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Elpi: rule-based extension language

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To Cinzia and Anna

Elpi: rule-based extension language

Résumé

Ce document présente Elpi, un langage par règles conçu pour étendre des applications telles que les assistants de preuve interactifs (notamment Rocq). Elpi est un mélange de λProlog, un langage de programmation logique d'ordre supérieur, et de Constraint Handling Rules, permettant une manipulation facile des arbres de syntaxe avec des lieurs et des trous.

Le manuscrit décrit Elpi comme un langage de programmation, en le comparant à Prolog et λProlog. Il détaille ensuite son implémentation, en soulignant les clés de son efficacité et de sa facilité d'intégration dans les applications hôtes. Il poursuit avec la description de l'intégration d'Elpi dans Rocq et conclut par une enquête qui recense les applications développées avec Elpi, en identifiant les fonctionnalités du langage importantes pour chacune.

Deux études de cas illustrent la puissance d'Elpi. La première est une version de l'algorithme d'inférence de types Hindley-Milner. La seconde est un outil de transfert de preuves pour Rocq.

Mots-clés : λProlog, CHR, Elpi, OCaml, Rocq.

Elpi: rule-based extension language

Abstract

This document presents Elpi, a rule-based language designed to extend applications such as interactive theorem provers, notably Rocq. Elpi combines λProlog, a higher-order logic programming language, with Constraint Handling Rules, enabling concise and expressive manipulation of syntax trees that include binders and holes.

The manuscript presents Elpi as a programming language and contrasts it with Prolog and λProlog. It then describes the implementation, highlighting the factors that contribute to its efficiency and the ease with which it can be integrated into host applications. The text continues with details on Elpi's integration with Rocq and concludes with a survey of applications built on Elpi, identifying the language features most relevant to each.

Two case studies illustrate Elpi's capabilities. The first presents a version of the Hindley-Milner type inference algorithm. The second describes a proof-transfer tool for Rocq.

Keywords: λProlog, CHR, Elpi, OCaml, Rocq.

Contents

CHAPTER 1

Introduction

1.1 Prologue

Life is full of unanswered questions. One of them, it seems, is the true purpose of this Habilitation to Direct Research (HDR) document, beyond fulfilling an administrative requirement. The University of Nice recommends a brief introduction and conclusion, wrapping the abstracts of papers published after the Ph.D.. While that approach would have been easier and shorter, it would also have resulted in a document that, ten years from now, I would have no interest in revisiting. Instead, I chose to use this opportunity to organize and reflect on the research I have conducted over the past decade, all centered around a single theme – the Elpi language. My aim is to put this work in perspective, drawing some conclusions about what succeeded and what did not. It will not leave a lasting mark, but I hope I will not be ashamed to have put it online either.

So, here is how it all begins.

1.1.1 Beyond the Odd Order Theorem

I was fortunate to participate in this major mechanization project that culminated in the formalization of a 250-pages long proof in Rocq [Gon+13]. After its completion, a natural question emerged: what made such an achievement possible, beyond the leadership of Georges Gonthier? More importantly, how could others be empowered to construct similarly impressive machine-checked proofs?

In my view, three main factors contributed to the project's success: (1) sound engineering practices—nothing new, but too often overlooked by the formal-proofs community; (2) the formalization technique known as boolean reflection (or small-scale reflection), which carves out a lightweight classical-mathematics framework in Rocq, making excluded middle, function extensionality, and proof irrelevance available when constructivism allows and using computation as a reliable form of automation; and (3) the innovative way of programming of the Rocq elaborator.

My focus was on the third point. The elaborator is the component that bridges the gap between the text input by the user and the formal terms understood by Rocq. By extending the elaboration process with small programs, we enabled Rocq to behave like an informed reader – one who knows some basic facts and can combine them using well-known rules. Previously, Rocq was often seen as a dumb reader: tireless, but requiring every detail to be spelled out. These new programs changed the game for the Odd Order Theorem, making it possible to use mathematical notations that convey a great deal of information implicitly, by convention.

Our goal became to help users craft such programs, since programming the elaborator as we did was challenging. Structuring the library so that knowledge could be organized by the author and retrieved by these programs was possibly even harder, but we started from the first one. The programming mechanism, called Canonical Structure inference [MT13], is closely related to the language of Unification Hints [Asp+09] I studied during my Ph.D. Around the same time, Rocq was extended with type classes [SO08], which were used for similar purposes and immediately raised similar programming issues.

To me, these programming languages were strikingly similar to Prolog. Our code was organized as rules and driven by unification. At that time, I had barely heard of Prolog; I even have passed over that course in my university curricula. Coincidentally, the Parsifal team at Inria Saclay was working on a language of that family, and thanks to Assia Mahboubi and Stéphane Graham-Lengrand, I was invited to give a talk at a workshop they were hosting.

In my talk, I explained some of the programs we had written and how they helped us write mathematics in Rocq. Dale Miller asked a few questions and suggested that his λProlog language [MIL91] was probably what I was looking for. Since I knew nothing about it, I could only nod and ask to continue the discussion offline.

Dale was kind to set up a meeting with Claudio Sacerdoti and myself to discuss a language well suited to describing the elaboration process of an interactive theorem prover.

1.1.2 A snowy day

On the day of the meeting, “Paris” was frozen – a supposedly apocalyptic snowstorm forced authorities to close all buildings and facilities in the Plateau de Saclay, reducing public transportation to a minimum. For some reason, only the PCRI building remained open, so the three of us agreed to buy sandwiches, walk up the hill (in boots), and meet in a deserted building.

After two hours of gentle introduction to λProlog, and after savoring the elegance of the `copy` predicate, I remember clearly saying "sold..." but immediately following with, “How do I do evars?”. λProlog could easily describe the rules my programs were made of and elegantly manipulate the data type of Rocq terms, which notably contain binders. But terms also contain holes – evars in Rocq slang – which are another beast to tame when implementing an elaborator or, more generally, the outer layers of a tool like Rocq. The answer to my question was not readily available, so this was likely the beginning of this line of research.

λProlog was not just an idealized language; it had a reference implementation in the Teyjus [NM99] system. During a visit to Bologna, Claudio Sacerdoti and I managed to write a toy elaborator for a dependent type theory in one day, but we could not find a good solution for evars. Our intuition was to try to reuse the unification variables of λProlog to represent evars, but that turned out to be extremely hard, since λProlog does not really give you a way to control their assignment. An elaborator takes as input a term with holes and returns a term where only some holes are assigned – not all of them. In interactive proofs, it is the user that drives the proof constructon, not the prover. One obvious option was to reify evars as is done in the code of Rocq: represent them with numbers and explicitly thread around a state associating a type and an assignment to each evar. But doing so destroyed the conciseness and elegance of the code we were writing.

I became convinced it was worth devising a variant of λProlog to overcome this limitation, but changing Teyjus or integrating it into Rocq seemed too difficult. Teyjus is a compiler that generates bytecode for a virtual machine written in C; adding constructs to the source language would have required mastering the entire pipeline, not to mention the complexity of using C code from OCaml, the language in which Rocq is written. So I wrote a proof of concept interpreter of λProlog in OCaml, both to better understand the language and to ease its integration with Rocq or its lightweight twin Matita [Asp+11]. On that toy interpreter, it was easy to experiment, evaluate hacks, etc. We managed to write an elaborator for the Calculus of Inductive Constructions with it, and had Matita run it on the statements of the theorems in its arithmetic library. It worked! Almost, at least. The performance was bad – very bad – something like 22,000 times slower than acceptable. Still, the experiments we could run on the prototype made it clear that most of the hacks could be explained in terms of constraints (as in constraint logic programming) and rules to manipulate these constraints.

Together with Claudio Sacerdoti, we rewrote the interpreter to make it reasonably fast and later extended the language with constraints. I named this piece of software ELPI, which stands for Embeddable Lambda Prolog Interpreter [Dun+15]. When the language started to differ from λProlog, i.e., when we mixed it with Constraint Handling Rules [Frü98], we started to use Elpi for the language itself, not just the implementation.

From the very beginning, Elpi was conceived to be embedded into a host application such as Rocq. It was never intended as a general-purpose programming language, nor to scale to large applications. It always had a niche domain of application compared to mainstream programming languages. In retrospect this clear focus was instrumental in building a working piece of software with the limited resources we had.

1.2 Elpi

Elpi is free software, distributed as a library under the LGPL license. It is developed openly at `https://github.com/LPCIC`, where we maintain the interpreter `elpi`, its integration in Rocq (`rocq-elpi`), and the Visual Studio Code extensions `elpi-lang` and `roc-elpi-lang`.

This document describes the research we have conducted over the past decade around and with this language.

The first two chapters cover Elpi “de A à Z”, both as a language and as an implementation. We credit Olivier Ridoux [Rid98] for the titles of these chapters and thank him for writing an HDR manuscript we enjoyed reading. The third chapter describes the integration of Elpi in Rocq, and the fourth surveys applications developed using Elpi, most of which are Rocq extensions.

Sections 2.7 and 4.5 contain full examples of Elpi and Rocq-Elpi programs, in particular the Hindley-Milner type inference algorithm and a proof transfer tool. The former algorithm is closely related to elaboration and demonstrates how Elpi can deal with terms containing holes. The latter takes advantage of the rule-based nature of Elpi to implement a smooth integration with Rocq. Both are tested with Elpi version 3.1 and Rocq-Elpi version 3.0. We follow these typesetting conventions for code blocks:

```
%
% Elpi code possibly intertwined with   {{ Rocq code }}
%
```

ELPI

```
(*
  Rocq code possibly intertwined with   lp:{{ Elpi code }}
*)
```

ROCQ

```
(*
   OCaml code
*)
```

OCAML

CHAPTER 2

Elpi the language: *de A à Z*

Elpi is a combination of two languages: λProlog [MIL91; MN12] and Constraint Handling Rules [Frü98; SNE+10]. The former is a higher-order variant of Prolog [Phi92; KÖR+22] based on Hereditary Harrop Formulas, a generalization of Horn clauses. The latter is a language designed to manipulate a store of constraints which, in the case of Elpi, consists of suspended λProlog goals – that is, sequents.

We briefly introduce all the components that constitute Elpi. Most of Elpi's syntax is inherited from λProlog, as is its type discipline [MN12]; similarly, most of its dynamic semantics is inherited from Prolog [BV89] and CHR [Duc+04]. We state this upfront, with no claim of novelty, and will simply refer to the language as Elpi in what follows. Nevertheless, we will highlight any novelties or differences from λProlog or Prolog as they arise.

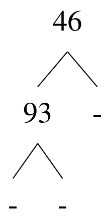
2.1 Elpi v.s. Prolog

For an introduction to Prolog, the reader is spoiled for choice [SS94; CM03]. Here, we limit ourselves to establishing terminology and highlighting the key characteristics of Elpi, as well as its differences from standard Prolog, from the programmer's perspective. We identify three main aspects: data types, rules, and search.

The first feature worth noting is the support for algebraic data types, such as lists and trees. Below is the declaration of binary trees of integeres in Elpi syntax:

```
kind tree type.
type leaf tree.
type node int -> tree -> tree -> tree.
```

The `leaf` and `node` symbols can be used to build terms and have arities 0 and 3, respectively. Elpi follows the tradition of typed functional languages by prescribing not only an arity but also a more precise type for `leaf` and `node`. In this document, we refer to them as term constructors, or simply constructors. Elpi follows the tradition of statically-typed languages: types play no role at run time but greatly contribute to the static analysis of the code, allowing typos and sometimes even true errors to be reported early. The term `(tree 46 (tree 93 leaf leaf) leaf)` represents the tree depicted on the right. For readers familiar with Prolog, note that Elpi adopts the syntactic convention of λ-calculus for function application, namely $(f\ x\ y)$, rather than the mathematical convention $f(x, y)$ used by Prolog.



The second feature we discuss is that code is organized into rules, and a rule constitutes the smallest code unit[1]. A rule consists of two parts: the head and the premises. Intuitively, the head drives the selection of a rule, reducing the current problem (the goal) to simpler ones. Operationally, the head is *unified* [Rob65] with the goal, possibly assigning values to the rule parameters, which are embodied by *unification variables*. When such assignment exists, it makes the goal and the rule's head syntatically equal. Moreover the unification algorithm guarantees that the computed assignment is the minimal one ensuring this property.

As an example, we provide the code for computing the size of a tree, together with its signature:

```
type size tree -> int -> prop.
size leaf 0.
size (node _ L R) N :- size L N1, size R N2, N is 1 + N1 + N2.
```

ELPI

Again, we favor a type assignment over a simple arity. Here `prop` is the type of executable code,[2] meaning that `size`, when applied to two arguments, becomes a goal and a program can be run to solve it. In contrast, `node`, when applied to any number of arguments, is just a piece of data; it is not a goal. Operations on built-in data types, such as integers, are provided via the built-in **`is`** operator, which expects its right-hand to be a ground term and evaluates it before unifying it with the left-hand side.

The following snippet depicts how to run a query, in particular how to ask Elpi to compute the size `N` of a given tree `T`.

```
goal> T = tree 46 (tree 93 leaf leaf) leaf, size T N.

Success
  N = 2
```

QUERY

In this code we used the infix `=` symbol to declare that `T` is equal to the tree we want to size. Operationally `=` unifies the two arguments, exactly as rules' heads are unified with the goal.

It is worth noting that variables do not need to be used linearly in the head of a rule: since the head is unified with the goal, it makes sense to write any term there, not just terms that happen to be patterns in the sense of functional programming languages. For example, the following piece of code succeeds only if the tree is empty or made of two identical subtrees.

```
type symmetric tree -> prop.
symmetric leaf.
symmetric (node _ T T).
```

ELPI

The third and last feature we discuss is that Elpi internalizes a form of search based on backtracking: all rules are potentially applied, and there is no commitment to the first rule whose head unifies with the goal. Here is an example of code that relies on backtracking to test the membership of an element in a tree.

[1] In this document we prefer the word rule to clause, but occasionally we shall use the latter.

[2] This type is called `o` in standard λProlog, following [Chu40], but Elpi uses `i` and `o` for input and output, rather than `i` for individuals and `o` for propositions.

```elpi
type mem tree -> int -> prop.
mem (node N _ _) N.
mem (node _ L _) N :- mem L N.
mem (node _ _ R) N :- mem R N.
```

Backtracking is a very powerful feature, but without control it can lead to unnecessarily poor computational complexity. In Elpi the only construct to control backtracking is the cut operator.

```elpi
mem (node N _ _) N :- !.
mem (node _ L _) N :- mem L N, !.
mem (node _ _ R) N :- mem R N.
```

Operationally, the cut is not a premise to be solved, but rather a directive to the runtime, indicating the intention to commit to the current rule and solution. The first cut indicates that if the number is found in the current node, we commit to this rule and discard the following ones: we do not care if the number also occurs deeper in the tree. The second cut indicates that if the number was found in the left subtree, we should not look in the right subtree, so we discard the last rule. To be precise, the second cut also discards any alternative solution (yet to be computed) to the preceding premises, i.e., once N is found once, we do not try to find it anymore.[3]

2.1.1 What really matters (to me)

Algebraic data types are an essential tool for writing succinct code to describe and process syntax trees. They are available in languages from the Prolog family, as well as in most functional languages and, nowadays, in some mainstream ones.[4] Thus, they do not constitute the feature that tips the balance in favor of a logic programming language.

By contrast, functional languages do not usually provide backtracking, whereas logic programming and backtracking are undeniably linked by a strong bond, and the ability to "compute relations" is a typical selling point. Still, according to our experience, the vast majority of code written in Elpi does not require this feature. In [FT25], we estimate that 96% of existing Elpi code computes functions!

In conclusion, we believe that the most interesting characteristic of Elpi is its rule-based nature. It trades the elaborate syntax offered by most programming languages for the notion of a smaller code unit. A code unit has meaning in itself, and the operation of adding or removing it from a program is well defined.

An assignment, a while loop, or a function definition do not constitute a code unit. An assignment is meaningless without a variable declaration; similarly, a while loop only acquires meaning in a larger context. Even the definition of a self-contained function fails to be a unit in this sense: while adding it to a program makes some sense (although, since nobody calls it, it serves little purpose), removing a used function simply results in a broken program.

A rule has meaning given by its logical interpretation. Adding or removing it from a logic program simply results in a different logic program. Typically, by adding a rule, the resulting program is able to handle one more case – a bit like a proof system lets you prove more theorems if you add more axioms, or becomes more and more incomplete as axioms are removed. From the

[3] Technically speaking, Elpi implements a *hard* cut.

[4] Sometimes algebraic data types are called variant types, with arguments, so to avoid scaring off some public.

parallel with proof systems, it is clear that adding “bad” rules can result in a program that gives unexpected answers, but the operation of adding or removing them is still meaningful, even if one admits non-logical constructs, like cut, and departs from the ideal world of logic. If we admit non-logical constructs, then the order of rules becomes relevant, and the insertion of a rule should be accompanied by a grafting directive, such as being added before or after another rule.

It is thanks to this rule-based nature that logic programming scales so naturally to the domain of syntax trees with binders (next section) and integrates so well with interactive provers (see chapter 4).

2.2 Elpi v.s. λProlog

The trees of interest here are more complex than the one depicted in the previous section. In particular we focus on those that constitute the foundational language of interactive provers, like Rocq. A characteristic common to all of them is the presence of binders, such as forall and exists in the world of specification, or (lambda) abstraction in the world of terms.

The “hello world” example in this domain is the simply typed lambda calculus. The syntax of terms and types is declared as follows:

```elpi
kind term type.
type app term -> term -> term.
type lam (term -> term) -> term.
```

The type declaration for terms introduces a pairing constructor `app`, which combines a function and its argument, and a function former `lam` that abstracts a term over a (bound) variable representing the formal argument. The declaration uses the function space of the Elpi programming language to represent abstraction: the function carried by `lam` can be applied to an actual argument to recover the body of the expression, where the formal argument is replaced by the actual argument.

This approach to representing data with binders is known as Higher Order Abstract Syntax (HOAS) [PE88], or more precisely, λ-tree syntax [Mil18]. The term HOAS is often used in contexts where the function space of the programming language is not well suited to the task, in contrast to λProlog. For the representation of syntax with binders to be faithful, the function space must permit only abstraction and substitution. In contrast, languages with first-class functions typically allow additional operations on variables, such as conditional expressions, which can result in so-called exotic terms.

In this approach, we refer to the language being encoded as the *object language* (in this case, the simply typed lambda calculus), while the language used for the encoding is called the *meta language*, or simply, and to our preference, the programming language. In this syntax, the function fst $= \lambda x.\lambda y.x$ is written as follows:

```elpi
Fst = (lam x\ lam y\ x)
%           ^^^^^^^^^^^ scope of x
%                  ^^^^ scope of y
```

The syntax `x\ t` is the programming language abstraction of the variables `x` over an expression `t`. Hence the term `(x\ lam y\ x)` is an Elpi function of `x`. We call variables such as `x` *bound variables*, not to be confused with unification variables: bound variables cannot be assigned by the unification algorithm, while unification variables can.

Note that the syntax tree of our object language, λ-calculus, does not feature a term constructor for variables, since we reuse the bound variables of Elpi for that. As sketched above, the body of the first abstraction can be recovered by applying the function to an argument – for example, the constant `c`: `((x\ lam y\ x) c)` evaluates to `(lam y\ c)`, thanks to the β-rule. For the sake of completeness we recall that the equational theory of λProlog also includes the η-rule, e.g., `f = x\ f x`.

Before implementing a type checker we need to give a syntax to types. The only type former we consider is the arrow, the function space. Even this time we do not provide a constructor for variables and we are going to use Elpi's unification variables for that.

```
kind ty.
type arr ty -> ty -> ty.
```

ELPI

The type checker for the simply typed λ-calculus can be implemented using the following two rules:

```
type of term -> ty -> prop.
of (app H A) T :- of H (arr S T), of A S.
of (lam F) (arr S T) :- pi c\ of c S => of (F c) T.
```

ELPI

The rule for application reads: to assign type `T` to the application of `H` to `A`, we need to assign the type `S` arrow `T` to `H` and the type `S` to `A`. The rule for abstraction assigns the type `S` arrow `T` if, given any fresh symbol `c` of type `S`, the body of the function `F` has type `T` when the bound variable is substituted by `c`. Fresh symbols like `c` are also called eigenvariables or fresh names.

