Synthetic Aperture Radar

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This report corresponds to the mathematical part of our work, you will find attached the computational part.

1 Question 1: SAR Data

We model the surround environment with a point-like reflector at z_{ref} with reflectivity ρ_{ref}. G follows the following equation :

$$\nabla_x \hat{G} + \frac{\omega^2}{c_0^2} * \hat{G} = -\rho_{ref}\delta(x - z_{ref}) \tag{1}$$

Based on the polycopié (page 32), we have the Lippmann-Schwinger equation with $c_0 = 1$:

$$\hat{G}(\omega, x, y) = \hat{G}_0(\omega, x, y) + \omega^2 \int \hat{G}(\omega, x, z)\rho(z)\hat{G}_0(\omega, z, y)dz. \tag{2}$$

Here, we set $\rho(x) = \rho_{\text{ref}}\delta(x - z_{\text{ref}})$. Substituting this into the equation, we obtain:

$$\hat{G}(\omega, x, y) = \hat{G}_0(\omega, x, y) + \omega^2 \hat{G}(\omega, x, z_{\text{ref}})\rho_{\text{ref}}\hat{G}_0(\omega, z_{\text{ref}}, y). \tag{3}$$

Approximating $\hat{G}$ by $\hat{G}_0$ (Born approximation), we get:

$$\hat{G}(\omega, x, y) = \hat{G}_0(\omega, x, y) + \omega^2 \hat{G}_0(\omega, x, z_{\text{ref}})\rho_{\text{ref}}\hat{G}_0(\omega, z_{\text{ref}}, y) \tag{4}$$

The first term of the right hand side ($\hat{G}_0(\omega, x, y)$) corresponds to the wave directly emitted by the source. This term is known and can be directly computed. The question is about the term reflected, which is the second term (in the case $x = x_n$ and $y = x_n$). This second term in fact corresponds to the evolution of the wave $\hat{f}_n(\omega)$ when it is reflected in z_{ref} and goes back to x_n. Thus :

$$\hat{r}_n(\omega) = \omega^2 \hat{G}_0(\omega, x_n, z_{\text{ref}})\rho_{\text{ref}}\hat{G}_0(\omega, z_{\text{ref}}, x_n)\hat{f}_n(\omega) \tag{5}$$

Because $\hat{G}_0$ is synmetric in space ($\hat{G}_0(\omega, x_n, z_{\text{ref}}) = \hat{G}_0(\omega, z_{\text{ref}}, x_n)$), we in fact get :

$$\hat{r}_n(\omega) = \omega^2 \rho_{\text{ref}}\hat{G}_0(\omega, x_n, z_{\text{ref}})^2 \hat{f}_n(\omega) \tag{6}$$

Which is the result we wanted to get, up to a factor ω^2 that we will get rid of for simplicity purposes (as ω is a parameter of the problem and does not directly impact the maximum of the Imaging function) :

$$\hat{r}_n(\omega) = \rho_{\text{ref}}\hat{G}_0(\omega, x_n, z_{\text{ref}})^2 \hat{f}_n(\omega) \tag{7}$$

2 SAR Imaging function

In inverse imaging, the matched filter process consists of correlating a known delayed signal with an unknown signal to detect the known signal (the template) in the unknown one. Here, the known signal is $\hat{r}_n(\omega)$, and the unknown signal we want to correlate with is $\hat{R}(\omega, x, x_n) = \hat{G}_0(\omega, x_n, z_{\text{ref}})^2 \hat{f}_n(\omega)$ (not caring about ρ_{ref}), which is the signal theoretically obtained would the signal bounce in $z_{\text{ref}} = x$. As this signal should be time-reversed, we conjugate it. We use all the experiences ($n = 1 \to N$), and thus we get :

$$I(x) = \sum_n \int \overline{\hat{R}(\omega, x, x_n)}\hat{r}_n(w)dw \tag{8}$$

Here is another justification with more mathematics :

