**Synthetic Aperture Radar**

The goal is to generate data numerically, to plot the images obtained by Synthetic Aperture Radar (SAR), and to study the resolution and stability properties of the method with respect to two types of noise: measurement noise and uncertainty in the antenna positions.

**1) Preliminaries.**
We assume that the medium has speed of propagation $c_0 = 1$. The homogeneous three-dimensional Green's function $\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y})$ is solution of

$$\Delta_{\boldsymbol{x}} \hat{G}_0 + \omega^2 \hat{G}_0 = -\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^3,$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \exp(i\omega |\boldsymbol{x} - \boldsymbol{y}|).$$

**2) SAR data.**
In Synthetic Aperture Radar (SAR), a unique antenna is used to collect data (in practice, the antenna is mounted on a moving platform that can be a satellite, a plane or a drone). This antenna can be used as a source and as a receiver. It takes successive positions $(\boldsymbol{x}_n)_{n=1,\dots,N}$. At time $T_n$, the antenna is located at position $\boldsymbol{x}_n$, it transmits a broadband pulse $f_n(t) = f(t - T_n)$ and records the backscattered signal $r_n(t)$.

We assume that the pulse $f(t)$ has central frequency $\omega_0$ and bandwidth $B$: $\hat{f}(\omega) = \mathbf{1}_{[\omega_0 - B, \omega_0 + B]}(\omega)$. We assume that there is one point-like reflector at $\boldsymbol{z}_{\rm ref}$ with reflectivity $\rho_{\rm ref}$. We use the Born approximation to generate the data:

$$\hat{r}_n(\omega) = \rho_{\rm ref} \hat{G}_0(\omega, \boldsymbol{x}_n, \boldsymbol{z}_{\rm ref})^2 \hat{f}_n(\omega).$$

Explain this formula.

**3) SAR imaging function.**
We build the imaging function by the backpropagation process (also called matched filter process):

$$\mathcal{I}(\boldsymbol{x}) = \sum_{n=1}^{N} \int \overline{\hat{R}(\omega, \boldsymbol{x}, \boldsymbol{x}_n)} \hat{r}_n(\omega) d\omega$$

with

$$\hat{R}(\omega, \boldsymbol{x}, \boldsymbol{x}_n) = \hat{G}_0(\omega, \boldsymbol{x}_n, \boldsymbol{x})^2 \hat{f}_n(\omega)$$

Explain this imaging function.

**4) Resolution analysis.**
Plot a two-dimensional image in the plane $z = 0$ obtained in the following configuration: $\boldsymbol{x}_n = (x_n, 0, 0)$ with $x_n = -a/2 + a(n-1)/(N-1)$ with $a = 20$,

$N = 64$; $\omega_0 = 2\pi$, $B = \pi/4$, $\boldsymbol{z}_{\rm ref} = (5, 100, 0)$. Discretize the frequency band $[\omega_0 - B, \omega_0 + B]$ into $M = 64$ regularly spaced frequencies.
Carry out a resolution analysis (along the $x$-axis and $y$-axis), theoretically and numerically (as in the sensor imaging configuration addressed in the lecture notes); for the theoretical analysis, it may be interesting to consider the framework $B \ll \omega_0$ and $\lambda_0 \ll a \ll L$ where $\lambda_0 = 2\pi c_o/\omega_0$ and $L = \|\boldsymbol{z}_{\rm ref} - \boldsymbol{x}_{N/2}\|$.

**4) Stability analysis.**

We want to study the stability of the image with respect to two types of perturbations: additive measurement noise and uncertainty in the antenna positions.

In the first case (measurement noise), the data is corrupted by an additive Gaussian noise. Each frequency sample $\hat{r}_n(\omega)$ on the frequency grid is corrupted by an independent additive complex zero-mean Gaussian noise.

In the second case (uncertain antenna positions), the successive positions of the antenna are only known approximately. Thus, the data are generated with the perturbed position $\boldsymbol{x}_n + \boldsymbol{e}_n$ where $\boldsymbol{e}_n$ are independent zero-mean Gaussian vectors in $\mathbb{R}^3$. The backpropagation is carried out numerically with the unperturbed positions $\boldsymbol{x}_n$.

In both cases, plot a few images with different noise levels (increase the noise level until the noise can be seen in the image). Plot the evolution of the localization error $\mathbb{E}[\|\hat{\boldsymbol{x}} - \boldsymbol{z}_{\rm ref}\|^2]$, where $\hat{\boldsymbol{x}} = {\rm argmax}_{\boldsymbol{x}} |\mathcal{I}(\boldsymbol{x})|$ as a function of the noise level (standard deviation of the additive noise or of the position perturbation). Find theoretical estimates of the localization error in the framework $B \ll \omega_0$ and $\lambda_0 \ll a \ll L$.

**References.** The first reference is a short version of the second one.

[1] M. Cheney, A mathematical tutorial on SAR, SIAM Review, SIAM Review 43 (2001) 301-312.
[2] M. Cheney and B. Borden, Fundamentals of radar imaging, SIAM, 2009.