We can now run the type checker on the fst function:

```
goal> Fst = (lam x\ lam y\ x), of Fst Ty.

Success
  Ty = arr S1 (arr S2 S1)
```

QUERY

The two rules above closely match the "inference rules" commonly used by the logic and programming language communities:

$$\frac{\Gamma \vdash e_1 : \sigma \to \tau \qquad \Gamma \vdash e_2 : \sigma}{\Gamma \vdash e_1\, e_2 : \tau} \qquad\qquad \frac{\Gamma, c : \sigma \vdash e[x/c] : \tau \quad c \text{ fresh in } \Gamma}{\Gamma \vdash \lambda x.e : \sigma \to \tau}$$

The main difference is that Γ is managed by Elpi: the `pi` operator guarantees the freshness of c, while `=>` locally augments the current program with an instance of the usual rule for variables:

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}$$

but specialized to c, namely

$$\frac{}{\Gamma \vdash c : T}$$

Rules added via implication are called *dynamic*, or *hypothetical*. Variables bounds by `pi` are said to be bound by the *program*, or by the λProlog *context*.

While Prolog finds its roots in Horn clauses, λProlog – and thus Elpi – are based on Hereditary Harrop formulas [MN12]. While the rule for application can be seen as a formula in the Horn fragment:

$$\text{of } H \text{ (arr } S\ T) \wedge \text{of } A\ S \Rightarrow \text{of (app } H\ A)\ T$$

the rule for lambda abstraction truly requires the ∀ quantifier and the use of (nested) implication:

$$(\forall c, \text{of } c\ S \Rightarrow \text{of } (F\ c)\ T) \Rightarrow \text{of (lam } F) \text{ (arr } S\ T)$$

Reading the rule for abstraction as a Hereditary Harrop formula makes it clear that the scope of c and its companion rule are limited to the subgoal of $(F\ c)\ T$, unlike the `assert/1` and `retract/1` Prolog built-ins.[5]

As previously mentioned, we want to use Elpi in the context of an interactive prover like Rocq, where syntax trees are often incomplete (i.e., they contain holes). For example, we would like to run the type checker on the goal `of (app F A) T` were `F` and `A` are unknown. This is where λProlog falls short, due to the lack of a mechanism to avoid divergence by suspending search on certain goals and the absence of a sublanguage to manipulate suspended goals. For example, the following query results in the rule for application to be applied indefinitely:

```
% never returns
goal> of X T.
^C % we interrupt the execution
```

QUERY

Indeed by applying the rule for application (that we recall with fresh parameter names):

```
  of (app H1 A1) T1 :- of H1 (arr S1 T1), of A1 S1.
% of X           T  % the goal
```

ELPI

one assigns `app H1 A1` to `X` and the runs the subgoals:

```
of H1 (arr S1 T1)
of A1 S1
```

ELPI

The same rule applies to the first subgoal, and that in turns assigns to `H1` a term `app H2 A2` and generates as a subgoal `of H2 (arr S2 T2)`, and this process never ends.

2.2.1 Incomplete terms

λProlog provides native support for syntax trees with bound variables by reusing the function space of the programming language to represent binders. The language also provides another kind of variable that we call *unification variables*.[6] These unification variables are equipped with a

[5] Although some Prolog implementations provide the notion of "transaction," which can be used to simulate scoping.

[6] Sometimes also called meta-variables

scope – a list of bound variables that can be used in their assignment. This is essential to avoid capturing variables by mistake, and it is well studied in [Mil92].

The natural question is: *can we reuse the unification variables of the programming language to represent holes in object language?* In other words, the runtime of our logic programming language already carries a substitution for these variables, undoes its application upon backtracking, avoids the generation of circular terms (i.e., not trees) by performing the occurs-check, and prevents variable capture. Can we reuse all that?

Elpi aims to answer this question positively. Moreover, it provides support to attach data to holes "on the side," similar to how hypothetical rules are used in the type checker we just wrote to attach a type to fresh constants.

From a programming perspective, the requirements are as follows:

control instantiation of holes In particular, it should be possible to opt out of the generative behavior of Prolog based on the order of rules. According to our experience, when a hole is encountered, one of the following two actions needs to be implemented: either suspend the computation and generate a constraint, or synthesize an assignment for the hole via a dedicated routine (more complex than unification).

attach data to holes This can range from a simple boolean flag, as we will do when implementing ML type inference with equality types (section 2.7), to a full typing sequent, as is necessary when holes represent missing Rocq terms.

manipulate the attached data For example, combine multiple pieces of data attached to the same hole or perform a consistency check when the hole is assigned.

2.3 Modes and Constraints

Elpi provides a notion of mode declaration that is uncommon in logic programming. In particular, arguments flagged as *input* are not required to be ground in the goal, but are matched against the rule "pattern" instead of being unified. This means that no unification variable occurring in the goal inside an input argument is instantiated when the rule is fired.

Mode declaration is accompanied by the `uvar` keyword, which denotes holes in the head of a rule.

In light of this, we can rephrase the type checker for simply typed λ-calculus as follows:

```
pred of i:term, o:ty.
of (app H A) T :- of H (arr S T), of A S.
of (lam F) (arr S T) :- pi c\ of c S => of (F c) T.
of (uvar as E) T :- declare_constraint (of E T) [E].
```

The `pred` directive combines a type and mode declaration, in this case the first argument is in input mode (the `i:` marker) while the second in output mode. The `uvar as E` syntax, reminiscent of ML, binds the unification variable `E` to a hole in the goal and is semantically equivalent to the following, where `var/1` is the standard Prolog predicate for testing if a variable is unbound.

```
of E T :- var E,
  declare_constraint (of E T) [E].
```

Many Prolog implementations feature a “delay directive,” that is, a sublanguage to prescribe when goals should be suspended or resumed. In Elpi, we opted for a lower-level but more flexible approach, where the programmer declares constraints or suspends goals using the `declare_constraint` built-in predicate. In the example above, `[E]` is the list of variables that “block” the suspended goal: as soon as one of these variables is assigned, the goal should be resumed. We say that `E` is the *trigger* of the constraint.

It is worth noting that the input mode described above can, in principle, be described via a program transformation, which we sketch below:

```
of E T :- not (var E), E = (app H A), of H (arr S T), of A S.
of E (arr S T) :- not (var E), E = (lam F),
  pi c\ (of E S :- not (var E), E = c) => of (F c) T.
of E T :- var E, declare_constraint (of E T) [E].
```

With this mechanism in place, the type checker can handle terms containing holes in a meaningful way:

```
goal> of (app H A) T.
Success:

Constraints:
  of A S   /* suspended on A */
  of H (arr S T)   /* suspended on H */
```

As expected, the incomplete term is well-typed under the constraint that `A` has a type compatible with that expected by `H`, which in turn is expected to be a function to `T`, the type of the term passed to `of`.

When `H` is instantiated, its corresponding constraint is resumed. Below, we set `H` to the identity function, thereby forcing its input and output types to be the same.

```
goal> of (app H A) T, H = (lam x\ x).

Success:

Constraints:
  of A T   /* suspended on A */
```

Notice how the remaining constraint now differs from before: the type of `A` is now the same `T` as that of the whole term. It is reassuring that the same result is obtained for the very similar goal where the value of `H` is given upfront: `of (app (lam x\ x) A) T`.

Unfortunately, the ability to suspend and resume goals alone is not enough to implement a satisfactory type checker. For example, the goal `of (app D D) T` succeeds with constraints:

```
of D S  /* suspended on D */
of D (arr S T)  /* suspended on D */
```

ELPI

These constraints are clearly incompatible; therefore, the type checker should have failed.

We need a language to combine the constraints attached to the hole D. For this, we chose Constraint Handling Rules.

2.4 Constraints Handling Rules

Constraint Handling Rules (CHR) is a declarative programming language based on rules that operate on a multiset of constraints [SNE+10]. A rule matches the set of constraints against a set of patterns, possibly equipped with a guard, and when it fires, it adds or removes constraints from the store. Elpi implements the so-called refined operational semantics [Duc+04] of CHR, which intuitively tests rules in the given order and commits to the first one that fires.

The following Elpi code complements the running example by imposing that each hole has only one typing constraint.

```
constraint of {
  rule (of X T1) \ (of X T2) <=> (T1 = T2)
}
```

ELPI

The rule reads: if the constraint of X T1 is present in the store together with of X T2, then remove the latter and generate a new goal T1 = T2. This is sufficient to make the type checker reject of (app D D) T, since the goal S = arr S T fails. The variable S occurring on the left also occurs under a rigid term constructor arr on the right: The equation fails the so-called *occurs check* and is rejected to avoid creating a cyclic term.

One detail to note is that constraints are *matched* against the constraint handling rules; that is, the non-linear occurrence of X does not risk merging two unrelated typing constraints: the rule above only applies if two constraints about the same variable are in the store. More precisely, the unification variables occurring in the constraints are *frozen*, i.e., replaced by fresh constants (that cannot be assigned by the unification algorith).

Another important aspect is that constraint handling rules can match multiple goals at once whereas logic programming rules only apply to one. While the execution of an Elpi program can be seen as a depth-first construction of a proof tree, with the possibility of suspending the exploration of a branch, the application of a constraint handling rule operates on all suspended branches. In a sense, it is a “meta” proof search step.

As mentioned before, Elpi automatically manages the context – that is, the set of fresh constants generated by pi and hypothetical rules introduced by **=>**. As a result, an Elpi goal is not just a predicate P, but rather a sequent $E \triangleright \Gamma \vdash P$, where E is the set of fresh constants (names) and Γ is the set of hypothetical rules.

The previous rule we used to model uniqueness of typing is therefore incomplete, since the same hole can occur under different contexts. The more elaborate version below accepts different contexts and different sets of fresh constants.

```
rule (E1 :> G1 ?- of (uvar X S1) T1)
   \ (E2 :> G2 ?- of (uvar X S2) T2)
 <=> (T1 = T2).
```

CHR

In the rule above, we use the keyword `uvar` to inspect the unification variables present in the constraint. More precisely, a frozen unification variable `F` with the bound variables `x` and `y` in scope is presented to constraint handling rules as `uvar f [x,y]` for a fresh constant `f`. This lightweight form of reification ensures that CHR rules are used to match, rather than unify, constraints, and that the programmer has access to the scope of unification variables. Moreover, each constraint is frozen to a distinct space of names; that is, `E1` and `E2` are made disjoint.

While correct for simple types, this rule does not scale to a type discipline like that of Rocq, where terms can occur in types. For example, the bound variables in `S1` may occur in `T1`, and the ones in `S2` may occur in `T2`: one needs to align `T1` with `T2` before equating them. We refer the interested reader to [GCT19] for details of a CHR rule modeling the uniqueness of typing property in Rocq's logic, and to [Joj01] for the role played by unification variable scopes in dependent type theory.

2.5 Elpi = λProlog + CHR

Before providing a precise semantics for Elpi, we offer an intuitive overview of how λProlog and CHR interact, as illustrated in fig. 2.1.

One can think of the execution of a λProlog program as the construction of a proof tree for the query, following a depth-first strategy (in the figure, this corresponds to bottom-up, left to right). This process treats λProlog rules as nodes in the proof tree, where the number of premises determines the branching factor. Action (1) operates on the leftmost open tree branch: the *active* goal g is circled in blue, while the other goals, circled with dashes, are *inert* until the tree branch corresponding to the active goal is completed or *suspended*. Suspension, depicted as action (2), forms the bridge to CHR and involves pausing the exploration of a tree branch, marking the unsolved goal as a constraint with a wakeup trigger (in the figure, moving it into the red rounded box).

By suspending a branch, the proof tree construction can proceed to the next inert goal, promoting it to active. During tree construction, action (1) may assign a unification variable, which serves as the trigger for a constraint. When this occurs, action (3) *resumes* the constraint by making it the next active goal (the previous active goal becomes inert). Finally, action (4) operates on the entire set of suspended branches. As a result of a constraint handling rule, a constraint can be removed – thus blocking any further work on the corresponding tree branch by (1) or (3) – or a new branch can be created and its goal made active.

Although we will provide a logical interpretation of the application of constraint handling rules, these rules sometimes cannot be easily described as a tree-building strategy. For example, a rule that identifies two identical branches and eliminates one is effectively building a DAG, at least in memory, by sharing the two subtrees. Another example is the detection of two suspended, incompatible goals such as `even X` and `odd X`, and the replacement of one by `false`. In this case, a theorem linking the two predicates is used to justify immediate backtracking, rather than allowing the proof search to eventually discover the unsatisfiability of the two suspended goals once `X` is instantiated, or concluding by producing an incomplete tree (which the theorem deems impossible to complete).

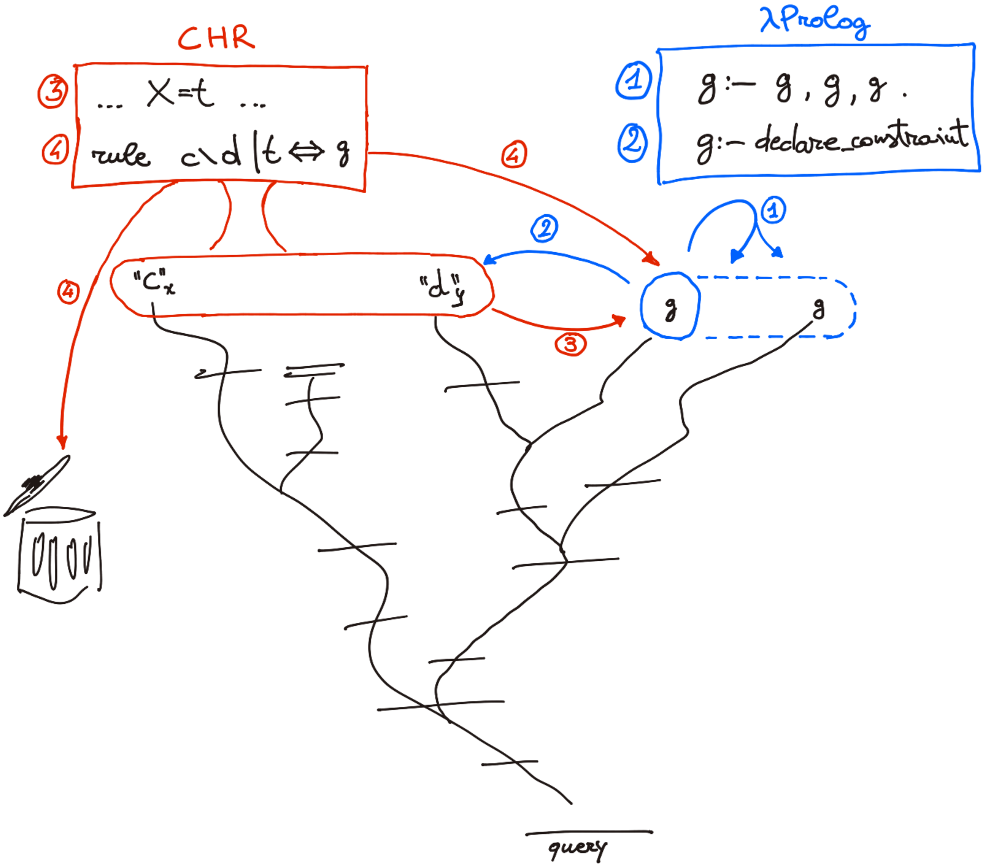


Figure 2.1: An Elpi computation

$$\begin{array}{rll}
\mathbb{F} & ::= p, q, \ldots, f, g, \ldots & \text{functors} \\
\mathbb{V} & ::= \mathrm{X}, \mathrm{Y} \ldots x, y, \ldots & \text{variable} \\
\mathbb{Ap} & ::= \mathbb{F}\ \overrightarrow{\mathbb{Tm}} \mid \mathbb{V}\ \overrightarrow{\mathbb{Tm}} & \text{applicative} \\
\mathbb{A} & ::= \texttt{Cut} \mid \texttt{Cst}\ \mathbb{Ap}\ \overrightarrow{\mathbb{V}} \mid \mathbb{Ap} \mid \texttt{pi}\ \mathbb{V}.\ \mathbb{A} \mid \mathbb{R} \texttt{ => } \mathbb{A} & \text{atom} \\
\mathbb{Tm} & ::= \mathbb{Ap} \mid \lambda\mathbb{V}.\ \mathbb{Tm} & \text{term} \\
\mathbb{R} & ::= \mathbb{Ap}\ \texttt{:-}\ \overrightarrow{\mathbb{A}} & \text{logic programming rule} \\
\mathbb{C} & ::= \overrightarrow{\mathbb{Ap}} \setminus \overrightarrow{\mathbb{Ap}} \mid \mathbb{Ap} \Leftrightarrow \overrightarrow{\mathbb{Ap}} & \text{constraint rule}
\end{array}$$

Figure 2.2: Abstract syntax

2.5.1 Abstract syntax and operational semantics

Conventions We write $\overrightarrow{t}$ to denote $t_1 \ldots t_n$ i.e. a (possibly empty) list of ts. Similarly, we write $\overrightarrow{tu}$ to denote the element-wise pairing (zipping) of $\overrightarrow{t}$ and $\overrightarrow{u}$ into a list of pairs. Finally write $t[x/y]$ for the usual, capture avoiding, operation of replacing the variable x with y inside t.

We use $\varnothing$ for the empty list and $x :: xs$ for prepending an element x to a list xs; $@$ for list concatenation; $[e \texttt{ if } p \mid x \in \vec{x}]$ for list comprehension (we omit the filter when p is true).

We denote with $\_ \times \_$, $\_ + \_$, $\overrightarrow{\_}$ and $\_ \rightarrow \_$ the product, disjoint union, list and function space of types, respectively, and with $\mathrm{B} = \{\top, \bot\}$ the booleans. By abuse of notation we use non-terminals as types, i.e., $\mathbb{A}$ is the type of atoms.

Syntax The abstract syntax of Elpi is given in fig. 2.2. We assume a set $\mathbb{F}$ of functors (term/predicate constructors), and a set $\mathbb{V}$ of term variables ($X, Y, \ldots$ for unification variables and $x, y, \ldots$ for bound variables).

In the syntax of atoms $\mathbb{A}$ we single out `Cut` and `Cst`. The `Cut` operator corresponds to `!` in the concrete syntax. The `Cst` one corresponds to `declare_constraint` in the concrete syntax and it carries the applicative term to suspend and the list of variables triggering its resumption. In the syntax for constraint rules $\mathbb{C}$ we find first a list of terms representing patterns to match, then a list of terms to match and remove, then a guard[7] and finally a list of new goals.

Semantics A substitution σ of type Σ is a partial mapping from unification variables $\mathbb{V}$ to terms $\mathbb{Tm}$. We write $\sigma\ t$ the application of the substitution to t. We write $\texttt{vars}\ \sigma$ for the set of variables occurring in the domain of σ, i.e. $\texttt{vars}\ \sigma = \{\ X \mid \sigma X \text{ is defined }\}$. The empty substitution is written ε.

We assume a function $\texttt{unify}$ of type $\mathbb{Tm} \times \mathbb{Tm} \times \Sigma \rightarrow \Sigma + \bot$ such that if $\texttt{unify}\ t_1\ t_2\ \sigma = \sigma' \neq \bot$ then σ' is the most general extension of σ such that $\sigma' t_1 = \sigma' t_2$ [Mil92]. We assume a function $\texttt{match}$ of type $\mathbb{Tm} \times \mathbb{Tm} \times \Sigma \rightarrow \Sigma + \bot$ such that if $\texttt{match}\ t\ p\ \sigma = \sigma' \neq \bot$ then σ' is the most general extension of σ such that $\sigma' p = \sigma t$ and $\sigma' t = \sigma t$, i.e. `match` does not assign variables in t but only in p that acts as the pattern.

A program π of type $\mathcal{P}$ is a pair $(\mathbb{N}, \mathcal{I})$ where $\mathbb{N}$ is a set of names and $\mathcal{I}$ an index mapping predicate names to an *ordered list* of rules. We write $x \# \pi$ to find a name fresh in $\mathbb{N}$. We write $\pi\ p$ for the rules of predicate p in $\mathcal{I}$ and we write $x + \pi$ to add the name x to $\mathbb{N}$ and we write $h + \pi$ to prepend the extra rules h to $\mathcal{I}$ (hence h has the highest priority in the new program).

[7] The guard defaults to the always true one in the concrete syntax

A constraint store κ of type $K = \overrightarrow{\mathcal{P} \times \mathbb{Ap} \times \mathbb{V}}$ is a multiset of triples (π, c, t) where π is a program, where c stands for a constraint (an applicative term in $\mathbb{Ap}$) and t of type $\overrightarrow{\mathbb{V}}$ is the trigger. The empty store is written ε; we use $k - (\pi, c, t)$ to remove a constraint from the store. The rules acting on the constraint store $\mathcal{R} = \overrightarrow{\mathbb{C}}$ are morally part of the program, but since they never change we omit them.

