0.23303
 Benin    :  18   BDI    :  18   Mean   :1999   Mean   :0.26126
 Botswana :  18   BEN    :  18   3rd Qu.:2004   3rd Qu.:0.34889
 Brazil   :  18   BFA    :  18   Max.   :2007   Max.   :2.48542
 (Other)  :1540   (Other):1540
 CivilWarDummy       OPENLag         GDPGrowthLag       POLITYLag
 Min.   :0.000   Min.   :  1.09   Min.   :-62.368   Min.   :-10.00
 1st Qu.:0.000   1st Qu.: 44.31   1st Qu.: -0.458   1st Qu.: -4.00
 Median :0.000   Median : 61.21   Median :  1.961   Median :  6.00
 Mean   :0.181   Mean   : 74.29   Mean   :  1.899   Mean   :  2.97
 3rd Qu.:0.000   3rd Qu.: 97.37   3rd Qu.:  4.428   3rd Qu.:  9.00
 Max.   :1.000   Max.   :456.56   Max.   : 88.748   Max.   : 10.00
                 NA's   :30       NA's   :32        NA's   :63
  POLITYSQLag     InterstateWarLag    PolityLag   BatDeaths1000Lag
 Min.   :  0.0   Min.   :0.00000   Min.   : 0   Min.   : 0.000
 1st Qu.: 25.0   1st Qu.:0.00000   1st Qu.: 6   1st Qu.: 0.000
 Median : 49.0   Median :0.00000   Median :16   Median : 0.000
 Mean   : 49.5   Mean   :0.00364   Mean   :13   Mean   : 0.264
 3rd Qu.: 81.0   3rd Qu.:0.00000   3rd Qu.:19   3rd Qu.: 0.000
 Max.   :100.0   Max.   :1.00000   Max.   :20   Max.   :30.239
 NA's   :63                        NA's   :63
    GDPLagK          LnDeathPM
 Min.   : 0.153   Min.   :1.57
 1st Qu.: 1.576   1st Qu.:4.97
 Median : 5.011   Median :5.45
 Mean   : 8.582   Mean   :5.35
 3rd Qu.:10.265   3rd Qu.:5.85
 Max.   :42.683   Max.   :7.82
 NA's   :30
```

Log? Si.


Deaths Per Thousand
ln(Deaths Per Million)
Density
Value
0.0
0.5
1.0
1.5
2.0
2.5
0
2
4
6
8

OLS Regression

```
> OLSfit<-with(HIV, lm(LnDeathPM~GDPLagK+GDPGrowthLag+
+                      OPENLag+POLITYLag+POLITYSQLag+CivilWarDummy+
+                      InterstateWarLag+BatDeaths1000Lag))
> summary(OLSfit)

Call:
lm(formula = LnDeathPM ~ GDPLagK + GDPGrowthLag + OPENLag + POLITYLag +
    POLITYSQLag + CivilWarDummy + InterstateWarLag + BatDeaths1000Lag)

Residuals:
   Min     1Q Median     3Q    Max
-3.940 -0.388  0.095  0.447  1.953

Coefficients:
                  Estimate Std. Error t value    Pr(>|t|)
(Intercept)       5.493740   0.044516  123.41     < 2e-16 ***
GDPLagK          -0.027965   0.002509  -11.15     < 2e-16 ***
GDPGrowthLag     -0.002261   0.002430   -0.93      0.3524
OPENLag           0.001972   0.000368    5.35 0.000000099 ***
POLITYLag         0.010009   0.003356    2.98      0.0029 **
POLITYSQLag      -0.002182   0.000734   -2.97      0.0030 **
CivilWarDummy     0.051862   0.047026    1.10      0.2703
InterstateWarLag  0.129922   0.283361    0.46      0.6467
BatDeaths1000Lag -0.024675   0.011732   -2.10      0.0356 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.651 on 1548 degrees of freedom
  (91 observations deleted due to missingness)
Multiple R-squared:  0.177,Adjusted R-squared:  0.173
F-statistic: 41.7 on 8 and 1548 DF,  p-value: <2e-16
```

Fixed Effects

```
> FEfit<-plm(LnDeathPM~GDPLagK+GDPGrowthLag+OPENLag+POLITYLag+POLITYSQLag+CivilWarDummy+
+                       InterstateWarLag+BatDeaths1000Lag,data=HIV,effect="individual", model="within",
+                       index=c("ISO3","year"))
> summary(FEfit)
Oneway (individual) effect Within Model