Not taking ρ_{ref} into account, we want to match $R(t,x,x_n)$ (the signal that would be capted by x_n if it bounced on x) with the signal that is in fact capted, which is $r_n(t)$. It is thus sensible to define :

$$I_n(x) = \langle R(\cdot, x, x_n), r_n\rangle^2 \tag{9}$$

As per Plancherel, we get :

$$I_n(x) = \langle \hat{R}(\cdot, x, x_n), \hat{r}_n\rangle^2 = \int \overline{\hat{R}(\omega, x, x_n)}\hat{r}_n(\omega)d\omega \tag{10}$$

The final imaging function being the sum of the imaging functions of the n experiments (it would be sensible to take the mean instead, but as we only want to get z_{ref}, we do not care about multiplicative constants), we get :

$$I(x) = \sum_n \int \overline{\hat{R}(\omega, x, x_n)}\hat{r}_n(\omega)d\omega \tag{11}$$

3 Resolution analysis

We compute I_n, which is the imaging function with only one sensor located in x_n. We have up to a multiplicative constant:

$$\begin{aligned}
I_n(x) &= \int \overline{\hat{R}(\omega, x, x_n)}\hat{r}_n(\omega)d\omega & (12)\\
&= \int \overline{\hat{G}_0(\omega, x, x_n)^2\hat{f}_n(\omega)}\hat{G}_0(\omega, x_n, z_{\text{ref}})^2\hat{f}_n(\omega)d\omega & (13)\\
&= \int_{\omega-B}^{\omega+B} \frac{1}{|x-x_n|^2|x_n-z_{\text{ref}}|^2}\exp\left(2i\omega(|x_n-z_{\text{ref}}|-|x_n-x|)\right)d\omega & (14)\\
&= \frac{1}{|x-x_n|^2|x_n-z_{\text{ref}}|^2}\int_{\omega-B}^{\omega+B}\exp\left(2i\omega A_n(x)\right)d\omega & (15)\\
&= \frac{1}{|x-x_n|^2|x_n-z_{\text{ref}}|^2}\frac{1}{A_n(x)}\exp(2iw_0A_n(x))\sin(2BA_n(x)) & (16)
\end{aligned}$$

where $A_n(x) = |x_n - z_{\text{ref}}| - |x_n - x|$.

We want to simplify $\frac{1}{|x-x_n|^2|x_n-z_{\text{ref}}|^2}$. We take x in a neighborhood of z_{ref} : $x = z_{\text{ref}} + \epsilon z$. Assuming $a << L$, we obtain $|x - x_n| \approx |x_n - z_{\text{ref}}| \approx L$. Thus :

$$\begin{aligned}
I_n(x) &\approx \frac{1}{L^4}\frac{1}{A_n(x)}\exp(2iw_0A_n(x))\sin(2BA_n(x)) & (17)\\
&\approx 2B\frac{1}{L^4}\exp(2iw_0A_n(x))\text{sinc}(2BA_n(x)) & (18)
\end{aligned}$$

The last line we changed sin to sinc.

Considering the first order approximation : $A_n(x) \approx \epsilon\frac{x_n-z_{\text{ref}}}{|x_n-z_{\text{ref}}|}\cdot z$, we then carry on with the resolution analysis on the x-axis and the y-axis.

Up to here, we have therefore the imaging function to be

$$\boxed{I(x) = \sum_n \exp\left(2i\omega_0A_n(x)\right)\,\text{sinc}\left(2BA_n(x)\right)} \tag{19}$$

3.1 x-axis

Here, we work with $z = (1,0)$. Thus, $A_n(x) \approx \epsilon\frac{x_n-z_{\text{ref}}}{|x_n-z_{\text{ref}}|}\cdot z = \epsilon\frac{(x_n-z_{\text{ref}})_1}{|x_n-z_{\text{ref}}|} \approx \epsilon\frac{-\frac{a}{2}+a\frac{n-1}{N-1}-5}{L}$ because $a << L$ and $z_{\text{ref}} = (5,100)$. Thus, as n changes, A_n changes from $\epsilon(-\frac{a}{2}-5)/L$ to $\epsilon(\frac{a}{2}-5)/L$, spanning a range of $\epsilon\frac{a}{L}$. Furthermore, up to a constant,

$$I(x) = \sum_n I_n(x) = \sum_n \exp(2iw_0A_n(x))\text{sinc}(2BA_n(x)) \tag{20}$$