A goal g of type $\mathcal{G} = \mathcal{P} \times \mathbb{A} \times \overrightarrow{\mathcal{A}}$ is a triple made of a program, an atom and a list of so called *cut-to* alternatives. An alternative a of type $\mathcal{A}$ is a triple of type $K \times \Sigma \times \overrightarrow{\mathcal{G}}$ made of a constraint store, a substitution and a list of goals.

The operational, big-step semantics of Elpi is defined by the function

$$\texttt{run} : \mathcal{A} \times \overrightarrow{\mathcal{A}} \to (K \times \Sigma \times \overrightarrow{\mathcal{A}}) + \bot$$

shown in Figures fig. 2.3 and fig. 2.4. We write $\texttt{run}\ (\kappa,\ \sigma,\ \overrightarrow{g}\,)\ a \rightsquigarrow (\kappa',\ \sigma',\ a')$ to indicate that a goal list $\overrightarrow{g}$ in a constraint store κ, under a substitution σ and alternatives a, terminates with an updated constraint store κ', a substitution σ', and a remaining list of (still unexplored) alternatives a'. We write $\texttt{run}\ (k,\ \sigma,\ \overrightarrow{g}\,)\ a \rightsquigarrow \bot$ when the execution halts, that is, when it fails to solve one of the given goals and runs out of alternatives.

The rules in fig. 2.4 are syntax directed, while the ones in fig. 2.3 are given top priority and are applied in order. Even if rules for CHR have precedence we start by explaining the ones in fig. 2.4 because they give a meaning to the cut-to alternatives.

Rule $\text{run}_\top$ applies when all goals are solved and returns the constraint store and the substitution along with the still unexplored alternatives. Rule $\text{run}_\bot$ applies when no rule matches the current goal and no alternative is available. Rule $\text{run}_\circlearrowleft$ implements (chronological) backtracking by simply popping the first alternative, returning to the most recent choice point.[8] Rule $\text{run}_!$ implements a hard cut by restoring the cut-to alternatives saved by $\mathcal{B}$ (see its description below). Rule $\text{run}_@$ backchains all applicable rules, generating new alternatives, the first of which becomes the new goal. Rule run_σ accounts for the higher-order programming aspect of λProlog: unification variables can be used to pass predicates around. For example `P = true, P` is a query that requires Rule run_σ to succeed. Rule $\text{run}_\forall$ introduces a fresh name into the program, while Rule $\text{run}_\Rightarrow$ introduces a new rule into the program.

Rules $\text{run}_\triangle$ and $\text{run}_\triangledown$ address the constraint aspect of the language: the former updates the constraint store when a new constraint is declared, while the latter resumes a constraint as soon as the substitution σ acts on its trigger. The second rule follows [MP93], which also investigates how to combine higher-order logic programming with constraint programming. It may look wrong that the rule uses the current alternatives as the cut-to alternatives for the resumed goal, but these cut-to alternatives will never be used since constraints are applicative terms, i.e., not the cut operator.

Definition of $\mathcal{B}$ (backchain) The role of $\mathcal{B}$ of type $K \times \mathcal{P} \times \mathbb{Ap} \times \overrightarrow{\mathcal{G}} \times \Sigma \times \overrightarrow{\mathcal{A}} \to \overrightarrow{\mathcal{A}}$ is to generate an alternative for each rule that applies to the given goal.

[8] A choice point is, intuitively, an anchor to a point in time we could backtrack to. When doing so assignments are undone, discarding the result of the failed computation.

$$\frac{\begin{array}{c}(\pi,\ c,\ t) \in k \quad \exists v \in t, v \in \texttt{vars}\ \sigma \\ \texttt{run}\ (k - (\pi,\ c,\ t),\ \sigma,\ (\pi,\ c,\ a) :: \overrightarrow{g}\,)\ a \rightsquigarrow r\end{array}}{\texttt{run}\ (k,\ \sigma,\ \overrightarrow{g}\,)\ a \rightsquigarrow r}\ \text{run}_{\bigtriangledown}$$

$$\frac{\mathcal{CHR}(k,\ \pi,\ \sigma\, c,\ t,\ a') = (\overrightarrow{g'}, k') \qquad \texttt{run}\ (k',\ \sigma,\ \overrightarrow{g'} \mathbin{@} \overrightarrow{g}\,)\ a \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ \texttt{Cst}\ c\, t,\ a') :: \overrightarrow{g}\,)\ a \rightsquigarrow r}\ \text{run}_{\bigtriangleup}$$

Figure 2.3: Semantics: constraints

$$\frac{}{\texttt{run}\ (\kappa,\ \sigma,\ \varnothing)\ a \rightsquigarrow (\kappa,\ \sigma,\ a)}\ \text{run}_{\top}$$

$$\frac{\mathcal{B}(k,\ \pi,\ p\ \overrightarrow{t},\ \overrightarrow{g},\ \sigma,\ \varnothing) = \varnothing}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (p\ \overrightarrow{t}\,),\ \_) :: \overrightarrow{g}\,)\ \varnothing \rightsquigarrow \bot}\ \text{run}_{\bot}$$

$$\frac{\mathcal{B}(k,\ \pi,\ p\ \overrightarrow{t},\ \overrightarrow{g},\ \sigma,\ a :: al) = \varnothing \qquad \texttt{run}\ a\ al \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (p\ \overrightarrow{t}\,),\ \_) :: \overrightarrow{g}\,)\ (a :: al) \rightsquigarrow r}\ \text{run}_{\circlearrowleft}$$

$$\frac{\texttt{run}\ (k,\ \sigma,\ \overrightarrow{g}\,)\ a \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\_,\ \texttt{Cut},\ a) :: \overrightarrow{g}\,)\ \_ \rightsquigarrow r}\ \text{run}_{!}$$

$$\frac{\mathcal{B}(k,\ \pi,\ p\ \overrightarrow{t},\ \overrightarrow{g},\ \sigma,\ al) = a' :: al' \qquad \texttt{run}\ a'\ (al' \mathbin{@} al) \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (p\ \overrightarrow{t}\,),\ \_) :: \overrightarrow{g}\,)\ al \rightsquigarrow r}\ \text{run}_{@}$$

$$\frac{y \# \pi \quad \texttt{run}\ (k,\ \sigma,\ ((y + \pi),\ g[x/y],\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (\texttt{pi}\ x.\, g),\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}\ \text{run}_{\forall}$$

$$\frac{\texttt{run}\ (k,\ \sigma,\ ((h + \pi),\ g,\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (h \texttt{=>} g),\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}\ \text{run}_{\Rightarrow}$$

$$\frac{\sigma(X\ \overrightarrow{t}\,) =_{\beta\eta} p\ \overrightarrow{u} \qquad \texttt{run}\ (k,\ \sigma,\ (\pi,\ p\ \overrightarrow{u},\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}{\texttt{run}\ (k,\ \sigma,\ (\pi,\ (X\ \overrightarrow{t}\,),\ a) :: \overrightarrow{g}\,)\ a' \rightsquigarrow r}\ \text{run}_{\sigma}$$

Figure 2.4: Semantics: higher-order logic programming

$$\mathcal{B}(k,\ \pi,\ p\,\vec{t}\,,\ \vec{g}\,,\ \sigma,\ a) =$$

$$\Big[\underbrace{(k,\ \sigma',\ ([(\pi, g, a) \mid g \in \vec{b}\,]\ @\ \vec{g}\,))}_{\text{of type } \mathcal{A}}\ \texttt{if}\ \mathcal{U}(\vec{tu},\ \sigma) = \sigma' \neq \bot\ \Big| \qquad \underbrace{(p\ \vec{u}\ \texttt{:-}\ \vec{b}\,)}_{\text{of type } \mathbb{R}} \in \pi\ p\Big]$$

It is worth noting that each new goal (in each alternative) carries the same cut-to alternative a, and that the value passed to $\mathcal{B}$ by rule run$_@$ is the list of alternatives prior to backchaining. In other words, it does not include any alternatives being created, nor any that will be created in the future by exploring the new alternatives.

Definition of $\mathcal{U}$ (select) The function $\mathcal{U}$ is used to filter rules, retaining only those that apply. Unlike standard Prolog, arguments marked by the user as input (denoted $\vec{tu_i}$) are matched, while those marked as output (denoted $\vec{tu_o}$) are unified.

$$\mathcal{U}(\vec{tu}, \sigma) = \textbf{fold}\ \texttt{unify}\ \vec{tu_o}\ (\textbf{fold}\ \texttt{match}\ \vec{tu_i}\ \sigma)$$

The list processing combinator **fold** is defined by the following equations:

$$\textbf{fold}\ \_\ \_\ \bot = \bot \qquad \textbf{fold}\ f\ (x :: xs)\ a = \textbf{fold}\ f\ xs\ (f\ x\ a) \qquad \textbf{fold}\ \_\ \varnothing\ a = a$$

Definition of $\mathcal{CHR}$ (simpagation) CHR rules do have a logical reading. In particular a rule $G_1 \setminus G_2 \mid T \Leftrightarrow G_3$ is to be understood as $T \wedge G_1 \Rightarrow (G_2 \Leftrightarrow G_3)$: when the guard T holds, we can replace G_2 by G_3 provided that G_1 still needs to be solved.

Constraint rule application in Elpi follows the refined operational semantics [Duc+04], which amounts to a precise nesting of iterations. The starting point is the so-called *active constraint*, which, in the context of Elpi, is the constraint just declared as `Cst` $c\ t$ by rule run$_\triangle$.

$$\mathcal{CHR}(k,\ \pi,\ c,\ t,\ a) = (g', k')\ \text{where}$$

1. add the active constraint $(\pi,\ c,\ t)$ to k to obtain k', and set g' to $\varnothing$
2. for each rule $P_1 \ldots P_x \setminus P_{x+1} \ldots P_n \mid Q \Leftrightarrow G$
3. for each position $0 < j \leq n$ in increasing order
4. for each permutation of $C_1 \ldots C_n$ constraints in k' having $C_j = (\pi,\ c,\ t)$ (C_j is the active constraint)
5. match all C_i with P_i and run the guard Q in the initial program π_0. If the guard succeeds, i.e., `run` $(\varnothing,\ \varepsilon,\ [\pi_0, Q, \varnothing])\ \varnothing \rightsquigarrow (\varnothing,\ \sigma,\ \_)$, then:
   - remove $C_{x+1} \ldots C_n$ from k'
   - remove all permutations involving $C_{x+1} \ldots C_n$

- add σG to g'

6. move to the next rule (point 2)

As anticipated in section 2.3, adapting CHR to Elpi – or more precisely, to its λProlog part – requires some extra care. In particular, a constraint $((\mathbb{N}, \mathcal{I}),\ c,\ t)$ represents a λProlog goal:

$$\mathbb{N} \triangleright \mathcal{I} \vdash c$$

where $\mathbb{N}$ is the set of nominal constants introduced by `pi`, and $\mathcal{I}$ is the set of rules available to solve c. Pragmatically, only hypothetical rules are stored in the constraint, since all static rules are common to all constraints.

For simplicity, the context $\mathcal{I}$ is presented as a list: if the user wants to compare these contexts as sets, they must implement the quotienting in the rule's guard Q. To simplify context management, the user can specify a filter to discard hypothetical rules for predicates not relevant to the constraint (typically, rules for predicates unrelated to c). For reference, the concrete syntax for a block of rules about constraints `p` and `q` under a context restricted to `r` and `s` is:

```
constraint r s ?- p q {
  % rules for sequents on p or q under p, q, r and s
}
```

Once more, pragmatically, unless the user explicitly opts out, step 4 only considers permutations of constraints with a trigger that overlaps with the trigger t of the active constraint. This lets the user group constraints into clusters, drastically reducing the number of permutations to consider. In order to opt out the user can add a designated, inert, trigger to all constraints as follows:

```
code :- declare_constraint (p X) [X, _].
%                                    ^ special trigger
```

The special trigger `_` is a designated, unnamed variable. It cannot be assigned, so it plays no role as a trigger. In the example above, the trigger `[X, _]` is equivalent to `[X]` with respect to $\text{run}_{\triangledown}$, but it is common to all triggers that specify it when computing whether two triggers overlap; i.e., `[X, _]` and `[Y, _]` do overlap.

Finally, the system enforces the disjointness of names used in each constraint; that is, step 1 really adds $((\mathbb{N}', \mathcal{I}'),\ c',\ t)$ to k, where each name in $\mathbb{N}'$ is fresh in k, and where $\mathcal{I}'$ and c' are obtained by substituting each name in $\mathbb{N}$ with the corresponding one in $\mathbb{N}'$.

We refer the interested reader to [GCT19] for a more formal description of the operational semantics of the $\mathcal{CHR}$ procedure.

2.5.2 Constraints v.s. global variables

Most programming languages, including variants of Prolog, offer the concept of global variables – a feature that, while best used sparingly, can be extremely convenient. Elpi does not natively support global variables, but it is easy to implement a global state on top of constraint handling rules, provided that the read/write performance of global variables is not critical for the application.

Here is an example of an integer global variable with a set/get API.

```
pred set i:int.
set I :- declare_constraint (set I) [_].

pred get o:int.
get I :- declare_constraint (get I) [_].

pred value i:int.
value I :- declare_constraint (value I) [_].

constraint set get value {
  rule \ (set I) (value _)    <=> (value I).
  rule (value J) \ (get I)    <=> (I = J).
  rule (set _)                <=> (halt "no global variable").
  rule (get _)                <=> (halt "no global variable").
  rule (value _) (value _)    <=> (halt "double initialization").
}
```

The code uses `value` to initialize the global variable, and `set` or `get` to overwrite or read its value.

```
goal> value 2, get N, set 3.

Success:
  N = 2

Constraints:
 value 3  /* suspended on X0 */
```

Note that the constraint store honors backtracking, making this global state simply a convenient way to avoid threading an explicit state, as one would do in a pure language.

2.6 Syntactic sugar

Elpi features some simple syntactic sugar that is eliminated by the compiler (see section 3.3) before running programs.

2.6.1 Namespaces

Elpi programs are constructed by accumulating files – that is, by concatenating lists of rules, which may be written by different authors. It is possible for two files, such as `file1` and `file2`, to contain declarations for a predicate with the same name, for example, `p`.

If the types for `p` do not match, Elpi fails immediately and issues an error. However, if the types do match, the union of the two sets of rules can change the meaning of the predicate or its computational complexity, particularly in cases involving failure. For example the following code does not take a linear time to fail (when the element is not in the list):

```
pred mem i:list A, i:A.     % from one file
mem [X|_] X.                % from one file
mem [_|XS] X :- mem XS X.   % from one file
mem [_|YS] Y :- mem YS Y.   % from another file
```

A straightforward solution is to include the file name as part of the predicate names, such as `file1_p` and `file2_p`.

To reduce syntactic overhead, Elpi provides two syntactic features: `namespace` and `shorten`. The former prefixes all predicate names within a block, while the latter allows predicates to be accessed using a short name inside a file or block. By convention `.` is used as the namespace separator.

In the following example, the predicate `p` lives in the namespace `n` and is therefore accessible via the long name `n.p` outside the namespace.

```
% directive to begin a namespace called n
namespace n {

  % predicate p belongs to the current namespace
  pred p.

  % predicates in the current name space have short names..
  p :- ... p ...

% ..until the end of the namespace
}

% outside the name space, p is called n.p
q :- n.p.
```

Long names can be shortened via the `shorten` directive.

```ELPI
% directive to shortens n.p by making n. implicit
shorten n.{ p }.

% this shorter code..
q :- p.

% ..desugars to the more verbose
q :- n.p.
```

2.6.2 Macros

Elpi provides hygenic macros to provide short syntax to recurrent patterns. For example:

```ELPI
%     macro  arguments   expansion
macro @pi-of T F      :- pi x\ of x T => F x.
```

allows one to write the type checker rule for lambda abstraction as follows:

```ELPI
of (lam F) (arr S T) :- @pi-of S c\ of (F c) T.

% desugnars to
of (lam F) (arr S T) :- pi x\ of x S => of (F x) T.
```

As in this case macros are mostly useful to describe idioms rather than making the code shorter.

2.6.3 Spilling

We argue that the names we give to the objects we manipulate are the most important form of documentation, and poorly chosen names are the primary cause of unreadable code.

The time a careless programmer saves by using TMP, AUX, or X' instead of thinking of an appropriate name is payed back, with interest, by any reader of the code – including the programmer herself.

Logic programming, even in its higher-order flavor, distinguishes commands from expressions (predicates from data). This characteristic is the main contributor to the proliferation of temporary variables – a phenomenon that functional programming languages reduce by placing function calls and data in the same syntactic category.

For example, the OCaml [Ler+] code List.rev (List.append l1 l2) or its equivalent List.append l1 l2 |> List.rev allows the programmer to avoid naming the result of append, while in logic programming one often reads:

```LP
code L1 L2 Result :- append L1 L2 TMP, rev TMP Result.
```

Elpi provides a syntactic facility that allows one to use partially applied predicates as function calls. By partially applied, we mean “applied to all the arguments flagged as input.” For example, the code above can be written as follows:

```
code L1 L2 Result :- Result = {rev {append L1 L2}}.
```

The curly braces identify a term to be *spilled* to the closest *execution point*, determined by walking the terms inside out and identifying the first expression of type `prop`. When spilling is nested as in the example above its elaboration is akin to the generation of the administrative normal form, which is commonly used in the compilation of functional programming languages.

2.6.3.1 Spilling under a binder

When the spilled expression needs to cross a binder to reach the execution point, the elaboration becomes more sophisticated. For example:

```
code Result :- Result = (lam x\ {mk-app f [x]}).

% is elaborated to
code Result :- (pi y\ mk-app f [y] (TMP y)), Result = (lam x\ TMP x).
```

To run the spilled code in a context with the bound variable `x`, the expression must be placed under a `pi y`. Similarly, the temporary variable `TMP` needs to be abstracted over the variable, so that we can identify the original `x` with the one abstracted by `pi y`.

2.6.3.2 Spilling and impliction

When working under binders of the object language it is often necessary to augment the context via the implication operator. For example consider a variation of the λ-calculus where the `tlam` term constructor carries the type of the bound variable:

```
type app        term -> term -> term.
type tlam ty -> (term -> term) -> term.
```

When crossing `tlam` one probably wants to keep that type at hand via a dedicated rule, for example to be able to call the type checker under the binder.

```
code Ty Result :-
  % we call the predicate of under the binder for x
  Result = (tlam Ty x\ tlam {of x Ty => of (app g x)} y\ ...)
  %                          ^^^^^^^^^^^ hypothetical context for spilling

% desugars to
code Ty Result :-
  % the spilled expression keeps the implication
  (pi y\ of y Ty => of (app g y) (TMP y))
  Result = (tlam Ty x\ tlam (TMP x) y\ ...)
```

When spilling is combined with quotations, as we will better describe in section 3.4.1, the hypothetical context for spilling may be implicitly added by the compiler when desugaring the quotation. For example we call an predicate `api` under the object language binder for `x`.

```
code Result :-
  Result = {{ λx : nat => lp:{{ app f {api x}  }}  }}
%              object lang

% desugars to
code Result :-
  (pi x\ of x {{ nat }} => api x (TMP x)),
  Result = {{ λx : nat => lp:{{ app f (TMP {{ x }})  }}  }}
```

ELPI

As we will discuss at length in section 3.4.2, the code of `api` needs the context of the object language in order to work properly, hence the spilled expression must be executed under the implication, that this time is not written by the programmer but rather generated out of the binder for `x` of the object language.

2.6.3.3 Spilling ambiguities

Not all that glitters is gold. Occasionally, the nearest execution point may not correspond to the programmer's intention.

```
code D TodoForLater  :-
  TodoForLater = [ this, that {rev D} ].
```

ELPI

If `this` and `(that {rev D})` are both predicates, then the latter expands to `(rev D TMP, that TMP)`, indicating that `D` is reversed when the items constituting `TodoForLater` are executed, rather than when the todo list is generated, which may have been the intended behavior.

Based on our experience, the closest execution point generally aligns with the programmer's expectations, but it has been undeniably a source of debate.