Call:
plm(formula = LnDeathPM ~ GDPLagK + GDPGrowthLag + OPENLag +
    POLITYLag + POLITYSQLag + CivilWarDummy + InterstateWarLag +
    BatDeaths1000Lag, data = HIV, effect = "individual", model = "within",
    index = c("ISO3", "year"))

Unbalanced Panel: n=117, T=1-18, N=1557

Coefficients :
                   Estimate Std. Error t-value Pr(>|t|)
GDPLagK          -0.0987550  0.0094605 -10.439  < 2e-16 ***
GDPGrowthLag      0.0045675  0.0020894   2.186    0.029 *
OPENLag           0.0077044  0.0009468   8.138 8.67e-16 ***
POLITYLag         0.0505600  0.0051147   9.885  < 2e-16 ***
POLITYSQLag      -0.0006743  0.0009589  -0.703    0.482
CivilWarDummy     0.0751139  0.0534712   1.405    0.160
InterstateWarLag -0.3030380  0.2396271  -1.265    0.206
BatDeaths1000Lag  0.0004229  0.0103239   0.041    0.967
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    445.6
Residual Sum of Squares: 378.6
R-Squared:      0.1505
Adj. R-Squared: 0.1384
F-statistic: 31.7023 on 8 and 1432 DF, p-value: < 2.2e-16
```

Random Effects (using `lmer`)

```
> REfit<-lmer(LnDeathPM~GDPLagK+GDPGrowthLag+OPENLag+POLITYLag+POLITYSQLag+CivilWarDummy+
+               InterstateWarLag+BatDeaths1000Lag+(1|ISO3),data=HIV,REML=FALSE)
> summary(REfit)
Linear mixed model fit by maximum likelihood  ['lmerMod']
Formula:
LnDeathPM ~ GDPLagK + GDPGrowthLag + OPENLag + POLITYLag + POLITYSQLag +
    CivilWarDummy + InterstateWarLag + BatDeaths1000Lag + (1 | ISO3)
   Data: HIV

     AIC      BIC   logLik deviance df.resid
  2698.9   2757.7  -1338.4   2676.9     1546

Random effects:
 Groups   Name        Variance Std.Dev.
 ISO3     (Intercept) 0.265    0.515
 Residual             0.270    0.520
Number of obs: 1557, groups:  ISO3, 117

Fixed effects:
                  Estimate Std. Error t value
(Intercept)       5.272156   0.086694    60.8
GDPLagK          -0.050509   0.005092    -9.9
GDPGrowthLag      0.002749   0.002077     1.3
OPENLag           0.004776   0.000706     6.8
POLITYLag         0.044502   0.004565     9.7
POLITYSQLag      -0.000964   0.000888    -1.1
CivilWarDummy     0.060362   0.052101     1.2
InterstateWarLag -0.251942   0.240937    -1.0
BatDeaths1000Lag -0.003502   0.010331    -0.3

Correlation of Fixed Effects:
            (Intr) GDPLgK GDPGrL OPENLg POLITYL POLITYS CvlWrD IntrWL
GDPLagK     -0.172
GDPGrowthLg -0.032 -0.051
OPENLag     -0.554 -0.222 -0.015
POLITYLag   -0.047 -0.222  0.002  0.017
POLITYSQLag -0.373 -0.341  0.000  0.054 -0.051
CivilWrDmmy -0.194 -0.002  0.076  0.074  0.126   0.060
IntrsttWrLg -0.005  0.014 -0.025 -0.009 -0.028   0.013   0.023
BtDths1000L -0.045 -0.013  0.129  0.044  0.056  -0.019  -0.105 -0.329
```

HLM with Random β for GDP

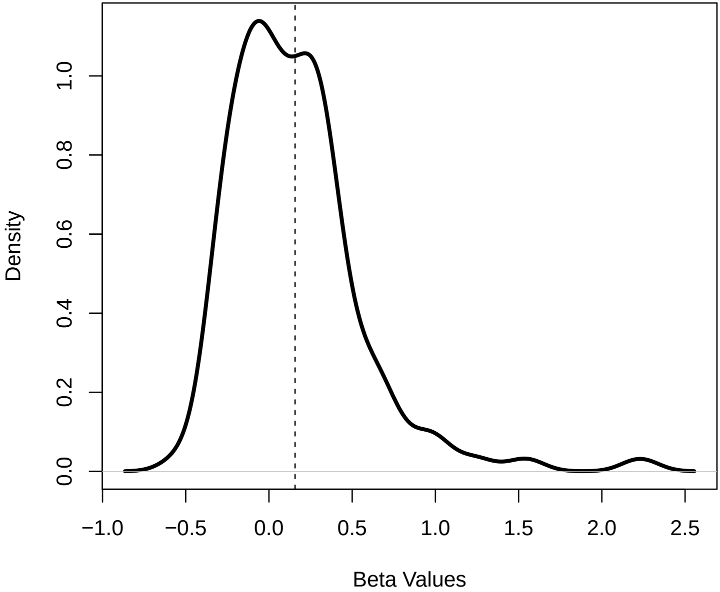
```
> HLMfit1<-lmer(LnDeathPM~GDPLagK+(GDPLagK|ISO3)+GDPGrowthLag+OPENLag+POLITYLag+POLITYSQLag+CivilWarDummy+
+                InterstateWarLag+BatDeaths1000Lag,data=HIV,REML=FALSE)
> summary(HLMfit1)
Linear mixed model fit by maximum likelihood  ['lmerMod']
Formula:
LnDeathPM ~ GDPLagK + (GDPLagK | ISO3) + GDPGrowthLag + OPENLag +  POLITYLag + POLITYSQLag + CivilWarDummy +
InterstateWarLag + BatDeaths1000Lag
   Data: HIV