Because $B << w_0$, and when there are a lot of captors, this is basically the integral of a smooth function (the sinus cardinal) against a complex exponential. If the range of the values taken by $2iw_0 A_n(x)$ is big, then $I(x) \approx 0$. Thus, one needs $w_0 \epsilon \frac{a}{L} \lesssim 2\pi$. Under this constraint, we get (up to a constant) :

$$I(x) = \sum_n \exp(2iw_0 A_n(x)) \tag{21}$$

Because $B << w_0$, and thus $B\epsilon\frac{a}{L} << 2\pi$, and the sinus cardinal are approximated by 1. As $A_n(x)$ spans a range of $\epsilon\frac{a}{L}$, the resolution in the x-axis is λ such that $2\lambda\frac{a}{L}\omega_0 = 2\pi$, which is $\lambda = \frac{L}{a}\frac{\pi}{\omega_0}$. We can go even further, and say that this sum of exponentials can be approximated as the integral of an exponential, to get :

$$I(x) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp(2iw_0\epsilon\frac{y-5}{L})dy \tag{22}$$

We discard $\exp(2iw_0\epsilon\frac{-5}{L})$ and consider this term constant because it doesn't vary much compared to $\exp(2iw_0\epsilon\frac{\epsilon}{L}$ on the wave length. The computation of the integral with the approximation gives:

$$I(x) = \frac{1}{\omega_0\epsilon}\sin(\omega_0\epsilon\frac{a}{L}) \tag{23}$$

And up to a constant, in the x-axis :

$$\boxed{I(x) = \text{sinc}\left(\frac{\omega_0 a}{L}\epsilon\right)} \tag{24}$$

And the corresponding resolution in the x-axis is

$$\boxed{\lambda = \frac{L}{a}\frac{\pi}{2\omega_0}} \tag{25}$$

Which confirms our guess for the resolution on the x-axis.

3.2 y-axis

Here, we work with $z = (0,1)$. Thus, $A_n(x) \approx \epsilon\frac{x_n - z_{\text{ref}}}{|x_n - z_{\text{ref}}|} \cdot z = \epsilon\frac{(x_n - z_{\text{ref}})_2}{|x_n - z_{\text{ref}}|} = \epsilon\frac{L}{\sqrt{L^2 + (-\frac{a}{2} + a\frac{n-1}{N-1} - 5)^2}} \approx \epsilon(1 - \frac{1}{2L}(-\frac{a}{2} + a\frac{n-1}{N-1} - 5))$ because $a << L$ and $z_{\text{ref}} = (5, 100)$. Thus, A_n spans the set $[1 - \frac{a}{4L}, 1 + \frac{a}{4L}]$Furthermore, up to a constant,

$$I(x) = \sum_n \exp(2iw_0 A_n(x))\text{sinc}(2BA_n(x)) \tag{26}$$

Because $B << w_0$, on this range of A_n (in which the exp does not vary that much), sinc is about constant. We can thus get it out of the sum, and get (with sum $\rightarrow$ integral) :

$$I(x) = \text{sinc}(2B\epsilon)\int_{-\frac{a}{4}}^{\frac{a}{4}} \exp(2iw_0\epsilon(1 + \frac{y}{L}))dy \tag{27}$$

Thus, we get :

$$I(x) = \text{sinc}(2B\epsilon)\exp(2iw_0\epsilon)\int_{-\frac{a}{4}}^{\frac{a}{4}} \exp(2iw_0\epsilon\frac{y}{L})dy \tag{28}$$

And finally, up to a constant :

$$I(x) = \text{sinc}(2B\epsilon)\exp(2iw_0\epsilon)\text{sinc}(\frac{w_0 a}{2L}\epsilon) \tag{29}$$

The complex exponential is of constant modulus, and thus we can get rid of it, and just say :

$$\boxed{I(x) = \text{sinc}(2B\epsilon)\text{sinc}(\frac{w_0 a}{2L}\epsilon)} \tag{30}$$

The resolution in the y-axis is thus $\boxed{\lambda = \min(\frac{1}{2}\frac{\pi}{B}; \frac{2L}{a}\frac{\pi}{w_0})}$. With the numerical parameters given in the project, the minimum of the two is in fact $\frac{1}{2}\frac{\pi}{B} \approx 2$, which is coherent with the results given by the code.