2.7 A complete example: Hindley-Milner type inference

The most well known type inference algorithm is the one crafted by Milner for the ML language [Mil78]. The version we study here covers equality type variables, the `''a` abstraction in standard ML syntax [Mil+97] for decidable types – that is, types that can be effectively compared, unlike the function space.

This makes a good use case for Elpi because:

- it manipulates syntax trees with binders, namely lambda abstraction and let binding;
- it manipulates types containing unification variables (i.e., holes). The key step in the algorithm is to turn (existentially quantified) unification variables into universally quantified type variables to form a schema;
- unification variables have an attribute, a flag that is set if the equality test is used on values of their type.

2.7.1 Hindley-Milner in a nutshell

Before delving into the implementation we recall the main idea behind this type inference algorithm. The algorithm is capable to assign universally quantified types, also called schemas, to let-bound functions, proviso certain conditions hold. Type schemas capture the notion of polymorphism, i.e. a function of type $\forall\alpha, \alpha \rightarrow \alpha$ can be run on argumets of any type, a bit like a universally quantified theorem applies to all individuals.

More precisely, the algorithm universally quantifies type variables that are used *only* in the function body. Whenever a type variable "leaks" from the body it is unsound to quantify it. For example, the function `f` below is polymorphic, while `h` is not:

```
let a = (* a : string × int *)
  let f x = (* f : ∀α,α → α *)
    x
  in
    (f "foo", f 1)   (* f is polymorphic: can take both string and int *)

let b y =
  let h x = (* h : α → bool *)
    x = y
  in
    (h "foo", h 1)   (* error: α = string ≠ int = α *)
```

OCAML

The typing algorithm proceeds much like the type checker for simply typed λ-calculus described in section 2.2. It uses existentially quantified type variables to represent unknown types and relies on unification to establish type equality. The clever step is to recognize when an existentially quantified variable used in a function body occurs nowhere else, indicating that the code makes no assumptions about that type. In this case, the function is made polymorphic by assigning to it a type schema.

In the example above, the body of `h` compares `x` with `y`, forcing their type to be the same α, which thus escapes the body of `h`. As a result, one cannot use `h` on arguments of types other than the type of x. If the definition b were allowed (by treating h as polymorphic), then running b on, say, 2, would result in a runtime type error when comparing 2 with "foo".

The algorithm we describe in the following section also tracks type variables that are used to type terms that are compared with the equality predicate (=). If abstracted in a schema, these variables are flagged so that the schema can only apply to types that admit an equality test (unlike the function space).

2.7.2 Syntax of terms and types

The syntax of terms is very similar to that of the simply typed λ-calculus, with the addition of let and the equality test. We omit conditionals, loops, and recursion, since they do not play an important role here. As in section 2.2, we use a HOAS encoding of terms (fig. 2.5). However, we add a glo term constructor to enable the setup of a shared environment for our examples.

$$\begin{array}{rcl}
e & = & x \\
 & | & e_1\; e_2 \\
 & | & \lambda\, x \,.\, e \\
 & | & \texttt{let}\; x = e_1\; \texttt{in}\; e_2 \\
 & | & e_1 = e_2 \\
\\
\tau & = & \alpha \\
 & | & \tau \rightarrow \tau \\
 & | & \texttt{int} \\
 & | & \texttt{bool} \\
 & | & \texttt{list}\; \tau \\
 & | & \texttt{pair}\; \tau\; \tau \\
\sigma & = & \tau \\
 & | & \forall\, \alpha \,.\, \sigma \\
 & | & \forall\, \bar{\alpha} \,.\, \sigma
\end{array}$$

```
kind term   type.
type glo    string -> term.
type app    term -> term -> term.
type lam    (term -> term) -> term.
type let    term -> (term -> term) -> term.
type eq     term -> term -> term.

kind tye   type.
type (-->) tye -> tye -> tye.
type int   tye.
type bool  tye.
type list  tye -> tye.
type pair  tye -> tye -> tye.

kind ty     type.
type all    bool -> (tye -> ty) -> ty.
type mono   tye -> ty.
```

Figure 2.5: Syntax of terms and types

Monomorphic type expressions τ and type schemas σ are depicted at the bottom of fig. 2.5. As in the original paper, we fix a few built-in type constructors instead of supporting a generic n-ary type former, but unlike the ML tradition, we place type constructors to the left of an application (i.e. we write list int rather than int list). We use the infix symbol --> for the function space of ML.

2.7.3 Typing rules

The typing routine works with a context and, as we did for the simply typed lambda calculus, we delegate its management to the programming language. Context entries are represented by hypothetical rules of the of predicate, the same predicate that will implement the Hindley-Milner typing algorithm.

$$\begin{array}{lrcl} \text{Context} & \Gamma & = & \epsilon\,(\mathtt{empty}) \\ & & \mid & \Gamma,\, x : \sigma \\ \text{Typing} & & = & \Gamma \vdash e : \sigma \end{array}$$

```elpi
% means |- Term : Type
pred of i:term, o:ty.
```

Below, we give a few examples of type assignments to global constants. It is worth highlighting the fact that the type schema for "undup", is special: it requires the element of the list to have an equality type.

```elpi
of (glo "1")       (mono int).
of (glo "true")    (mono bool).
of (glo "plus")    (mono (int --> int --> int)).
of (glo "nil")     (all ff x\ mono (list x)).
of (glo "cons")    (all ff x\ mono (x --> list x --> list x)).
of (glo "pr")      (all ff x\ all ff y\ mono (x --> y --> (pair x y))).
of (glo "fst")     (all ff x\ all ff y\ mono (pair x y --> x)).
of (glo "size")    (all ff x\ mono (list x --> int)).
of (glo "undup")   (all tt x\ mono (list x --> list x)).
```

The specialization of a type schema allows one to plug type expressions in place of quantified variables.

$$\frac{\tau' = \{\alpha_i \mapsto \tau_i\}\,\tau \qquad \beta_i \notin \text{free}(\forall\alpha_1 ... \forall\alpha_n.\tau)}{\forall\alpha_1 ... \forall\alpha_n.\tau \sqsubseteq \forall\beta_1 ... \forall\beta_m.\tau'}$$

When the substituted variable stands for an equality type, one must check that the type expression supports an equality test.

$$\frac{\tau' = \{\bar{\alpha}_i \mapsto \tau_i\}\,\tau \qquad \beta_i \notin \text{free}(\forall\alpha_1 ... \forall\alpha_n.\tau) \qquad \overline{eq}(\tau_i)}{\forall\alpha_1 ... \forall\alpha_n.\tau \sqsubseteq \forall\beta_1 ... \forall\beta_m.\tau'}$$

The Elpi code departs from the pen-and-paper presentation by instantiating all schema variables at once with fresh unification variables, possibly constraining them to be equality types.

```elpi
pred specialize i:ty, o:tye.
specialize (mono T) T.
specialize (all ff F) T :-
  prune Fresh [], specialize (F Fresh) T.
specialize (all tt F) T :-
  prune Fresh [], specialize (F Fresh) T, eqbar Fresh.
```

The only detail worth describing is the use of the prune built-in to restrict the scope of Fresh to the empty one. While this is not necessary for the algorithm to work properly, it simplifies the output that we shall comment at the end of the section. Finally, it is not hurting either since the types we deal with in this example are not dependent – binders are for terms only but Fresh represents a type.

$$\begin{array}{ll}
\overline{eq}(\text{bool}) & \\
\overline{eq}(\text{int}) & \\
\overline{eq}(\text{list } \tau) & \text{if } \overline{eq}(\tau) \\
\overline{eq}(\text{pair } \tau_1 \ \tau_2) & \text{if } \overline{eq}(\tau_1) \wedge \overline{eq}(\tau_2) \\
\overline{eq}(\bar{\alpha}) &
\end{array}$$

```
pred eqbar i:tye.
eqbar bool.
eqbar int.
eqbar (list A) :- eqbar A.
eqbar (pair A B) :- eqbar A, eqbar B.
eqbar (uvar as A) :-
  declare_constraint (eqbar A) [A,_].
```

Figure 2.6: Equality types

The `eqbar` predicate, fig. 2.6, traverses concrete type expressions and flags variables via a constraint, so that we can distinguish equality type variables from the others.
The auxiliary constraint `theta` is used to gather the (duplicate-free) list of unification variables constrained to be equality types.

```
constraint of ?- gammabar eqbar theta rm-eqbar {
  rule (eqbar V) \ (theta L) | (not(mem L V)) <=> (theta [V | L]).
}

pred theta i:list tye.
theta L :- declare_constraint (theta L) [_].
```

The use of constraints here is not essential, but is really convenient. The alternative is to thread a state – the list of all type variables used so far paired with their equality type status – *everywhere*. The constraint store offers a global state carried alongside the goals, and constraint handling rules provide the possibility to organize the state, such as grouping all entries into a list. Since constraints are only combined via a CHR rule when they share a trigger, we list the designated trigger _ in all of them.

The declarative code for the type inference algorithm is given in fig. 2.7 alongside the corresponding Elpi code. The rules for lambda abstraction and application are the same as for the simply typed lambda calculus section 2.2. The rule for variables finds a schema in Γ and assigns to the variable any valid instance. The rule for `let` pushes into the context a type $\overline{\Gamma}(\tau)$ for a let-bound expression of type τ in Γ. Finally, the rule for equality tests whether the type of the expressions admits an equality test. The only difference between the declarative rules and the Elpi code the rule for variables, is intentionally extended to any term, so that it can apply to global constants as well.[9]

2.7.4 Type generalization

We have already described how `specialize` relates to $\sqsubseteq$. What remains to be described is the $\overline{\Gamma}$ operation in charge of synthesizing type schemas.

$$\overline{\Gamma}(\tau) = \forall\, \hat{\alpha} \,.\, \tau \qquad \hat{\alpha} = \text{free}(\tau) - \text{free}(\Gamma)$$

[9] One could have asserted `name X`, i.e., forced `X` to be a `pi`-introduced name

$$\frac{\Gamma \vdash e_0 : \tau \to \tau' \qquad \Gamma \vdash e_1 : \tau}{\Gamma \vdash e_0\, e_1 : \tau'}$$

```
of (app H A) (mono T) :-
  of H (mono (S --> T)),
  of A (mono S).
```

$$\frac{\Gamma,\ x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : \tau \to \tau'}$$

```
of (lam F) (mono (S --> T)) :-
  pi x\ of x (mono S) =>
    of (F x) (mono T).
```

$$\frac{\Gamma \vdash e_0 : \tau \qquad \Gamma,\ x : \bar{\Gamma}(\tau) \vdash e_1 : \tau'}{\Gamma \vdash \texttt{let}\ x = e_0\ \texttt{in}\ e_1 : \tau'}$$

```
of (let E B) (mono TB) :-
  of E (mono T),
  gammabar (mono T) PT,
  pi x\ of x PT =>
    of (B x) (mono TB).
```

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau \quad \overline{eq}(\tau)}{\Gamma \vdash e_1 = e_2 : \texttt{bool}}$$

```
of (eq LHS RHS) (mono bool) :-
  of LHS (mono T),
  of RHS (mono T),
  eqbar T.
```

$$\frac{x : \sigma \in \Gamma \quad \sigma \sqsubseteq \tau}{\Gamma \vdash x : \tau}$$

```
of X (mono T) :-
  of X (all E Poly),
  specialize (all E Poly) T.
```

ELPI

Figure 2.7: Typing algorithm

Intuitively, one must compute the free type variables in Γ and τ and abstract only those not free in Γ – that is, those used only locally in the expressions being bound by the let.

This operation is beyond the capabilities of λProlog alone, even with the mode extensions discussed in section 2.3, since Γ is not first-class. This is where the “meta” status of constraint handling rules becomes useful: suspended goals are sequents.
The code to implement $\overline{\Gamma}(\tau)$ given in fig. 2.8 performs two tasks:

- It abstracts `T` into a schema `POLYT` and returns it by assigning `TS`;
- It cleans up the store of constraints by removing the `eqbar` constraints on variables that are now bound in the schema, and updates `theta` by recomputing it.

The first task is necessary, while the second is mainly to explain and motivate the scheduling choices described in section 2.5. In particular, it is crucial that all unneeded `eqbar` constraints are removed before `theta` is recomputed by combining the new constraint with all `eqbar` constraints still in the store. It is also worth noting how the (declarative) semantics of CHR ensures that the new `theta` constraint is combined with all `eqbar` ones, regardless of the order in which they are added to the store: up to now, `theta` was present and was updated at every `eqbar` addition; now all `eqbar` constraints are present, and `theta` gets added and combined with them all.

The definition of free type variables, fig. 2.9, is not surprising, but it is worth reminding two details from section 2.4 of the corresponding Elpi code.

The first detail is that the context is a list of propositions. The `prop` type is a union type of all predicates, not just `of`. We do not need to add a rule to skip

```
pred gammabar i:ty, o:ty.
gammabar (mono T) TS :- declare_constraint (gammabar (mono T) TS) [_].

constraint of ?- gammabar eqbar theta rm-eqbar {
  rule  \ (theta L)
          (G ?- gammabar T TS)              % matched and removed
        | (generalize L G T POLYT ToRemove) % guard + synthesis
      <=> (TS = POLYT, rm-eqbars ToRemove, theta []). % new goal
  rule \ (eqbar X) (rm-eqbar X).
}

pred rm-eqbars i:list tye.
rm-eqbars [].
rm-eqbars [X|XS] :- rm-eqbar X, rm-eqbars XS.

pred rm-eqbar i:tye.
rm-eqbar X :- declare_constraint (rm-eqbar X) [X, _].

pred generalize i:list tye, i:list prop, i:ty, o:ty, o:list tye.
generalize Theta Gamma (mono T) PolyT ToQuantify :-
  free-ty (mono T) [] VT,
  free-gamma Gamma [] VGamma,
  filter VT (x\ not(mem VGamma x)) ToQuantify,
  bind ToQuantify Theta T PolyT.
```

ELPI

Figure 2.8: Core for $\overline{\Gamma}(\tau)$

any unrelated predicate the program may have loaded with implication, because the **constraint** of ?- gammabar eqbar theta rm-eqbar directive tells Elpi to filter the context of suspended goals (constraints) and keep only entries for the of predicate.

The second is that, at the meta level of constraint handling rules, the **uvar** keyword becomes a regular syntax tree node and has two arguments. The first one is the unique identifier of the unification variable; the second is its scope, represented as a list of names.

The last bit of code necessary to implement $\overline{\Gamma}(\tau)$ is the code to bind (locally used) unification variables to universal quantifications.

```
pred bind i:list tye, i:list tye, i:tye, o:ty.
bind [uvar X _|XS] Theta T (all E x\ T1 x) :-
  if (mem Theta (uvar X _)) (E = tt) (E = ff),
  pi c\ (copy (uvar X _) c :- !) => bind XS Theta T (T1 c).
%       ^^^^^^^^^^^^^^^^^^^^^^^^^^^   load the substitution
bind [] _ T (mono T1) :- copy T T1. % fire the substitution
```

ELPI

The key is the copy predicate, given below, which implements substitution by traversing an expression. Since this substitution is not the built-in one (i.e., not function application), bind can alter its behavior by loading into the program some ad hoc rules. In particular, for each variable **uvar** X _ to abstract, we postulate a fresh symbol c and load a rule to copy **uvar** X _ to c.

$$
\begin{array}{lcl}
\text{free}(\ \texttt{int}\ ) & = & \emptyset \\
\text{free}(\ \texttt{bool}\ ) & = & \emptyset \\
\text{free}(\ \texttt{pair}\ \tau_1\ \tau_2\ ) & = & \text{free}(\ \tau_1\ ) \cup \text{free}(\ \tau_2\ ) \\
\text{free}(\ \texttt{list}\ \tau\ ) & = & \text{free}(\ \tau) \\
\text{free}(\ \tau_1 \to \tau_2\ ) & = & \text{free}(\ \tau_1\ ) \cup \text{free}(\ \tau_2\ ) \\
\text{free}(\ \alpha\ ) & = & \{\alpha\} \\
\text{free}(\ \forall\, \alpha\, .\, \sigma\ ) & = & \text{free}(\ \sigma\ )\ -\ \{\alpha\} \\
\text{free}(\ \Gamma\ ) & = & \bigcup\limits_{x:\sigma\in\Gamma} \text{free}(\ \sigma\ )
\end{array}
$$

```
pred free-ty i:ty, i:list tye, o:list tye.
free-ty (mono X) L L1 :- free X L L1.
free-ty (all _ F) L L1 :- pi x\ free-ty (F x) L L1.

pred free-gamma i:list prop, i:list tye, o:list tye.
free-gamma [] L L.
free-gamma [of _ T|X] L L2 :- free-ty T L L1, free-gamma X L1 L2.

pred free i:tye, i:list tye, o:list tye.
free int L L.
free bool L L.
free (list A) L L1 :- free A L L1.
free (pair A B) L L1 :- L1 = {free B {free A L}}.
free (A --> B) L L1 :- L1 = {free B {free A L}}.
free (uvar _ _ as X) L L1 :- if (mem L X) (L1 = L) (L1 = [X|L]).
```

ELPI

Figure 2.9: Free type variable

```
pred mem i:list tye, i:tye.
mem [uvar X _|_] (uvar X _) :- !.
mem [_|XS] X :- mem XS X.

pred filter i:list A, i:(pred i:A), o:list A.
filter [] _ [].
filter [X|XS] F [X|YS] :- F X, !, filter XS F YS.
filter [_|XS] F YS :- filter XS F YS.
```

ELPI

Figure 2.10: Auxiliary code

```
pred copy i:tye, o:tye.
copy int int.
copy bool bool.
copy (list A) (list B) :- copy A B.
copy (pair A B) (pair C D) :- copy A C, copy B D.
copy (A --> B) (C --> D) :- copy A C, copy B D.
copy (uvar _ _ as V) V.
```

ELPI

For completeness, the remaining code is given in fig. 2.10. It may be worth explaining the signature of the polymorphic, higher-order `filter` predicate. In particular, `i:(pred i:A)` stands for an input predicate over `A`, where the predicate itself expects an input.

2.7.5 Execution example

We can run the type inference algorithm on the following term, which defines a function f that tests whether the input list is empty and then applies f to two different expressions: `let` $f = (\lambda x.x = [])$ in $(f[\text{true}], f[1])$.

```
goal> theta [],
  of (let (lam x\ eq (glo "nil") x) f\
       app (app (glo "pr")
        (app f (app (app (glo "cons") (glo "true")) (glo "nil"))))
        (app f (app (app (glo "cons") (glo "1"))    (glo "nil")))
     ) Ty.
```

QUERY

As expected, the inferred type is $\text{bool} \times \text{bool}$.

```
Success
  Ty = mono (pair bool bool)

Constraints:
  theta [int, bool]  /* suspended on X0 */
```

QUERY

It is important to note that `f` is used at two different types and that `theta` contains two types, both of which used to be a variable representing an equality type. Both were assigned to the types at which `f` is used.[10] Note that `f` uses the comparison operator on the input list, so it is expected to require this list to be an equality type. [11]

It may be instructive to run the example on a term that does not completely constrain the type at which f is used, i.e., by applying `f` on an empty list: `let` $f = (\lambda x.x = [])$ in $(f[], f[1])$.

[10] For simplicity, we did not write code to remove assigned items from `theta`, since `bind` simply ignores items that are not variables.

[11] In this specific case, `f` compares the input list with the empy one, so the requirement is not strictly necessary, but the type discipline is not precise regarding the behavior of the equality test.

```
goal> theta [],
  of (let (lam x\ eq (glo "nil") x) f\
        app (app (glo "pr")
          (app f (glo "nil")))
          (app f (app (app (glo "cons") (glo "1")) (glo "nil"))))
     Ty.

Success
  Ty = mono (pair bool bool)

Constraints:
 theta [int, X0]
   /* suspended on X1 */
 {f} :> of f (all tt c1 \ mono (list c1 --> bool)) ?- eqbar X0
   /* suspended on X0, X1 */
```

This time, `theta` contains an unassigned variable `X0`. On this same variable, we also have an `eqbar` constraint. Since constraints store their context, and the variable is generated under the binder for `f`, we see the type of `f` as synthesized by the algorithm: `''a list -> bool`, i.e., a polymorphic function from lists to bool, requiring the type parameter to be an equality type.

The code amounts to about 100 lines of Elpi. It is a toy example, in the sense that it lacks code for error reporting, but its size is quite remarkable: it roughly matches the length of a pen-and-paper presentation of the same algorithm.

2.7.6 A word on bidirectional type inference

Perhaps the most unexpected feature of the running example is that it is a bidirectional algorithm. In particular, the rule for application pushes down the expected type of the argument when type checking it.