     AIC      BIC   logLik deviance df.resid
  2298.8   2368.4  -1136.4   2272.8     1544

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 ISO3     (Intercept) 9.168    3.028
          GDPLagK     0.200    0.447    -0.74
 Residual             0.136    0.369
Number of obs: 1557, groups:  ISO3, 117

Fixed effects:
                  Estimate Std. Error t value
(Intercept)       4.791024   0.302393   15.84
GDPLagK           0.155304   0.048233    3.22
GDPGrowthLag      0.000872   0.001555    0.56
OPENLag           0.005995   0.000834    7.19
POLITYLag         0.039930   0.003959   10.09
POLITYSQLag      -0.003896   0.000770   -5.06
CivilWarDummy     0.009747   0.040489    0.24
InterstateWarLag -0.261331   0.178583   -1.46
BatDeaths1000Lag  0.013020   0.007920    1.64

Correlation of Fixed Effects:
            (Intr) GDPLgK GDPGrL OPENLg POLITYL POLITYS CvlWrD IntrWL
GDPLagK     -0.686
GDPGrowthLg  0.018 -0.067
OPENLag     -0.120 -0.085  0.002
POLITYLag   -0.018 -0.033 -0.007 -0.074
POLITYSQLag -0.084 -0.055  0.002 -0.019  0.039
CivilWrDmmy -0.041 -0.004  0.080  0.025  0.101   0.052
IntrsttWrLg -0.009  0.005 -0.020  0.018 -0.039   0.017   0.019
BtDths1000L -0.009 -0.008  0.101  0.065  0.063  -0.052  -0.095 -0.353
```

Testing

```
> anova(REfit,HLMfit1)

Data: HIV

Models:
REfit: LnDeathPM ~ GDPLagK + GDPGrowthLag + OPENLag + POLITYLag + POLITYSQLag +
REfit:     CivilWarDummy + InterstateWarLag + BatDeaths1000Lag + (1 |
REfit:     ISO3)
HLMfit1: LnDeathPM ~ GDPLagK + (GDPLagK | ISO3) + GDPGrowthLag + OPENLag +
HLMfit1:     POLITYLag + POLITYSQLag + CivilWarDummy + InterstateWarLag +
HLMfit1:     BatDeaths1000Lag

        Df  AIC  BIC logLik deviance Chisq Chi Df Pr(>Chisq)
REfit   11 2699 2758  -1338     2677
HLMfit1 13 2299 2368  -1136     2273 404.1      2     <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

```
> Bs<-data.frame(coef(HLMfit1)[1])
>
> head(Bs)
    ISO3..Intercept. ISO3.GDPLagK ISO3.GDPGrowthLag ISO3.OPENLag
AGO          3.96339    0.3234238       0.000869237   0.00598492
ARG          3.57905    0.1164726       0.000869237   0.00598492
ARM          5.07487    0.1142131       0.000869237   0.00598492
AUS          9.97544   -0.1999752       0.000869237   0.00598492
AUT          7.08153   -0.0845660       0.000869237   0.00598492
AZE          3.80985    0.0133378       0.000869237   0.00598492...
>
>
> mean(Bs$ISO3.GDPLagK)
[1] 0.156798
```

Random Coefficients (Plotted)



Wrap-Up & Extensions

- Can expand to 3- and 4- and higher-level models (e.g., students in classrooms in schools in districts)
- Cross-Level Interactions...
- Widely used in education, psychology, ecology, etc. (less so in economics, political science)
- There are many, many excellent books, websites, etc. that address HLMs