4 Stability analysis

4.1 Additive gaussian noise

Instead of receiving $\hat{r}_n(\omega)$, we do receive $\hat{r}_n(\omega)+\hat{b}_n(\omega)$ with b_n a standard complex Gaussian noise $N_C(0,\sigma^2)$.

As per the part about the imaging function, our noisy imaging function given by frequency ω is thus :

$$J(\omega,x)=\sum_n \overline{\hat{R}(\omega,x,x_n)}(\hat{r}_n(\omega)+\hat{b}_n(\omega))=I(\omega,x)+N(\omega,x) \tag{31}$$

Where $I(\omega,x)$ is the noiseless image and $N(\omega,x)=\sum_n \overline{\hat{R}(\omega,x,x_n)}\hat{b}_n(\omega)$ is the image of noise. $N(w,.)$ is a complex gaussian random field with mean 0, with covariance function :

$$\begin{aligned}
c_w(x,x') &= E((\sum_n \overline{\hat{R}(\omega,x,x_n)}\hat{b}_n(\omega))\overline{(\sum_n \overline{\hat{R}(\omega,x',x_n)}\hat{b}_n(\omega))}) && (32)\\
&= E(\sum_n \overline{\hat{R}(\omega,x,x_n)}\hat{R}(\omega,x',x_n)|\hat{b}_n(\omega)|^2) && (33)\\
&= \sigma^2\sum_n \overline{\hat{R}(\omega,x,x_n)}\hat{R}(\omega,x',x_n) && (34)
\end{aligned}$$

His relation function can be computed likewise;

$$Q(x,x')=\sum_n \overline{\hat{R}(\omega,x,x_n)}\,\overline{\hat{R}(\omega,x',x_n)}\,\mathbb{E}\left(\hat{b}_n(\omega)*\hat{b}_n(\omega)\right)=0$$

Because $\hat{b_n}$ is a complex standard Gaussian noise

We compute his variance function $\sigma_b^2(w)n\frac{\sigma^2}{(4\pi L)^4}$ (with σ the noise of measurement),

$$\begin{aligned}
\sigma_b^2(\omega) &= \sigma^2\sum_n \overline{\hat{R}(\omega,x,x_n)}\,\hat{R}(\omega,x',x_n)\\
&= \sigma^2\sum_n \frac{1}{(4\pi|x_n-x|)^2(4\pi|x_n-x'|)^2}\cdot 1_{[\omega_0-B,\omega_0+B]}(\omega) && (35)\\
&\approx \sigma^2 n\frac{1}{(4\pi L)^4}\cdot 1_{[\omega_0-B,\omega_0+B]}(\omega)
\end{aligned}$$

We approximate $J(x)=I(x)+N(x)=\int_w\sum_n \overline{\hat{R}(\omega,x,x_n)}(\hat{r}_n(\omega)+\hat{b}_n(\omega))dw$ by :

$$J(x)=\sum_w\sum_n \overline{\hat{R}(\omega,x,x_n)}(\hat{r}_n(\omega)+\hat{b}_n(\omega)) \tag{36}$$