The monodirectional version could look like the following code:

```
of (app H A) (mono T) :-
  of H (mono TH),
  of A (mono TA),
  assert H TH (TA --> T).

pred assert i:term, i:tye, i:tye.
assert _ TY ETY :- TY = ETY, !.
assert T TY ETY :-
  halt "Error: term" T "has type" TY "but its context expects" ETY.
```

Note how `TH` is not related to `TA` before the call to `of A (mono TA)`, and how the code only checks after the fact that `TH = (TA --> T)`.

2.8 Pitfalls

Over the years, we have put Elpi in the hands of a good number of users. Here are some pitfalls that have tricked everyone (ourselves included).

2.8.1 Misleading precedence of implication

When one writes `A, B => C, D`, what he really means is `A, B => (C, D)`. This becomes clear when one has a program `A, B => D` and wants to sneak in a debug print before `D`. For example `A, B => print D, D` reads as `A, (B => print D), D`.

We argue that `=>` should bind more strongly than "," on the left but not on the right – something that parsing technology does not support out of the box but that can be implemented with a bit of elbow grease.

To improve on this we introduced a new syntax for implication, namely `==>`, that parses with these precedences, i.e., `A, B ==> C, D` means `A, (B => (C, D))`. We also have added a warning whenever `A, B => C, D` is used. This warning can be silenced by adding explicit parentheses as in `A, (B => C), D`.

2.8.2 Treacherous one-rule anonymous predicates

The higher-order capabilities of Elpi let one write code like

```
code L L' :- map L (x\r\ p x A, q A r) L'.
```

where `A` is arguably quantified (allocated) in the wrong place. By convention, variables are considered parameters of the entire rule, not of the inlined sub-rule.

The correct way to write the code above is

```
code L L' :- map L (x\r\ sigma A\ p x A, q A r) L'.
```

where each call to the anonymous rule has a different `A`. The `sigma` operator binds `A` and declares it as a unification variable when it is executed. It is standard in λProlog and fully supported by Elpi, but it is also too easy to forget it.

We believe it would be safer to allow only (partially applied) predicates as higher arguments. The syntactic overhead is often compensated by the extra clarity a (well-)named predicate provides.

```
p-then-q X R :- p X A, q A R.
code L L' :- map L p-then-q L'.
```

Although we have not implemented this idea (and thus not validated it), it seems reasonable to warn whenever a variable *only* occurs in the body of an anonymous rule (a term of function type to `prop`).

2.8.3 Scope error as a (recondite) failure

While `pi x\ F = x` logically fails because `x` is out of `F`'s scope, its practical, legitimate, use cases are rare. According to our experience, most of the time when a unification fails because of a scope problem, it is due to a user mistake.

Although we have not tried to implement any measures to help newcomers, we wonder if making this cause of unification failure fatal (i.e., aborting the program) would help newcomers quickly identify the true mistake. In particular, Elpi already provides built-in predicates to perform bound-variable occur-checking or pruning, allowing the user to avoid resorting to higher-order unification for these tasks.

2.8.4 Symmetric CHR rules

The CHR semantics described in section 2.5.1 behaves intuitively most of the time: rules are scanned in order and applied when applicable. It becomes a bit surprising, according to our experience, when the multiset nature of the constraint store plays a role, particularly when the permutations considered by step 5 contain duplicate constraints and when rules have some symmetry in their patterns.

```
pred c i:int.

main :-
  print "declare c 1",
  declare_constraint (c 1) [_],
  print "declare c 2",
  declare_constraint (c 2) [_].
```

ELPI

```
rule (c N) (c M) <=>
  (print "rule 1 on" N M).
rule (c N) <=>
  (print "rule 2 on" N).
```

CHR

```
declare c 1
rule 2 on 1
declare c 2
rule 1 on 2 1
rule 1 on 1 2
rule 2 on 2
```

QUERY

```
rule (c N) \ (c M) <=>
  (print "rule 1 on" N M).
rule (c N) <=>
  (print "rule 2 on" N).
```

CHR

```
declare c 1
rule 2 on 1
declare c 2
rule 1 on 2 1

rule 2 on 2
```

QUERY

According to the rule set on the left-hand side of the figure, rule 1 triggers twice because the second `c` constraint is active in both positions 1 and 2. This also occurs in the second rule set, but since the rule removes one (copy) of `c` from the store – and thus from the set of permutations – the rule cannot trigger twice.

Symmetry can also be broken by using a guard, e.g., requiring N < M, as depicted below:

```
rule (c N) (c M) | (N < M) <=>
  (print "rule 1 on" N M).
```

CHR

Since symmetry can be broken by using an antisymmetric predicate in the guard, it seem difficult to implement a sensible warning for this issue.

Chapter 3

Elpi the software: *de A à Z*

3.1 The first prototype

The first version of Elpi was written by me, with the objective of understanding the λProlog language and some of the techniques adopted to implement it, such as the suspension calculus [Nad02]. This proof of concept, that we call Elpi-POC, was written in OCaml [Ler+] in about three months.

Elpi-POC served as an easy playground to understand which features were needed to manipulate terms with holes: a safeguard against instantiating them by mistake (modes), and a way to attach metadata to holes (constraints).

Elpi-POC was functional enough to let us write an elaborator for dependent type theory and plug it into Matita [Asp+11]. It worked, but it was extremely slow – about 20,000 times slower than reasonable.

The main reason was that terms were purely functional, to facilitate backtracking: unification variables were encoded with numbers, and the execution would carry a companion purely-functional map carrying variable assignments made by unification. In this setting, backtracking, a feature with a reputation for making things difficult, becomes trivial, since holding a pointer to a previous version of the assignment map is enough to time travel. At the same time decorrelating terms from the assigment map, a heap really, means one cannot take advantage of the programming language automatic memory management: when a unification variables disappears from a term, no mechanism removes it from the map, and worse, the OCaml garbage collector has to scan this evergrowing map.
Introducing instructions to clear memory (in an unsafe way) was enough to see one order of magnitude of slowness disappear.

Also, the suspension calculus did not seem to help. Suspension calculus looked similar to explicit substitutions, but until the very last paper in the pile on my desk, it was not clear it was actually isomorphic to λ_σ [Aba+91]. While explicit substitutions are often presented as a way to obtain better complexity, in practice they fail to deliver better performance. In Rocq, two people tried using explicit substitutions in many places with similar conclusions.[1] In Elpi-POC, in our limited tests, disabling them resulted in a slightly faster runtime. Our analysis is that, in practice, the extra machinery to delay substitutions and pack them together rarely triggers (as an optimization to achieve better complexity), but it has a constant memory overhead. Most of the time, all redexes are fired in one go, and the result is immediately inspected, practically defeating the advantage of a lazy substitution.

[1] To my knowledge there is no written trace of these two independent experiments, that were reported to me by Thomas Braibant and Matthias Puech. To be honest, the so called lazy reduction machine by Bruno Barras currently employed in Rocq's kernel uses explicit shifts, i.e., one part of the job substitution performs is explicit, but not all of it.

Still, it was clear that the operation of crossing a binder and replacing the bound name with a fresh constant (i.e., a fresh name) was very frequent and required a fast implementation.

3.2 Runtime

The runtime of Elpi was developed in close collaboration with Claudio Sacerdoti Coen and Cvetan Dunchev around 2015, and is described in [Dun+15].

In this section, we summarize the main ingredients of that runtime.

3.2.1 Imperative terms, frames, and trail

In logic programming, computation makes progress by assigning unification variables. Therefore, the operations of assigning and retrieving the assignment, if any, must be efficient. Even if not very frequently – at least not in our use cases – assignments must be undone to implement backtracking.

Our experience with the first Elpi prototype suggested that purely functional terms and an explicit assignment map are not a viable solution, and pushed us toward an implementation closer to standard Prolog runtimes [Aït91]. Although we did not even consider targeting the Warren Abstract Machine (WAM) as the system Teyjus does [NM99], since we felt it was beyond our engineering capabilities, Elpi terms are mutable, and we store a log of assignments to undo them, as is done in the WAM.

The data type of terms is given in fig. 3.1. Pure terms represent the λ-calculus part of the language, where the **`Const`** node is used for both globally and locally bound names (i.e., global constants and bound variables). We detail the representation in terms of De Bruijn levels in section 3.2.2. Optimized terms are simply lists and the discard placeholder _. The Foreign Function Interface (FFI) part is for calling built-in predicates and for opaque data, which we detail in section 3.4.2. What remains are nodes for unification variables, and we postpone the description of **`UVar`** and **`Arg`** to section 3.2.4, since they are optimized versions of the nodes **`AppUVar`** and **`AppArg`**.

The nodes **`AppUVar`** and **`AppArg`** represent applied unification variables, with the difference that the former, which we call *uvar*, is heap-allocated, while the latter, which we call *arg*, is stack-allocated and is used to represent the parameters (i.e., arguments) of rules.

Each uvar contains a mutable slot for a term, a unique identifier used by the FFI, and a boolean flag to signal if the variable acts as a trigger for a constraint. Each arg contains the index of a slot in the activation frame (a term array) that is allocated each time a rule is used. The decision to distinguish between uvars and args comes from the WAM, which is itself inspired by standard programming languages: using a rule corresponds to a function call, which is usually implemented via the allocation of an activation frame on the stack. In our runtime, everything is allocated in the OCaml heap, but the stack frame becomes garbage (collectable by the garbage collector) as soon as the subgoals for a rule are generated. Because it is so short-lived, the stack frame is handled efficiently by the OCaml garbage collector (much like the stack is a piece of memory efficiently reclaimed).

As in the WAM, we keep the invariant that no heap term points to the stack; i.e., no term contained in a `uvar` has an **`Arg`** or **`AppArg`** node. Heap terms are longer lived, they are typically kept alive by a uvar present in the initial query, while stack frames are volatile.

The *vardepth* of a uvar is an integer representing the number of `pi`-generated fresh names existing when the uvar is created. We will write X^n for the uvar X with vardepth equal to n. This

```
type constant = int (* De Bruijn levels *)
type term =
  (* Pure terms *)
  | Const of constant
  | Lam of term
  | App of constant * term * term list

  (* Optimizations *)
  | Cons of term * term
  | Nil
  | Discard

  (* FFI (Foreign Function Interface) *)
  | Builtin of builtin_predicate * term list
  | CData of CData.t

  (* Heap terms: unif variables in the query *)
  | UVar    of uvar * int (* number of arguments *)
  | AppUVar of uvar * term list (* arguments *)

  (* Stack terms: unif variables in the rules *)
  | Arg    of int * int (* number of arguments *)
  | AppArg of int * term list (* arguments *)

and uvar = {
  vardepth : int; (* depth at creation time *)
  mutable contents : term;
  mutable uid : int; (* negative if it blocks a constraint *)
}
```

OCAML

Figure 3.1: The term data type

is already an optimized representation of unification variables, since one could simply use the list of explicit arguments to put these `pi`-generated names in scope. This representation is also used in Teyjus [Qi09].

When the `contents` field of a `uvar` is assigned by the unifier, the same `uvar` is also logged in the trail.

OCAML

```
type trail_item =
| UVarAssignement of uvar
| ConstraintAddition of cst
| ConstraintRemoval  of cst
type trail = trail_item list
```

The trail also features two other kinds of events that may need to be undone: the addition or removal of a constraint to the store. In particular when an uvar is assigned and its `uid` is negative some constraint may be resumed, hence removed from the store. Conversely the `declare_constraint` built-in predicate adds a constraint to the store.

However, we will not provide further details on this. Instead, we focus on the higher-order part of the language, the `Const` node.

3.2.2 Levels, not indexes

In his seminal paper [Bru94], De Bruijn introduced two dual naming schemes for λ-terms: one based on depth indexes (DBI) and one based on levels (DBL). In the former, a variable is named n if its binder is found by crossing n binders going toward the root. In the latter, a variable named n is bound by the n-th binder encountered in the path from the root to the variable occurrence.

Below, we write the term $\lambda x.(\lambda y.\lambda z.f\ x\ y\ z)\ x$ and its reduct in the two notations. We subscript the variable name by the index or level that represents it.

$$\begin{array}{lll} \text{DBI:} & \lambda x.(\lambda y.\lambda z.f\ x_2\ y_1\ z_0)\ x_0 \ \to_\beta & \lambda x.\lambda z.f\ x_1\ x_1\ z_0 \\ \text{DBL:} & \lambda x.(\lambda y.\lambda z.f\ x_0\ y_1\ z_2)\ x_0 \ \to_\beta & \lambda x.\lambda z.f\ x_0\ x_0\ z_1 \end{array}$$

Indexes are the most widely adopted representation, since most programming languages based on the λ-calculus perform weak reduction. This operates at the root of the term and always substitutes a closed term for the variable bound by the topmost lambda. In DBI notation, closed terms are invariant by lifting, unlike the “free” x_0 and x_2 that had to become x_1 in the example above. With levels, x_0 did not require any shifting, but the variable z, bound under the redex, had to be adjusted since the number of binders before it decreased. Free variables – or, more precisely, variables bound by the context under which reduction takes place – are invariant by shifting in DBL notation, while variables bound deeper are not. For example, $\mathcal{I} = \lambda x.x$ has a different representation in DBL per depth: under n binders, $\mathcal{I}$ becomes $\lambda x.x_{n+1}$.

By convention, we name bound variables in De Bruijn Level notation c_i, where i is the level, starting from 0. Global constants, like f, are represented by negative numbers (and are never shifted).

In λProlog, β-reduction plays two very different roles, even though they are coalesced into a single concept: substitution.

When evaluating a redex $(\lambda x.M)\ N$, one can substitute an arbitrary term N into M, and this powerful operation is very handy for implementing substitution for the object language. For example, the following two lines of code implement weak head reduction for the λ-calculus as encoded in section 2.2, by simply recognizing a redex and delegating to the β-reduction of λProlog the work of plugging `Arg` inside `F` wherever needed:

```
whd (app Hd Arg) Reduct :- whd Hd (lam F), !, whd (F Arg) Reduct.
whd X Reduct :- Reduct = X.
```

Still, according to our experience, the most widely used form of substitution is that of replacing one name with another, i.e., the one called *binder mobility* [Mil18]. For example, the type checker in section 2.2, displayed below with an explicit η-expansion around `F`, replaces the name `x` with the name `c`.

```
of (app H A) T :- of H (arr S T), of A S.
of (lam (x\ F x)) (arr S T) :- pi (c\ of c S => of (F c) T).
```

As per [MIL91], this restricted form of β-reduction is called β_0-reduction, and it is sufficient to execute λProlog code in the L_λ fragment (section 3.2.3), and in particular the code of the `of` predicate above.

If we represent the second rule in DBL notation at depth n, i.e., we imagine we have crossed n object-level binders with that rule, we have:

```
of (lam (x\ $F^n$ $x_n$)) (arr $S^n$ $T^n$) :-
  pi (c\ of $c_n$ $S^n$ => of ($F^n$ $c_n$) $T^n$).
```

Note how the first bound variable in the data, x, is represented with the same De Bruijn level as the first `pi`-quantified variable in the rule body. Once `pi` and `=>` have been executed, obtaining the β_0-normal form of `F` c_n amounts to replacing (in the term bound to `F`) all occurrences of x_n with c_n (note the *same* n). This is simply the identity, i.e., it is for free!

This observation convinced us to choose DBL as the notation for variables and to identify the L_λ^β fragment, a further restriction of the L_λ one.

3.2.3 The L_λ fragment

In [MIL91], Miller identifies a stable fragment of λProlog in which higher-order unification is decidable and generates most general unifiers. This fragment is large enough to cover many λProlog programs and is the basis for both Teyjus (version 2) and Elpi.

Definition 1 (L_λ fragment)**.** *A term is in the L_λ fragment if and only if every occurrence of a unification variable X^j is applied to $c_{a_1} \dots c_{a_n}$, where $c_{a_i} \neq c_{a_j}$ and $j \leq a_i$, with i and j distinct and i ranging between 1 and n.*

In other words, unification variables must be applied only to a duplicate-free list of names. In this fragment, all β-reductions are in fact β_0-reductions that, in some cases, can be implemented in constant time when terms are represented in DBL notation.

It is natural to further investigate which cases admit the efficient β_0-reduction implementation described above. For example, the following piece of code does not admit it:

ELPI

```
of (lam (x\ $F^n$ $x_n$)) (arr $S^n$ $T^n$) :-
  pi (d\ pi (c\ of $c_{n+1}$ $S^n$ => of ($F^n$ $c_{n+1}$) $T^n$)).
```

Intuitively, the program postulates more fresh names than there are bound ones in the data being traversed, causing x_n to be replaced by c_{n+1} in the recursive call. The optimization can only be applied when the bound variables in the data perfectly align with the fresh names postulated to move the binding from the data to the program. In this respect, (F^n c_{n+1}) presents an anomaly.

3.2.4 The L_λ^β fragment

In [Dun+15], we further restrict L_λ as follows:

Definition 2 (L_λ^β fragment)**.** *A term is in the fragment L_λ^β if and only if every occurrence of a unification variable X^j is applied to the k arguments $\underbrace{c_j \dots c_{j+k-1}}_{k \text{ items}}$ for $0 \leq k$, where $k = 0$ means no arguments.*

Note that X^j $c_j \dots c_{j+k-1}$ admits a space-efficient representation (the **UVar** node), since one can simply store j and k, two integers.

OCAML

```
type term =
  ...
  | UVar of uvar * int (* number of arguments *)
  | Arg  of int  * int (* number of arguments *)
```

These nodes are leaves, i.e., their size is constant in the number of binders that were crossed (and put in scope of the uvar/arg), while the nodes **AppUVar** and **AppArg** are linear in that number.

Also note that X^j $c_j \dots c_{j+k-1}$ can be rewritten as a Y^{j+k} by assigning to X^j the term $\lambda c_j \dots c_{j+k-1}.Y^{j+k}$. Moreover, for a unification variable in the compact form X^n, checking if a name c_m is in scope is as trivial as checking $m < n$; there is no need to look for c_m in the list of (distinct) arguments that the L_λ fragment allows for. When scope checking fails for a variable, it is easy to prune it; i.e., $X^n = Y^m$ with $n < m$ can be solved by assigning to Y^m a fresh Z^n. Finally, all global symbols, being represented by negative integers, naturally pass the scope check.

It is also worth pointing out that, in DBL notation, the variables bound by the program are invariant by shifting. Hence, "moving around" an **UVar** node is always efficient, unlike the more general case of **AppUVar**, where the arguments may need shifting (their type is `term`; they can be anything, even $\mathcal{I}$).

For quite some time, the Elpi runtime only had higher-order unification implemented for the L_λ^β fragment (although it always had full β-reduction). The Elpi test suite, which includes many λProlog programs taken from the Teyjus examples or λProlog-related publications, was mostly passing. Our intuition is that, in practice, higher-order unification is used either to simply pass data around or to ensure proper scoping of bound names (e.g., making a unification step fail). It is more rarely used to reconcile different scopes, i.e., finding the permutation to send one list of distinct names to another, or computing the intersection of two for pruning flexible terms and keeping execution going. It was only around 2016 that I extended the unification procedure to the full L_λ fragment.

3.2.5 Indexing

If one looks back at the $\mathcal{B}$ function at page 19, one immediately sees room for optimization. The `unify` function is run on all the rules for the current predicate that compose the program, one after the other. It is well known that one can *index* multiple heads into tree-like structures and, in a single pass, filter out heads that have no chance to unify.

Over time, the runtime was equipped with three indexing algorithms, but before presenting them, it is worth pointing out another advantage of the DBL notation. Let us observe once more the rule to type check a lambda abstraction:

```
of (lam (x\ $F^n$ $x_n$)) (arr $S^n$ $T^n$) :-
  pi (c\ of $c_n$ $S^n$ => of ($F^n$ $c_n$) $T^n$).
```

ELPI

Once `pi` is executed, c_n enters the program and becomes invariant by shifting, i.e., wherever it occurs it is represented as the number n. This simplifies indexing, since the hypothetical rule `(of` c_n S^n`)` can be indexed on c_n with no further complication.

3.2.5.1 Map

As in standard Prolog, the Map index only considers one predicate argument at depth 1. The user can choose which argument to index (usually the first one).

```
:index(_ 1) % second argument at depth 1 using "Map" (the default)
pred example i:term, i:term, o:term.
```

ELPI

Functors are represented as integers, from `-1` to `min_int`, and these integers are indexed using a Patricia tree implementation by Jean-Christophe Filliatre[2], that in turn is based on a paper by Okasaki and Gill [OG98]. Each integer is seen as a "list" of bits, and these lists are organized into a trie (tree of prefixes). The main idea is that common sublists are compressed to a single tree node; i.e., two lists differing only in the last bit are represented as a tree of depth two.

The only detail worth mentioning is that, in order to answer queries where the indexed argument is flexible, or when it must be flexible (the **`uvar`** keyword for input arguments), the index for each predicate is accompanied by two lists of rules: those to be returned in these two corner cases.

This index proved to be fast, but does not work well when the discriminating functor is deep, which is quite frequent when the rules talk about object-language terms. For example, the Rocq term `nat` is represented as, roughly, `global (indt "nat")`, while the term `x + 0` is represented by a tree where the head node, `"plus"`, is at depth 5!

```
% 1   2 3       4     5
  app [ global (const "plus"), x, global (indc "O")].
```

ELPI

[2] https://usr.lmf.cnrs.fr/~jcf/software.fr.html

3.2.5.2 Hash map

If more than one argument needs indexing, or if an argument needs to be indexed deeper, Elpi provides an index based on the idea of unification hashes. Such an index was implemented by Dunchev in 2015 and is based on a 2013 blog post by Hendricks.[3]

The hash value, a list of bits, is generated hierarchically up to the specified depth and is indexed using the Patricia tree implementation mentioned in the previous section. Unification variables in a rule head are mapped to a sequence of `1`s, while they are mapped to a sequence of `0`s when they are part of the query. Constants are mapped to a hash value – a sequence of both `1`s and `0`s. If the bitwise conjunction `&` of the hash of the query and the hash of the head of a rule is equal to the hash of the query, then the rule is a match. Intuitively:

- in a rule, `1` means "I provide this piece of info"
- in the query, `1` means "I demand this piece of info"

A flexible argument in the query is made of `0`s; hence, it demands nothing, since `0 & x = 0` for any bit `x` of the rule. Conversely, a flexible argument in the rule is made of `1`s; hence, it provides anything, since `x & 1 = x` for any bit `x` of the query.

For example, consider the following code featuring a fast path (rule 2):

```
kind nat type.
type o nat.
type s nat -> nat.

:index (2) "Hash" % index the first argument at depth 2 using hashes
pred mult i:nat, o:nat, o:nat.
mult o X o.                                  % rule 1
mult (s (s o)) X C :- plus X X C.            % rule 2
mult (s A) B C :- mult A B R, plus B R C.    % rule 3
```

Also, imagine a word size of 8 bits, and that the hash of `o` is `1001 1011`, while the hash of `s` is `1011 0010`.

Hashes for each argument are obtained by combining the hashes of the subterms, somewhat trimmed to fit the word size.

rule	argument	hash
1	`o`	1001 1011
2	`s (s o)`	0010 0010 (4 bits for each `s`)
3	`s A`	0010 1111

Now we consider the following three query arguments, their hashes, and their matches.

argument	hash	matched rules
s o	0010 1011	3
s X	0010 0000	2, 3
X	0000 0000	1, 2, 3

[3] http://blog.ndrix.com/2013/03/prolog-unification-hashes.html

In our testing, unification hashes are on par with the standard indexing technique, but they provide greater flexibility. The only downside is that it is hard to predict when collisions will happen, since hashes are trimmed to fit one OCaml word (63 bits). This limitation is imposed by the underlying Patricia tree implementation. Extending the implementation to work on multiple words is surely possible, but we did not explore it.

3.2.5.3 Discrimination tree

Fissore integrated into Elpi 3.0 an index based on discrimination trees, implemented following Pfenning's notes on automated theorem proving.[4]

```
:index(50 50) "DTree" % both arguments capping depth at 50
pred example i:term, i:term, o:term.
```

ELPI

The discrimination tree is essentially a trie of paths, and a path is a linearization of a term. Path items are symbols equipped with their arity, so that one can easily, although not with $O(1)$ complexity as in skip lists, ignore an entire subtree. By concatenating paths, one can easily apply the index to multiple arguments.

Fissore implemented two main optimizations. The first is to record the maximum depth (path length) needed to index program rules (also capped by a user-given bound). This value is used to cap the depth at which the query is inspected in order to generate its corresponding path, which is then searched in the trie. Queries are typically much larger than the rules' heads.
The second optimization is to flatten lists, i.e., a tree like the following:

```
app [global (const "plus"), x, global (indc "O")]
```

LP

is represented as if the `app` node had three children, i.e., the leaves `"plus"` and `"O"` are at the same depth. Special care is taken to handle open lists, i.e., when the tail of a list is a unification variable.

For the majority of λProlog programs, the discrimination tree is slower than the simpler, shallower ones presented before. Many ubiquitous predicates, like the list processing ones `map` or `mem`, have very few rules. Any index does a good job discriminating them, so the one with less constant overhead wins.

Still, the discrimination tree finds a good number of applications. Some tools, like Hierarchy-Builder (see section 5.2), make use of the Elpi program as a database, accumulating into it tens of thousands of rules for the same predicate, with arguments not very different from the term above. All these rules can be discriminated efficiently only by a deep index.

Another application is the determinacy checker recently added by Fissore [FT25]. In particular, the test for overlapping rules can be performed easily and efficiently by indexing all rules' heads with a discrimination tree at unbounded depth.

Another application, yet to be explored, is to use discrimination trees to implement a caching mechanism in the spirit of tabling [SSW94; Pie05a; SUM20], i.e., to index *all* the goals that are generated during an expensive computation. It is hard to imagine a shallow index performing well in this scenario.

[4] https://www.cs.cmu.edu/~fp/courses/99-atp/lectures/lecture28.html

3.3 Compiler

First of all, it should be noted that the Elpi compiler is not a traditional compiler generating (virtual) machine code. Still, before being interpreted, Elpi programs go through several steps typical of compilation pipelines. The Elpi compiler does:

0. massage the AST recovering structure
1. elaborate away the syntactic sugar, i.e. namespaces and spilling
2. run a pretty standard type checker
3. run a determinacy checker (since Elpi 3.0)
4. optimize the rules
5. allocate unique numbers for global symbols
6. link together (i.e. index) all rules

Task zero is really about recovering structure in the parsed text, such as grouping together signature declarations, rule declarations, constraint handling rules, etc. Finally, terms go through a scope analysis to separate locally bound names from global ones.

Task one eliminates syntactic sugar. Namespaces are a common concept in real-world programming languages, even if we could not really find a canonical reference to point to. Elaborating them away amounts to always using the long name for global identifiers, even when the user wrote a short one. Spilling is less common. It resembles the generation of the administrative normal form [SF92], where large expressions are decomposed into atomic ones via the introduction of intermediate binders. A related work is [CGM18], even if it considers a reasoning logic for λProlog and not the language itself.

Task two is inspired by [NP92], even if Elpi performs no inference of polymorphic types. Elpi supports overloading symbols: their type-driven resolution is inspired by the type inference procedure implemented by Harrison in Hol-light [Har09].

Task three is documented in [FT25], which was written too late to be made an integral part of this document. In a few words, Elpi 3.0 introduces more precise signatures for predicates, letting the user specify if the predicate leaves choice points or not. If it does not, it is called a function. A function is made of rules that can be statically deemed mutually exclusive, and that only call other functions, or call predicates but discard their choice points via a cut. Signatures for higher-order predicates like `map` are interpreted as conditional statements: if the higher-order argument `F` passed to `map` is a function, then so is `(map F)`.

The compiler performs very few optimizations in task four. In particular, it pre-processes clauses by identifying the variables, so as to know in advance the size of the activation frame (as mentioned in section 3.2.1). This procedure is also needed at runtime when hypothetical rules are loaded via implication; hence, this pass has to be lightweight: hypothetical rules are rarely used many times. One small optimization, fully performed statically, is to recognize lists and represent them with dedicated, more compact nodes in the syntax tree. Lists are ubiquitous.

A component that, instead, received quite a few optimizations is the linker, which is in charge of the last two tasks. Elpi supports separate compilation, i.e., programs can be made of units that are compiled at different points in time. In particular, most Elpi programs for Rocq generate new

rules (see, for example, section 4.5) that are then compiled and linked to the program the next time it is executed. In this sense, the efficiency of the linker directly impacts the startup time of programs; hence, by optimizing it, one speeds up the perceived execution time.
Some relevant Elpi applications, like Hierarchy-Builder (see section 5.2), accumulate over time more than 40,000 rules. We had to craft a linker that is as incremental as possible, i.e., its computational complexity depends (mainly) on the unit extending the base program, and does not depend on the size of the base program itself.

Another aspect of the compiler worth mentioning is that, since version 2.0, it carries the source-code location of each term from the parser to the static analyzers. In turn, this makes it possible to report precise error messages and attach to each subexpression some metadata, like the inferred type of a compound expression or the declared signature of a symbol. This information can be exploited by modern editors to improve navigation of the source code.

3.4 The API

The APIs to drive the interpreter essentially let one parse text, compile it into a unit, assemble units into an executable, and run it on a query. These APIs are not very interesting, so we focus on the other APIs that are instrumental for integrating Elpi with the host application, i.e., the ones that actually make it an *extension* language.

3.4.1 Quotations

λProlog fits in the galaxy of Logical Frameworks [HHP87], i.e. languages designed to describe and manipulate another language, usually called the *object language*. The object language is encoded using the constants and constructs (like variable binding) belonging to the framework.

An important aspect when mixing two languages, in practice, is to be able to use both syntaxes in a non-ambiguous way and with very little overhead. In particular, when Rocq is the object language, one would like to use Rocq's syntax as much as possible to write Rocq terms: Rocq's parser understands many nice, compact notations.

Elpi's parser provides a notion of quotation and anti-quotation: text delimited by matching-in-number curly braces is lexed/parsed as a verbatim piece of text, to later be processed by an object-language-specific parser, which in turn can call back the Elpi parser. The object-language parser must ultimately generate an Elpi syntax tree corresponding to the encoding of the object-language expression that was parsed.

The following code is an example mixing the two languages.

```
  rocq.say {{ 1 + lp:{{ app[global S, {{ 0   }} ]  }}    }}
% elpi....   rocq..     elpi........... rocq  elpi  rocq
```

Here, "1 + ..." and "0" are Rocq syntax delimited by the quotation markers, and `lp:{{ ... }}` is Rocq's way to escape back to Elpi.

There is only one rule governing quotations: a variable is bound in one language and only one. That is, variables bound in Rocq are not available in Elpi without entering a quotation. The following code results in an error, since `x` is undeclared (in Elpi).

```elpi
  rocq.say {{ forall x : nat, lp:{{ x }} }}
%                                   ↑
%                                   error: x is undeclared
```

Conversely, the following code is valid, since x is bound and used in Rocq's space.

```elpi
rocq.say {{ forall x : nat, lp:{{... {{ x }} ...}} }}
```

In practice, this rule amounts to completely delegating to each language the lexical analysis of its source code, with no interference.

Although important, syntax alone is not sufficient to provide tight integration with the host application. In particular, one wants to be able to call existing code from Elpi, either to access the internal state of the application or just to use some existing routine, like rocq.say to display a message to the user.

3.4.2 FFI: Foreign Function Interface

The Foreign Function Interface (FFI) of Elpi is composed of two parts. One is used to describe how *data* can be exchanged between Elpi and the host application. The other is used to describe the signature of host routines that are made available to Elpi programs as *built-in predicates*. We cover both in order.

3.4.2.1 First-order data

First-order data is data whose exchange does not depend on the Elpi context of hypothetical rules and constraints. Such data must be equipped with a Conversion.t declaration for its type.

```ocaml
module Conversion : sig

type 'a embedding =
  depth:int ->
  state -> 'a -> state * term * extra_goals

type 'a readback =
  depth:int ->
  state -> term -> state * 'a * extra_goals

type 'a t = {
  ty : ty_ast;                         (* type declaration *)
  pp : Format.formatter -> 'a -> unit; (* for printing/debugging *)
  embed : 'a embedding;                (* 'a -> term *)
  readback : 'a readback;              (* term -> 'a *)
}
```

A conversion consists of an embedding function that converts the data to an Elpi term and a readback function that goes the other way around. The type AST ty is used to generate a type declaration in the Elpi syntax, i.e., to document the existence of a data type.

The `depth` argument is present mainly because most of the other APIs work for the general case of terms with binders, and in DBL notation it is of paramount importance to know under how many binders we are operating. When the data has no binders, it serves no real purpose in the conversion code, but passing a meaningful value to the other APIs is still a sensible things to do. The `state` part is a convenience: as described in section 3.4.2.4, the programmer can store any pure datum, and Elpi takes care to keep this state in sync with backtracking.
Extra goals are typically of the form `X = t`, i.e., to communicate an assignment to Elpi. For example, one may want to craft a datum with a default value; when omitted in a call to a built-in predicate, it gets assigned by generating an extra goal for the Elpi runtime to solve.

Opaque data The simplest form of data is that without a syntax, i.e., opaque to Elpi. Elpi programs can pass such data around and operate on it via built-in predicates only. Strings, numbers, and file descriptors are examples of this. Elpi provides a high-level API to craft a conversion for such data.

```
module OpaqueData : sig

  type 'a declaration = {
    name : string;
    doc : string;
    pp : Format.formatter -> 'a -> unit;
    compare : 'a -> 'a -> int;
    hash : 'a -> int;
    hconsed : bool;               (* do hashcons *)
    constants : (name * 'a) list; (* global constants of that type *)
  }

  val declare : 'a declaration -> 'a Conversion.t
```

OCAML

The main requirement for an opaque datum is a comparison function, which is used by Elpi's unifier when a `CData` node is encountered, as well as a hash function for indexing. For convenience, the declaration API also allows the datum to be hash-consed and enables the declaration of a list of constants for it. From this declaration, the API can generate Elpi code describing the data type and its known constants.

Algebraic data Even without binders, that we cover in the next section, algebraic data types are a powerful programming abstraction, and usual OCaml code makes extensive use of them. An algebraic data type consists of a list of constructors (i.e., function symbols that build the data type) and works in tandem with pattern matching to destructure it.

Elpi's API to declare an algebraic data type is based on OCaml's Generalized Algebraic Data Types (GADTs) [SP08]. GADTs are a simple version of dependent types, where the indexes, and hence the equations considered for type checking, can only range over types. Below, we present the declaration for the `option` type.

```
let option (a : 'a Conversion.t) : 'a option Conversion.t =
  let open AlgebraicData in
  declare {
    ty = TyApp("option", a.Conversion.ty, []); % type AST
    doc = "The option type (aka Maybe)";
    pp = (fun fmt o ->
      Format.fprintf fmt "%a" (Util.pp_option a.Conversion.pp) o);
    constructors = [
      K("none","",N,
        B None,
        M (fun ok ko -> function None -> ok | _ -> ko ()));
      K("some","",A(a,N),
        B (fun x -> Some x),
        M (fun ok ko -> function Some x -> ok x | _ -> ko ()));
    ]
  }
```

OCAML

We focus on the `K` constructor, which encapsulates a constructor name, a first-class description of its signature, and two functions that serve to build and to recognize-and-destructure the OCaml constructors. For the `None` constructor, the Elpi symbol is `"none"`, which takes no arguments (the `N` signature). The `B` component provides the function to build `None` from no arguments, while `M` is the function that recognizes and destructures it. The `ok` continuation is typed according to the constructor's signature and takes the same arguments. The case for `Some` is more interesting, as it takes an argument of a type described by the signature `a`, which is a parameter of the entire declaration. Thus, `B` expects a function of one argument, and similarly, the continuation `ok` can only be invoked by passing it an argument of the type described by `a`.

The resulting Elpi code is shown below.

```
% The option type (aka Maybe)
kind option type -> type.
type none option A.
type some A -> option A.
```

ELPI

An OCaml expert might wonder why we did not write a PPX (pre-processing extension) for the OCaml compiler to synthesize an equivalent `Conversion.t` from the option type declaration. In fact, we did, but we never found the time to finish it[5], especially because we wanted it to also support higher-order data, which is substantially more complex.

3.4.2.2 Higher-order data

Higher-order data, especially when viewed through the lens of λProlog and its role as a logical framework, appears to be intimately related to the notion of context. Although we will discuss and motivate this further in chapter 4, we can provide a simple example here.

Suppose the host application encodes binders using De Bruijn Index (DBI) notation; that is, its numerical value n represents the distance of the variable from its binder (in the context Γ). In Elpi, the same variable is represented in De Bruijn Level (DBL) notation by the number m, which

[5] https://github.com/LPCIC/elpi/pull/63

is equal to the position of the variable in Γ. Thus, $m = |\Gamma| - n$: one cannot convert n to m or vice versa without knowing Γ (or, in this case, its length).

A `ContextualConversion.t` provides access to the embedding and read-back functions for a context and constraint store. The corresponding data types are determined by the `ctx_readback` companion function.

```
module ContextualConversion : sig

type ('a,'hyps,'constraints) embedding =
  depth:int -> 'hyps -> 'constraints ->
  state -> 'a -> state * term * extra_goals

type ('a,'hyps,'constraints) readback =
  depth:int -> 'hyps -> 'constraints ->
  state -> term -> state * 'a * extra_goals

type ('a,'h,'c) t = {
  ty : ty_ast;
  pp : Format.formatter -> 'a -> unit;
  embed : ('a,'h,'c) embedding;    (* 'a -> term *)
  readback : ('a,'h,'c) readback; (* term -> 'a *)
}

type ('hyps,'constraints) ctx_readback =
  depth:int -> hyps -> constraints ->
  state -> state * 'hyps * 'constraints * extra_goals
```

Operationally, the `ctx_readback` function is executed first to obtain the host application context, after which the datum is read back with that context available.

3.4.2.3 Built-in predicates

The signature and OCaml code of a built-in predicate are encapsulated by the `Pred` constructor, as shown in fig. 3.2. The signature is a GADT that specifies the type of the latter. The arguments of a built-in predicate can be of three kinds: input, output, or both.

- An input argument of type `A` requires that the (OCaml) type of the predicate `P` is of the form `A -> P'` for some `P'`.

- An output argument of type `B` requires that the type of the predicate `P` is of the form `bool -> P' * option B`. The boolean indicates whether the caller is binding the result (i.e., passing _ or an actual unification variable). This allows the programmer to optimize the code by not generating the output if it is not requested. As a result, the number of built-ins can be kept small without sacrificing efficiency.[6]

- An input-output argument of type `B` requires that the type of the predicate `P` is of the form `option B -> P' * option B`. In this case, the programmer is informed whether the user has provided some data or not.

The three types of arguments also have their contextual counterparts. The list of arguments can terminate with either **`Easy`** or **`Full`**. The latter requires a context read-back function that fixes a context and constraint data type used by the contextual arguments.

Here we present an example of a simple built-in predicate; an example using contextual data is postponed to section 4.2.

```
module BuiltInData : sig
  val string : string Conversion.t
...

let getenv : BuiltInPredicate.t =
  Pred("getenv",
    In(string,  "VarName",
    Out(option string, "Value",
    Easy ("Like Sys.getenv"))),
     (fun s _ ~depth ->
        try !:(Some (Sys.getenv s)) (* !:x  stands for  (), Some x *)
        with Not_found -> !: None))
```

This declaration produces the following Elpi code, which declares and documents the built-in predicate.

```
% [getenv VarName Value] Like Sys.getenv
external pred getenv i:string, o:option string.
```

Built-in predicates, like conversions, are allowed to access and update a state that is maintained by Elpi and synchronized with respect to backtracking, but is not directly visible from the programming language. This state serves to carry information specific to the host and is sometimes necessary to implement sophisticated embedding/read-back functions, as discussed in section 4.3.

3.4.2.4 Extensible state

The extensible state API is not particularly surprising. The only notable detail is that it assumes the data is pure; that is, there is no need to call any routine from the host system to restore a previous state upon backtracking. This limitation could be lifted, but so far there has been no real need for it.