N is a complex gaussian random field with variance $\sigma_b^2=n\frac{\sigma^2}{(4\pi L)^4}$, and covariance function $c(x,x')=\frac{\sigma^2}{1}\sum_w\sum_n \overline{\hat{R}(\omega,x,x_n)}\hat{R}(\omega,x',x_n)$ and relation function $Q(x,x')=0$. We assume that $\sigma_b << I(z_{\text{ref}})$, for the estimated argmax $\hat{z}$ (with noise) to be near z_{ref} with probability really close to 1. For x close to z_{ref}, thanks to last question, we have :

$$I(x)=I(z_{\text{ref}})(1+\frac{1}{2}(x-z_{\text{ref}})^T\Gamma(x-z_{\text{ref}})+O((x-z_{\text{ref}})^4)) \tag{37}$$

with $\Gamma=\frac{-1}{3}\begin{pmatrix}(\frac{w_0 a}{L})^2 & 0\\ 0 & 4B^2\end{pmatrix}$ (obtained by twice derivating the results obtained in last question for x axis and y axis around 0, taking for the y-axis $I(x)=\text{sinc}(2B\epsilon)$, because in this numerical application, it is this sinc that does vary strongly). Similarly :

$$N(x)=I(z_{\text{ref}})(\frac{N(z_{\text{ref}})}{I(z_{\text{ref}})}+\frac{1}{I(z_{\text{ref}})}\nabla N(z_{\text{ref}})^T(x-z_{\text{ref}})+O((x-z_{\text{ref}})^2\frac{N}{I})) \tag{38}$$

Given that $J=I+N$, J we have that

$$\boxed{\hat{z}=z_{\text{ref}}-2\frac{1}{I(z_{\text{ref}})}Re(\Gamma^{-1}\nabla N(z_{\text{ref}}))} \tag{39}$$

Let's compute first the covariance function of $\hat{z} = z_{\text{ref}}$:

$$E\left[(\hat{z} - z_{\text{ref}})(\hat{z} - z_{\text{ref}})^T\right] = \frac{2}{I(z_{\text{ref}})}^2 \Gamma^{-1} E\left[\operatorname{Re}(\nabla N(z_{\text{ref}})) \operatorname{Re}(\nabla N(z_{\text{ref}}))^T\right] \Gamma^{-1}$$

We used the fact that Γ is diagonal and real valued.

We need to find the law followed by $Re(\nabla N(z_{ref}))$. Close to z_{ref}, this a $C-$ valued gaussian process with covariance function

$$c(x, x') = \frac{\sigma^2}{1} \sum_w \sum_n \overline{\hat{R}(\omega, x, x_n)} \hat{R}(\omega, x', x_n) \tag{41}$$

But close to z_{ref} we have

$$\begin{aligned} R(\omega, x, x_n)\hat{R}(\omega, x', x_n) &= \frac{1}{(4\pi)^4 |x - x_n|^2 |x_n - z_{\text{ref}}|^2} \exp\left(2i\omega(|x_n - z_{\text{ref}}| - |x_n - x|)\right) \\ &\approx \frac{1}{(4\pi L)^4} \exp\left(2i\omega(|x_n - z_{\text{ref}}| - |x_n - x|)\right) \approx \frac{1}{(4\pi L)^4} + o(|x_n - z_x|) \end{aligned} \tag{42}$$

which is real-valued. Which means that $Re(N(x))$ is a real gaussian process of mean 0 and covariance function $c(x, x')/2$. Therefore $Re(\nabla N(x))$ is a real gaussian process too with a covariance function $\frac{\tilde{c}(x,x')}{2}$ with $\tilde{c}$ the covariance function of ∇N

To compute it now we have that $E(\nabla N(x) \nabla N(x')^T) = \delta^2_{1,1'} E(N(x)N(x')^2) = \delta^2_{1,1'} c(x, x')$. But :

$$\delta^2_{1,1'} c(z_{\text{ref}}, z_{\text{ref}}) = \sigma^2 \sum_w \sum_n \overline{\delta_{z_{\text{ref},1}} \hat{R}(\omega, z_{\text{ref}}, x_n)} \delta_{z_{\text{ref},1'}} \hat{R}(\omega, z_{\text{ref}}, x_n)^T \tag{43}$$

We recall that $\hat{R}(w, z_{\text{ref}}, x_n) = \hat{G}_0(w, z_{\text{ref}}, x_n)^2$ (because here we suppose $w \in \operatorname{supp}(\hat{f}_n)$). Furthermore, $\hat{G}_0(w, z_{\text{ref}}, x_n) = \frac{1}{4\pi L} \exp(iw|x_n - z_{\text{ref}}|)$ (including our previous approximation). We can easy compute the gradient now :

$$\delta_{z_{\text{ref}},1} R(\omega, x, x_n) = 2i\omega \frac{1}{(4\pi L)^2} \frac{z_{\text{ref}} - x_n}{|z_{\text{ref}} - x_n|} \exp 2i\omega |x_n - z_{\text{ref}}| \tag{44}$$