[6] A single built-in can serve the purpose of a dedicated built-in for each output. For example, `rocq.env.indt` provides access to all information about an inductive type, from the number of parameters (an integer) to the types of each constructor (a much larger data structure). If only the former is needed, one can simply pass _ as the second argument and incur no performance penalty compared to a dedicated `rocq.env.indt-num-params` that would compute only the former.

```
module Conv = Conversion
module CConv = ContextualConversion
module BuiltInPredicate : sig

type ('function_type, 'inernal_outtype_in, 'internal_hyps, 'internal_constraints) ffi =
(* Arguemnts that are translated independently of the program context *)
| In    : 't Conv.t * doc * ('i,'o,'h,'c) ffi -> ('t -> 'i,'o,'h,'c) ffi
| Out   : 't Conv.t * doc * ('i,'o * 't option,'h,'c) ffi -> (bool -> 'i,'o,'h,'c) ffi
| InOut: 't Conv.t * doc * ('i,'o * 't option,'h,'c) ffi -> ('t option -> 'i,'o,'h,'c) ffi

(* Arguemnts that are translated looking at the program context *)
| CIn    : ('t,'h,'c) CConv.t * doc * ('i,'o,'h,'c) ffi -> ('t -> 'i,'o,'h,'c) ffi
| COut   : ('t,'h,'c) CConv.t * doc * ('i,'o * 't option,'h,'c) ffi -> (bool -> 'i,'o,'h,'c) ffi
| CInOut: ('t,'h,'c) CConv.t * doc * ('i,'o * 't option,'h,'c) ffi -> ('t option -> 'i,'o,'h,'c) ffi

(* All arguments are context independent *)
| Easy: doc -> (depth:int -> 'o, 'o, unit, unit) ffi

(* Arguments are context dependent, here we provide the context readback function *)
| Full: ('h,'c) CConv.ctx_readback * doc ->
    (depth:int -> 'h -> 'c -> state ->
      state * 'o * extra_goals, 'o,'h,'c) ffi
...

type t = Pred : name * ('a,unit,'h,'c) ffi * 'a -> t
```

Figure 3.2: Predicate declaration API

```
module State : sig

  val new_state_descriptor : unit -> Setup.state_descriptor

  type 'a component

  val declare_component :
    ?descriptor:Setup.state_descriptor ->
    name:string ->
    pp:(Format.formatter -> 'a -> unit) ->
    init:(unit -> 'a) ->
    (* run just before the query *)
    start:('a -> 'a) ->
    unit ->
      'a component

  type t
  val get : 'a component -> t -> 'a
  val set : 'a component -> t -> 'a -> t
```

OCAML

It is worth noting that the API allows one to update the state just before executing a query, indeed the state is carried by the entire compilation phase.

3.5 Debugging

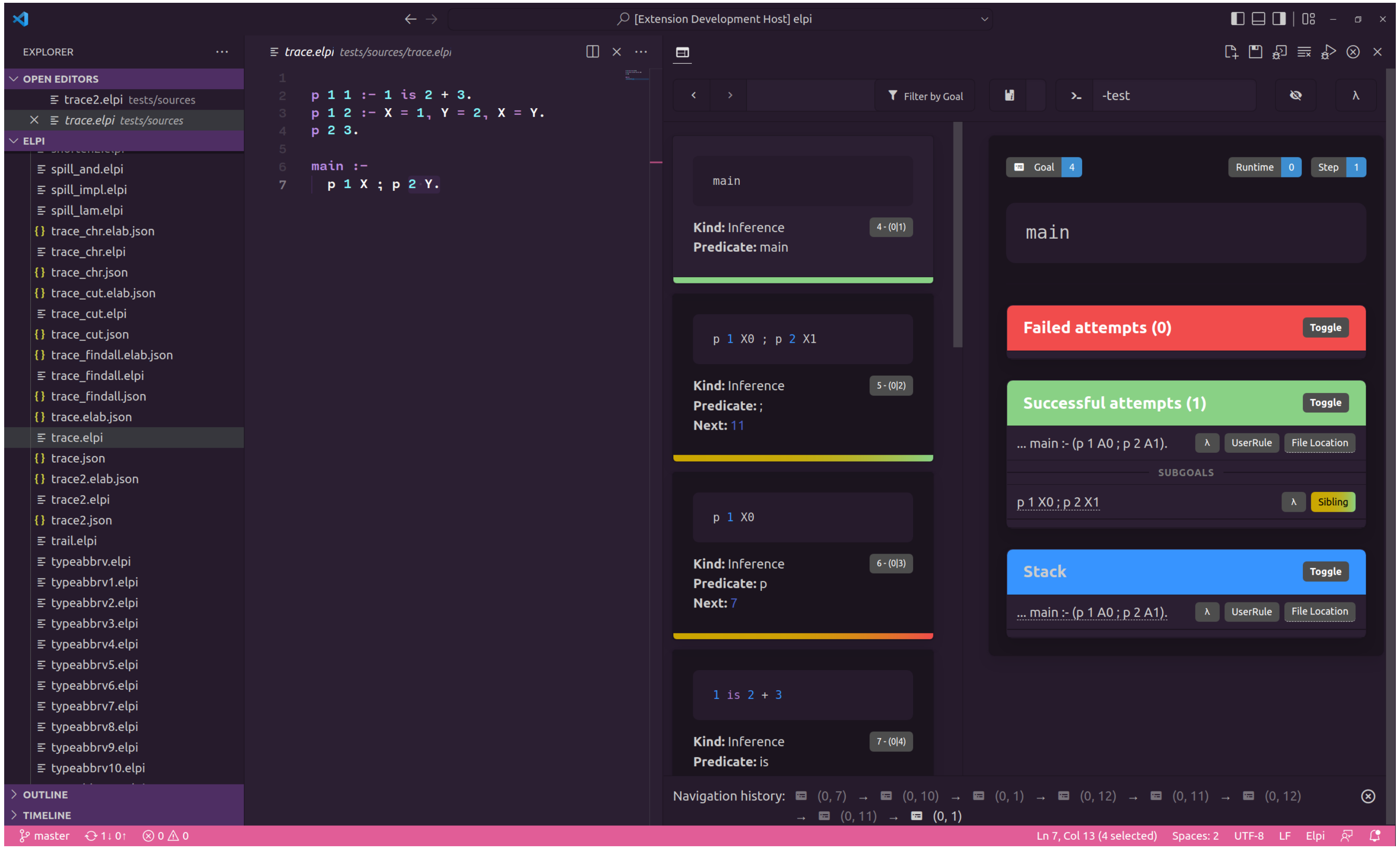
Although we are personally quite fond of "debugging using printf," we found it rather cumbersome, especially when Elpi code is shipped as a Rocq library; that is, when code is used far from its definition point.

Since the beginning, the Elpi runtime has included a debugging facility based on conditional compilation. The runtime is compiled twice: once with no instrumentation, for speed, and another time with (slow) printing instrumentation. This instrumentation was crucial for debugging the runtime itself, such as identifying De Bruijn level-related bugs. As the runtime became more robust, we began to add entry points to debug Elpi programs themselves, such as printing the current goal and the selected rules.

It then became possible to post-process the trace to reconstruct the stack (note that the operational semantics in section 2.5.1 is stackless) and to generate metadata suitable for interactive trace navigation, such as indicating whether a branch is ultimately successful or not.

Julien Wintz developed a trace browser (fig. 3.3) based on web technologies, making it easy to integrate into Visual Studio Code. It displays the trace as a mailbox of events. Each event corresponds to the application of a rule of the operational semantics section 2.5.1, and has pointers forward to all subgoals and backward to the ancestor (via the stack). The trace is also searchable.

The main limitation is that the implementation currently handles only "small" traces (below 20 megabytes), as it loads the trace upfront and web apps are run in a memory constrained sandbox by VSCode. Removing this limitation is possible and planned.



[Extension Development Host] elpi

trace.elpi tests/sources/trace.elpi

```
p 1 1 :- 1 is 2 + 3.
p 1 2 :- X = 1, Y = 2, X = Y.
p 2 3.

main :-
  p 1 X ; p 2 Y.
```

main
Kind: Inference
Predicate: main

p 1 X0 ; p 2 X1
Kind: Inference
Predicate: ;
Next: 11

p 1 X0
Kind: Inference
Predicate: p
Next: 7

1 is 2 + 3
Kind: Inference
Predicate: is

main

Failed attempts (0)

Successful attempts (1)

... main :- (p 1 A0 ; p 2 A1).

SUBGOALS

p 1 X0 ; p 2 X1

Stack

... main :- (p 1 A0 ; p 2 A1).

Navigation history: (0, 7) → (0, 10) → (0, 1) → (0, 12) → (0, 11) → (0, 12) → (0, 11) → (0, 1)

Ln 7, Col 13 (4 selected) Spaces: 2 UTF-8 LF Elpi

Figure 3.3: Trace browser

```
of (app M N) B :- of M (arr A B), of N A.
of (lam F) (arr A B) :- pi x\ of x A => of (F x) B.

copy (app M N) (app M2 N2) :- copy M M2, copy N N2.
copy (lam F) (lam F2) :- pi x\ copy x x => copy (F x) (F2 x).

cbv (lam F) (lam F2) :- pi x\ cbv x x => copy x x => cbv (F x) (F2 x).
cbv (app M N) R2 :- cbv N N2, cbv M M2, beta M2 N2 R2.

beta (lam F) T R2 :- !, subst F T R, cbv R R2.
beta H A (app H A).

subst F N B :- pi x\ copy x N => copy (F x) B.

cbn (lam F) (lam F2) :- pi x\ cbn x x => copy x x => cbn (F x) (F2 x).
cbn (app (lam F) N) M :- !, subst F N B, cbn B M.
cbn (app M N) R :- cbn M (lam F), !, cbn (app (lam F) N) R.
cbn (app X Y) (app X2 Y2) :- cbn X X2, cbn Y Y2.
```

Figure 3.4: Source code of the second group of benchmarks

3.6 Benchmarks

In [Dun+15] we evaluate Elpi on a collection of synthetic benchmarks and on a representative application. The synthetic tests are grouped into three categories: first-order programs drawn from the Aquarius test suite [Hay89] (crypto-multiplication, the μ-puzzle, a generalized eight-queens instance, and the Einstein zebra puzzle); higher- order programs that lie inside the fragment L_λ^β (see fig. 3.4); and a higher-order program outside the fragment, taken from the Teyjus test suite for normalizing SKI-expressions.

Benchmarks within L_λ^β include a type checker for λ-terms (the `of` program discussed in section 2.2) and two evaluators that compute expressions such as 5^5 using Church numerals: one that follows the call-by-value strategy and one the call-by-name one. The type-checking benchmark is designed to measure the cost of traversing binders: the test terms are largely projections and therefore contain many `lam` nodes.

Test	Elpi		Teyjus		Elpi/Teyjus	
	time (s)	space (Kb)	time (s)	space (Kb)	time	space
crypto-mult	3.48	27,632	6.59	18,048	0.52	1.53
μ-puzzle	1.82	5,684	3.62	50,076	0.50	0.11
queens	1.41	108,324	2.02	69,968	0.69	1.54
zebra	0.85	7,008	1.89	8,412	0.44	0.83
of	0.27	8,872	5.64	239,892	0.04	0.03
cbv	0.15	7,248	11.11	57,404	0.01	0.12
cbn	0.33	8,968	0.81	102,896	0.40	0.08
SKI	1.32	15,472	2.68	8,896	0.49	2.73

The results reported in the table show that Elpi performs particularly well on programs inside L_λ^β and remains competitive on the other benchmarks. The variable behaviour of Teyjus on the reduction tests can be explained by its explicit-substitutions machinery [Nad02] when crossing binders. Explicit substitutions naturally suit a lazy strategy such as call-by-name, although they incur some space overhead. By contrast, explicit substitutions are detrimental in call-by-value: when the redex argument is fully traversed the implementation tends to push suspended substitutions to the term leaves, which defeats the goal of delaying substitution. Forcing Teyjus to push explicit substitutions in the call- by-name case reduces memory use but increases running time (about ten seconds in our experiments), suggesting that the call-by-value timings are dominated by the overhead of explicit substitutions.

Elpi avoids substitution when crossing binders; this design choice yields both faster and more predictable behaviour. In the 5^5 reduction test, call-by-value outperforms call-by-name because it avoids duplicating non-normal terms.

In [Dun+15] we also compare Elpi and Teyjus on Helena, a type checker for the formal system $\lambda\delta$ [Gui09; Gui15]. Helena validates the proof terms of Landau's "Grundlagen" [Ben79] formalized in Automath. The reference Helena checker is implemented in OCaml; our Elpi implementation follows its algorithm closely and falls within L_λ^β. The Elpi implementation is, however, substantially shorter than the equivalent OCaml program: it is expressed in roughly fifty rules.

Grundlagen is a corpus of 6911 items (about 8MB). When the benchmarks were run (around 2015), Teyjus appeared to be limited by a fixed heap size of 256MB, which restricted it to verifying the first 2615 items. The tables below report pre-processing time (Pre), which includes parsing and compilation/elaboration, and verification time (Ver). We compare Elpi with Helena, Teyjus, and Rocq. Rocq implements a type checker for a λ-calculus that is strictly more expressive than $\lambda\delta$; thus Rocq can type-check the proof terms directly but possibly with additional overhead. We use Rocq timings as a reference for the order of magnitude when comparing Elpi against a state-of-the-art interactive prover. Where applicable we contrast native and interpreted executions.

Time (s) for 2615 items only			
	Elpi	Teyjus	Elpi/Teyjus
Pre	2.55	49.57	0.05
Ver	3.06	203.36	0.02
RAM (Mb)	91,628	1,072,092	0.09

Time (s) for all 6911 items					
Task	Helena		Elpi	Rocq 8.3	
	interp.	comp.	interp.	interp.	comp.
Pre	2.42	0.41	9.04	49.28	8.83
Ver	4.40	0.33	13.90	7.21	1.19

The data indicate that Elpi behaves similarly to Teyjus on first-order programs. The observed advantage of Elpi is likely attributable to the more efficient garbage collector available in OCaml. The same behaviour is seen for higher-order programs that lie outside L_λ^β.

Within L_λ^β Elpi is noticeably more efficient. Teyjus only matches Elpi when the program benefits from a lazy treatment of substitutions; even in that case Teyjus incurs a substantial memory overhead.

Because Elpi is designed to serve as an extension language for Rocq, the most relevant comparison is with Rocq's native OCaml implementation. Elpi is roughly an order of magnitude slower than Rocq, which is expected for an interpreter; we consider this penalty acceptable in light of the benefits of rapid prototyping and extensibility. Moreover, Elpi programs can invoke efficient Rocq routines when necessary. In short, Elpi aims to provide a high-level language for manipulating syntax trees while remaining fast enough for practical use.

CHAPTER 4

Rocq-Elpi

While Elpi is a standalone language and software project, it was designed to serve as an *extension language* for Rocq [The25]. The glue between Elpi and Rocq is called Rocq-Elpi.

My definition of this role, *extension language*, is largely influenced by the Lua programming language [Ier06]. Lua is an extension language for applications written in C and is widely used in the open source world and the gaming industry. Its purpose is to provide an easy way to extend the host application. Lua is easy to host because its FFI is well curated; exposing the internals of the application to the extension language requires limited effort from the application developer. It is easier to program in Lua than in C, because Lua is a higher-level language: for example, it features automatic memory management and provides dictionaries as a built-in data structure. Finally, it is easy for a user to get started with Lua because no special development environment is needed – the host application is sufficient, since Lua is an interpreter. Elpi aims to do the same for OCaml [Ler+], with Rocq as the host application of interest.

In academia, the same role is often called a meta-programming framework. Rocq's main data type, terms, are programs, and Elpi programs manipulate Rocq programs. In this sense, Elpi programs operate at the meta level.

4.1 Why extending Rocq in OCaml is hard

Fortunately, OCaml features automatic memory management, types, and algebraic data. It is much, much higher level than C. So why is it hard to extend Rocq? Why do we need another language?

The first difficulty is that the complexity of Rocq's core language, *Gallina*, is not completely hidden by the algebraic data types provided by the OCaml programming language. The missing features are encoded, and it is hard to completely hide them from the programmer via curated APIs. In particular, Gallina terms feature binders and holes.

Binders pose two problems of their own and interact poorly with holes. The first problem is that they are typically encoded with numbers (De Bruijn indexes), and it is all too easy to forget to shift a term. The second problem is that when a binder is crossed, one must always remember something about it, typically the type of the bound variable. Hence, programs have to pass around typing contexts (aka environments). It is tempting not to do this upfront, but expecting all developers – especially casual ones – to be disciplined seems a lost battle. We deeply agree with Mc Bride on this [McB00, page 20]: "Mantra: Γ is with me, wherever I go."

Holes are missing subterms and come with metadata: a typing sequent. Since the same hole can have multiple occurrences, a single sequent is stored on the side of terms. Moreover, since binders may be reduced away, each occurrence has an explicit substitution. So there is a state that that is associated with the terms – a state to be threaded in a functional setting. This state also gathers universe constraints, as required by Rocq's typical ambiguity (writing **Type** with no explicit level). Assignments are also part of this state, and the bugs that stem from the programmer forgetting it even gave rise to odd concepts among Rocq developers like "evar sensitive" (evar is the name for holes). If one forgets to look at terms through the right assignment (e.g., uses the empty one), the behavior is hard to debug: it is like OCaml code that behaves differently depending on whether a reference `x` was originally given the value 42, or if it was later updated to that value. The example is not an exaggeration: the state accompanying holes is really like a heap, and if one has to manage it by hand, it is like going back to manual memory management, pointer dereferencing (with no types), and no garbage collection. It does not sound that different from C.

The second difficulty is not inherent in OCaml, but rather an artifact of the history of Rocq. The APIs are not well curated. Exposing some of them to the extension language is a very good opportunity to incrementally produce a coherent set of APIs.

Finally, there is the challenge of setting up the development environment and process. Even when a user is past the inevitable frustration of setting up the right compiler and tools, an embedded interpreter still offers some advantages – especially if the language is rule-based. One can iterate faster, without even restarting the host application, and sometimes even modify the code at a distance: if the code is defined far away, in a library being used, one can replace bits of its code by replacing a rule defined there with a new one defined here, where the problem shows up, and in the end conduct a much faster investigation.

4.2 Encoding Rocq terms and contexts

The main data type that Rocq-Elpi programs work with is the one of Rocq terms. The first aspect of terms worth describing is the pointers to global objects. While `constant`, `inductive`, and `constructor` are opaque data (as in section 3.4.2.1), their nature is exposed as an algebraic data type.

```
kind gref type.
type const constant -> gref.          % Nat.add, List.append, ...
type indt  inductive -> gref.         % nat, list, ...
type indc  constructor -> gref.       % O, S, nil, cons, ...
```

ELPI

In the examples part of this document we shall use the string notation for these three opaque data types, althought in real code one needs to resolve a `string` to a `gref` via the `rocq.locate` API.

Global objects are injected into the term data type via the `global` term constructor and its variant `pglobal`, which is dedicated to universe-polymorphic global objects.

```
kind term type.
type global  gref -> term.
type pglobal gref -> univ-instance -> term.
type sort    sort -> term.                 % Prop, Type@{i}, SProp
```

ELPI

We omit the code for the `sort` data type and just remind it represents Rocq sorts such as **Prop** and **Type**.

Instead we focus on the three binders: `fun` and `prod` for abstraction, and `let` for abbreviation. In addition to the higher-order argument, the detail common to all of them is the `name` argument.