As $B << \omega_0$ we have $w \approx \omega_0$ for the considered bandwith

$$\delta_{z_{\text{ref}},1} R(\omega, x, x_n) = 2i\omega_0 \frac{1}{(4\pi L)^2} \frac{z_{\text{ref}} - x_n}{|z_{\text{ref}} - x_n|} \exp 2i\omega |x_n - z_{\text{ref}}| \tag{45}$$

So we have

$$\delta^2_{1,1'} c(z_{\text{ref}}, z_{\text{ref}}) = \sigma^2 \omega_0^2 M \sum_n 4 \frac{1}{(4\pi L)^4} \left(\frac{z_{\text{ref}} - x_n}{|z_{\text{ref}} - x_n|}\right)\left(\frac{z_{\text{ref}} - x_n}{|z_{\text{ref}} - x_n|}\right)^T \tag{46}$$

Because $a << L$, we have $\frac{z_{\text{ref}} - x_n}{|z_{\text{ref}} - x_n|} \approx (0, 1, 0)$. We finally get

$$E(Re(\nabla N(x)) Re(\nabla N(x'))^T) = \frac{E(\nabla N(x) \nabla N(x')^T)}{2} = \sigma^2 \omega_0^2 M n 2 \frac{1}{(4\pi L)^4} (0, 1, 0)^T (0, 1, 0) \tag{47}$$

We denote temporirarily

$$E(Re(\nabla N(z_{ref})) Re(\nabla N(z_{ref}))^T) = A$$

Therefore we find the covariance matrix to be

$\mathrm{E}\left[(\hat{z} - z_{\text{ref}})(\hat{z} - z_{\text{ref}})^T\right] = \frac{4}{I(z_{\text{ref}})^2} \Gamma^{-1} A \Gamma^{-1}$

(48)

We eventually find the relative error by using the following formula:

$$\boxed{E[|\hat{z}-z_{\text{ref}}|^2] = E[Tr(\hat{z}-z_{\text{ref}})(\hat{z}-z_{\text{ref}})^T)] = \frac{4}{I(z_{\text{ref}})^2}\frac{1}{(4B^2)^2}\sigma^2\omega_0^2 MN\frac{1}{(4\pi L)^4} * 9 * 2} \tag{49}$$

We can approximate $I(z_{\text{ref}}) \approx \frac{N\rho_{ref}}{(4\pi L)^4}$, which gives :

$$\boxed{E[|\hat{z}-z_{\text{ref}}|^2] = 18\frac{1}{4B^4}\sigma^2 M\frac{1}{N\rho_{ref}^2}(4\pi L)^4\omega_0^2} \tag{50}$$

Remark: This result is not fully consistent with the numerical observations, as the error appears to be distributed in all directions, whereas our approximation assumes motion exclusively along the y-axis.

4.2 Antenna Noise

We suppose now that the data is generated with the perturbed positions$x_{n'} = x_n + \epsilon_n$ with ϵ_n $N(0, \sigma^2 id)$.
The first part of the perturbed solution is an integral over ω so it doesn't change anything. Therefore we have in this case :

$$\boxed{I(x) = \sum_n \exp\left(2i\omega_0 A_n(x)\right)\,\text{sinc}\left(2BA_n(x)\right)} \tag{51}$$

but with $\boxed{A_n(x) = |x_{n'} - z_{\text{ref}}| - |x_n - x| = |\epsilon_n + (x_n - z_{\text{ref}})| - |x_n - x|}$

The general idea would be to look at the first and second derivatives of the imaging function inz_{ref} along the x and y axes, to find a local maximum of this function around z_{ref}. One possibility would have been to compute the integral using now gaussian processes. Unfortunately, we were not able to conduct the computation to its end.