It stands for pretty-printing information, e.g., the bound variable name as written by the user. It is important to note that `name` is an opaque data type with a trivial equality test. That is, the names “x” and “y” are equal in Elpi, although they hold different values in OCaml. This is key to aligning the equational theory of the meta-language Elpi with a sensible notion of equality for the object language Gallina, namely α-equivalence.

```
type fun  name -> term -> (term -> term) -> term.          % λ
type prod name -> term -> (term -> term) -> term.          % ∀
type let  name -> term -> term -> (term -> term) -> term.  % let
```

Application is n-ary, with the head and arguments held in a list. This choice, over binary application as in section 2.2, eases access to the head: according to our experience, the vast majority of rules are discriminated by the head constant of the terms they operate on.

```
type app       list term -> term.
```

The Rocq pattern matching constructor packs together the scrutinee, a term used to uniformly assign a type to branches, and a list of branches.

```
type match     term -> term -> list term -> term.
```

The fixpoint operator is a binder for the recursive call and carries the index of the decreasing argument.

```
type fix       name -> int -> term -> (term -> term) -> term.
```

Finally, a single constructor holds all primitive values, e.g., 63-bit integers, floating point numbers, and primitive record projections.

```
type primitive primitive-value -> term.
```

What is worth focusing on is the representation of contexts since it defines a convention to be used when crossing binders. In turn this convention enables APIs to be called deep inside terms. Two context entries need to be available: one for abstraction and one for abbreviation.

```
pred decl i:term, o:name, o:term.                  % Var Name Ty
pred def  i:term, o:name, o:term, o:term.          % Var Name Ty Bo
```

The first argument of each is, by convention, a bound variable, and is expected to be passed as input in order to retrieve the associated pretty printing information, the type, and, in the case of abbreviation, the body. In light of this, code that crosses a binder is expected to declare either a `decl` or a `def` hypothetical rule for the bound variable as follows:

```elpi
some-pred (fun N T F) :-
  pi x\ decl x N T => some-pred (F x).
```

And convenience macros help the programmer obey the discipline. Note how the macro arguments are the same of the term constructor it crosses:

```elpi
some-pred (fun N T F) :-
  @pi-decl N T x\ some-pred (F x).
```

As a result of this convention, one can call `rocq.typecheck` deep inside a term. As described in section 3.4, a built-in predicate can be equipped with a `ctx_readback` function, such as `proof_context` in the code below. The `term` contextual conversion can thus access a Rocq context when operating its readback or embedding code.

```ocaml
MLCode(Pred("rocq.typecheck",
  CIn(term, "T",
  CInOut(B.ioargC term, "Ty",
  InOut(B.ioarg B.diagnostic, "Diagnostic",
  Full (proof_context, "typchecks a term T ...")))),
    (fun t ety diag _ proof_context _ state ->
      let sigma = get_sigma state in
      let sigma, ty = Typing.type_of proof_context.env sigma t in
      let state, assignments = set_sigma ~depth state sigma in
      ...
```

Here, the `t` and `ety` values are read back in the `proof_context.env` Rocq context, which is also passed to the type checker to synthesize the type of `t`, and later (in the omitted code) to unify the inferred type with the expected one `ety`, when provided. The `B.diagnostic` datum is either the Elpi value `ok` or `error "message"`; thus, a call `rocq.typecheck T Ty ok` is the idiom to assert that `T` is well-typed of type `Ty`.

The only missing piece is `sigma`, the environment assigning type judgements to holes, i.e., the call to the `get_sigma` and `set_sigma` APIs, which we focus on in the next section.

4.3 Encoding holes

Each Rocq hole has an entry in `sigma` that resembles a sequent. We show in fig. 4.1 simplified code from the sources of Rocq 9.0.

In Elpi, we represent this data – referred to as `sigma` – as a set of `evar` constraints. Such a constraint links a *raw* Elpi unification variable `RawEv`, a type, and the *elaborated* Elpi unification variable `Ev` (their meaning is clarified below). The constraint should live in a context of hypothetical rules `Ctx` assigning types to bound variables. In other words, each evar is represented by the result of suspending a query as follows:

```
module Evar : sig

  type t = int                             (* hole handle    *)
  module Map : Map.s with type key = t     (* map on holes   *)
  ...

module Evd : sig

  type evar_body =
    | Evar_empty                      (* no inhabitant yet   *)
    | Evar_defined of constr          (* proof term          *)

  type evar_info = {
    evar_concl : constr;              (* sequent conclusion  *)
    evar_hyps  : named_context_val;   (* sequent context     *)
    evar_body  : evar_body;           (* optional proof term *)
    ...
  }

  type evar_map = {
    (* Evars, split in two for performance reasons *)
    defn_evars : evar_info Evar.Map.t;
    undf_evars : evar_info Evar.Map.t;

    (* Universe constraints *)
    universes  : UState.t;
    ...
  }
```

OCAML

Figure 4.1: Rocq's evar map

```
pi x1\ .. pi xn\ Ctx =>
  evar RawEv Ty Ev
```

ELPI

in a program with the following suspension rules:

```
pred evar i:term, i:term, o:term. % RawEv Ty Ev
evar X Ty R :- var X, var R,      !,
  declare_constraint (evar X Ty R)  [X, R].
evar X Ty R :- var X, not(var R), !,
  rocq.typecheck R Ty ok, X = R.
evar X Ty R :- not(var X), var R, !,
  rocq.elaborate-skeleton X Ty R ok.
```

ELPI

The code above enforces the following invariants:

- each unassigned Rocq evar has a corresponding Elpi constraint

- Ev can be assigned to a term R only if R has type Ty in context Ctx
- RawEv can only be assigned to a term X if X elaborates to a term R such that the previous condition holds

In the code above, rocq.elaborate-skeleton is an API very similar to rocq.typecheck, but that does not modify its input in place. Instead, it uses its input as a skeleton term that is copied. This allows for changes to the structure of the term, such as inserting explicit coercions to implement type casts. The API corresponds to the code that Rocq runs on any input term written by the user, while rocq.typecheck corresponds to code internally used by Rocq tactics to build well-typed terms.

In order to declare a new evar constraint at any time, even if the current λProlog program has more names and hypothetical rules than needed, we use the following predicate:

```
pred declare-evar i:list prop, i:term, i:term, i:term.
declare-evar Ctx RawEv Ty Ev :-
  declare_constraint (declare-evar Ctx RawEv Ty Ev) [_].

constraint declare-evar evar def decl {
  rule \ (declare-evar Ctx RawEv Ty Ev) <=> (Ctx => evar RawEv Ty Ev).
}
```

ELPI

Note how declare-evar uses a CHR rule to craft a new goal with a precise context Ctx, regardless of the context under which declare-evar is run. Indeed, a Rocq API that is called deep inside a term and encounters a new hole will add an entry for it to Rocq's evar map, but also generate a declare-evar, and hence an evar constraint in Elpi. This is possible thanks to the type of contextual conversions (section 3.4.2.2), which includes a list of extra goals to be generated in response to the conversion.

The only missing pieces are the get_sigma and set_sigma APIs. Their role is to synchronize updates to sigma in both directions. The former, get_sigma, is the simplest, since a Rocq datum of type evar_map is stored in the extensible state of Elpi.

To implement the other, we take advantage of an API offered by Elpi: a bidirectional map linking Rocq unification variables and Elpi holes.

```
module FlexibleData : sig
  module Elpi : sig type t ... end
  module type Host = sig type t ... end
  module Map : functor(Host : Host) -> sig
    type t
    val empty : t
    val add   : Elpi.t -> Host.t -> t -> t
    val elpi  : Host.t -> t -> Elpi.t
    val host  : Elpi.t -> t -> Host.t
    val fold  :
      (Host.t -> Elpi.t -> term option -> 'a -> 'a) -> t -> 'a -> 'a
```

OCAML

The extensible state contains two instances of the bidirectional map described above: one for raw holes and one for elaborated holes. A Rocq **Evar**.t is linked to one of each. To propagate changes to sigma performed by Rocq, we fold over the set of linked evars to check if they have been assigned. If so, we remove the corresponding evar constraint and generate the corresponding assignment as an Elpi unification problem between the hole and the Rocq term.

4.4 Vernacular language integration

The vernacular language of Rocq allows one to organize formalized knowledge, in particular to give names to Gallina terms and attach metadata to them. Rocq-Elpi extends the vernacular language with a few commands to declare, run, and modify Elpi programs.

For convenience, programs are divided into commands and tactics. Both run Elpi code, but the entry point is different; moreover, tactics can only be executed in Rocq proof mode.

4.4.1 Commands

A command is declared with **Elpi Command** followed by a name. Initially, a command only contains the glue code for Rocq, such as the HOAS data type of terms and the built-in predicates. Extra code can be added to a command via **Elpi Accumulate**. Accumulated code can be either static (e.g., from a file or a string) or dynamic (from a database; see section 4.4.3).

A command hello is executed with **Elpi** hello followed by arguments. The entry point is the main predicate that receives a list of arguments. The argument data type has constructors for built-in data types such as strings or numbers, as well as Gallina terms.

```
Elpi Command hello.
Elpi Accumulate hello lp:{{
  main [str X] :- rocq.say "Hello" X.
}}.
Elpi hello "reader".     (* prints "Hello reader" *)
Fail Elpi hello 46.      (* fails *)
```

The simple program above calls the rocq.say built-in whenever the command is passed a string. Since the entry point main has no rule for arguments of the form int X, the second calls fails.

Interested readers can consult the comprehensive tutorial on Elpi commands [Tasa], which is part of the Rocq-Elpi software distribution.

4.4.2 Tactics

Tactics are built in a similar way, with the main difference being that the entry point is called solve, which receives as arguments not only the arguments passed to the tactic, but also the goal on which the tactic operates.

```
Elpi Tactic intro.
Elpi Accumulate intro lp:{{
  solve (goal _ _ _ _ [str ID] as G) GS :- !,
    std.assert! (rocq.ltac.id-free? ID G) "name already taken",
    rocq.id->name ID N,
    refine (fun N _ _) G GS.
}}.
Goal ∀P, P -> P /\ P.
elpi intro "p".
Fail elpi intro "p". (* Error: name already taken *)
elpi intro "Hp".
```

In the example above, the tactic expects to receive a string `ID`; it checks that no Rocq proof variable is already using the name `ID`, and finally refines[1] the goal with a lambda abstraction, which is the proof term for the logical rule of introduction of universal quantification or implication. Note that the check for the name being free is just for convenience: a collision would not result in name capture in Elpi, and when translated back to Rocq, the deepest variable would be forcibly renamed to avoid capture in Rocq too.

Interested readers can consult a longer tutorial on Elpi tactics [Tasb], as well as look into eltac [Tas+], a collection of basic Rocq tactics rewritten in Elpi for didactic purposes.

4.4.3 Databases

A database is, in essence, an Elpi program, but unlike commands and tactics, it cannot be executed directly. It constitutes a piece of code that is accumulated into commands or tactics *by name*. This means that whenever a program runs, it sees the current contents of the databases it accumulates, not their contents at the time the command was initially declared.

A typical database consists of a single predicate, possibly with some default rules. The **Elpi Db** command declares a database.

```
Elpi Db age.db lp:{{
  pred age o:string, o:int.

  % the Db is empty for now, we put a rule giving a
  % descriptive error and we name that rule "age.fail".
  :name "age.fail"
  age Name _ :- rocq.error "I don't know who" Name "is!".
}}.
```

Elpi rules can be assigned a name via the `:name` attribute. Named rules serve as anchor points for new rules when they are added to the database.

A command to print the age of a given name can be declared as follows:

[1] The `refine` API is akin to Rocq's **refine** tactic: it makes progress on one goal by providing a term containing holes, holes for which new goals are generated

```
Elpi Command age.
Elpi Accumulate Db age.db.
Elpi Accumulate lp:{{

  main [str Name] :-
    age Name A,
    rocq.say Name "is" A "years old".

}}.

Fail Elpi age bob. (* I don't know who bob is! *)
```

We can add entries to the database manually as follows:

```
Elpi Accumulate age.db lp:{{

  :before "age.fail"      % we place this rule before the catch all
  age "bob" 24.

}}.

Elpi age bob. (* bob is 24 years old *)
```

We do not expect users of Rocq commands written in Elpi to know the syntax of Elpi, nor the commands to manipulate its programs, and even less to actually know that the Rocq commands they run are implemented in Elpi.

For this reason, Rocq-Elpi provides APIs to extend databases from Elpi. Here, we craft a new command `set_age` that uses this API.

```
Elpi Command set_age.
Elpi Accumulate Db age.db.
Elpi Accumulate lp:{{

  main [str Name, int Age] :-
    TheNewRule = age Name Age,
    rocq.elpi.accumulate _ "age.db"
      (clause _ (before "age.fail") TheNewRule).

}}.

Elpi set_age "alice" 21.
Elpi age "alice". (* alice is 21 years old *)
Elpi set_age "mallory" 24.
Elpi age "mallory". (* mallory is 24 years old *)
```

The same databases can be accumulated by commands and tactics in order to share a state – a common knowledge base that typically grows as the commands or tactics are used in a Rocq library.

4.4.3.1 Homoiconicity and rules

There are two characteristics of Elpi that make databases both possible and convenient to use.

The first is that Elpi is a homoiconic language: it can manipulate its own syntax. The following line illustrates this property:

```
% build code
TheNewRule = age Name Age,
```

ELPI

This is the simplest evidence of it: TheNewRule contains a piece of Elpi code built from the age predicate and the two arguments passed to main. Since construction and inspection are conflated into unification in logic programming, the same line can be used to analyze the syntax of a piece of code in order to extract the name and age. Moreover, since expressions and commands are not confused in logic programming, there is no confusion in the user's intention: she wants to build/inspect a piece of code, not to run it.[2]

The second helpful characteristic of Elpi is that code is organized into rules, meaning that age Name Age is an entire code unit that one can build, inspect and run. In comparison, the OCaml code for the age function is a single expression (a single unit):

```
let age s =
  match s with
  | ("bob" | "mallory") => 24
  | "alice" => 21
  | _ => failwith ("I don't know who " ^ s ^ " is!")
```

OCAML

While it is certainly possible to traverse the syntax tree above and add a pattern match branch before the last one, such manipulation is not entirely trivial. To make it so, one should opt out of most syntactic sugar and commit to using a syntactic form that is as close as possible to that of a rule-based language:

- making all variable bindings local to the pattern matching branch
- making all rules syntactically independent

The code below respects these points. Note that the s binding is now local to the last branch, and there is a clear separation between the bob and mallory branches:

```
let age = function
  | "bob" => 24
  | "alice" => 21
  | "mallory" => 24
  | s => failwith ("I don't know who " ^ s ^ " is!")
```

OCAML

The syntactic price one pays when writing Elpi code is thus partially compensated by the ease of extending existing code. The example in the following section makes good use of this possibility.

[2] In other words the quotation operator ' of the Lisp family of languages is not needed.

4.5 Example: proof transfer

As a simple example, we present a tactic and a command that communicate via a database. The tactic to_bool transfers a goal from the realm of propositions to that of computable (boolean) tests (i.e., effectively decidable propositions). The command register_decision takes a proof of equivalence between a proposition and a boolean test and compiles it into an Elpi rule. The database contains all known rules for translating propositions into tests: it is queried by to_bool and populated by register_decision.

It is customary to organize Rocq-Elpi code in this way. The key advantage is that to_bool is extensible: when a new Rocq predicate is declared and a new corresponding test is defined, it is sufficient to call register_decision to extend to_bool.

We assume a Rocq environment with the following lemmas. The reflect predicate [MT22] links a proposition with a boolean test: reflect P b means P holds if and only if b = true.

```
Lemma evenP n : reflect (is_even n) (even n).

Lemma andP  {P Q : Prop} {p q : bool} :
  reflect P p -> reflect Q q -> reflect (P /\ Q) (p && q).

Lemma elimT {P b} :
  reflect P b -> b = true -> P.
```

ROCQ

The database holds rules for the tb predicate. For simplicity, we only deal with one direction of the equivalence: we relate a proposition with its equivalence proof (disregarding the associated boolean test).

```
Elpi Db tb.db lp:{{

% [tb P R] finds [R : reflect P b] for some b
pred tb i:term, o:term.

:name "tb:fail"
tb Ty _ :- rocq.error "Cannot solve" {rocq.term->string Ty}.

}}.
```

ROCQ

Initially, the database contains only one rule, which aborts the search with an error message. All additional rules must be placed before this one.

The to_bool tactic is built as follows, on top of the database.

```ROCQ
Elpi Tactic to_bool.
Elpi Accumulate Db tb.db.
Elpi Accumulate lp:{{

solve (goal Ctx _ Ty _ [] as G) [G1] :-
  tb Ty P,
  refine {{ elimT lp:P _ }} G [G1].

}}.
```

The tactic simply queries the database for a proof `P` that the goal is related to a boolean test `b`, then refines the goal. with `elimT lp:P _`. The hole stands for the proof of `b = true`, i.e., the subgoal `G1`.

The most interesting part of this example is how the database is populated. We shall write a `compile` procedure that, given `evenP` and `andP`, synthesizes the following Elpi code:

```ELPI
tb {{ is_even lp:N }} {{ evenP lp:N }} :- !.
tb {{ lp:P /\ lp:Q }} {{ andP lp:PP lp:QQ }} :- !, tb P PP, tb Q QQ.
```

As sketched in section 4.4.3, the two rules above correspond to the following (closed) Elpi terms:

```ELPI
pi N\ tb {{ is_even lp:N }} {{ evenP lp:N }} :- [!]
pi P Q PP QQ\ tb {{ lp:P /\ lp:Q }} {{ andP lp:PP lp:QQ }} :-
  [!, tb P PP, tb Q QQ].
```

It is worth unfolding quotations to see that a few more quantifications are necessary for the second rule:

```ELPI
pi P Q p q PP QQ\
  tb (app[global (indt "and"), P, Q])
     (app[global (con "andP"), P, Q, p, q, PP, QQ]) :-
  [!, tb P PP, tb Q QQ].
```

Now that all quantifications are explicit, we can look at the types of `evenP` and `andP` and observe how each `pi` Elpi quantification corresponds to a Rocq quantification (dependent or not), and how Rocq premises correspond to recursive calls in Elpi.

```ROCQ
evenP : ∀n, reflect (is_even n) (even N).
andP : ∀P Q p q,
  reflect P p -> reflect Q q -> reflect (P /\ Q) (p && q).
```

We are now set to write our little rule compiler. The `compile` predicate has three inputs and one output. The first input is the type of a lemma, and the second is the proof of that lemma; the relation between the first two arguments is a key invariant. The third argument is a list of recursive calls, while the output is the code of the Elpi rule being generated.

```
pred compile i:term, i:term, i:list prop, o:prop.

compile {{ reflect lp:Ty _ }} P Hyps (tb Ty P :- [! | Hyps]).

compile {{ reflect lp:S _ -> lp:T }} P Hyps (pi h\ C h) :- !,
  pi h\ compile T {{ lp:P lp:h }} [tb S h | Hyps] (C h).

compile {{ ∀x, lp:(T x) }} P Hyps (pi h\ C h) :-
  pi x\ compile (T x) {{ lp:P lp:x }} Hyps (C x).
```

The first rule is the base case. When we reach the conclusion of the lemma `reflect lp:Ty _`, we know that the second argument `P` is a proof of that fact; hence, `tb Ty P` is a correct head for the rule. The premises of the rule are the `Hyps` recursive calls, prefixed with a cut.

The second rule handles premises that are turned into recursive calls. In this case, the rule being synthesized features an extra `pi` quantification, and the `Hyps` list grows longer. The `h` variable stands for the proof obtained by the recursive call – a proof of `reflect lp:S _` – hence a value we can pass to `P` obtaining the Rocq term `lp:P lp:h` that is a proof of `T`.

The last rule is a simpler version of the second; in this case, the quantification is not a premise. In this case we do not extend `Hyps`.

The code of the `register_decision` command locates the given lemma, finds its type, and generates the new rule `C`. Finally, it adds the rule to the database before the `"tb:fail"` anchor point.

```
Elpi Command register_decision.
Elpi Accumulate Db tb.db.
Elpi Accumulate lp:{{

main [str S] :-
  rocq.locate S GR,
  rocq.env.typeof GR Ty,
  compile Ty (global GR) [] C,
  rocq.elpi.accumulate _ "tb.db" (clause _ (before "tb:fail") C).

}}.
```

We can now populate the database and test our tactic.

ROCQ

```
Elpi register_decision andP.

Lemma test : is_even 6 /\ is_even 4.
Proof.
elpi to_bool. (* Error: Cannot solve is_even 6 *)
Abort.

Elpi register_decision evenP.

Lemma test : is_even 6 /\ is_even 4.
Proof.
elpi to_bool. (* even 6 && even 4 = true *)
simpl.        (* true = true *)
trivial.
Qed.
```

The Rocq snippet calls the `to_bool` tactic in a scenario where a predicate is not (yet) registered, resulting in an explanatory error message. Then, it registers a test for the missing predicate and successfully solves the goal.

We believe this simple example shows how natural it is to build commands and tactics around the concept of data bases and how a rule based language like Elpi is a good fit for that.

Chapter 5

Applications written in Elpi

In this chapter, we review several tools developed either on top of Rocq-Elpi or using Elpi alone. We highlight the Derive and Hierarchy-Builder Rocq packages, to which we have personally contributed. Subsequently, we briefly survey additional tools developed by our colleagues, some of which do not rely on Rocq. At the end of each section, we discuss which features of Elpi and Rocq-Elpi are beneficial to the tool under consideration.

5.1 Derive

The Derive application is a framework for automatic code generation, typically triggered by the declaration of a new inductive data type.

Rocq itself is known for generating induction principles and equality tests. However, it is also known to produce suboptimal induction principles when containers (e.g., lists) are involved, and it often fails to generate useful equality tests.

We identified an opportunity for improvement, beginning with the L3 internship of Luc Chabassier [1], who developed the first prototype during the summer of 2017. Within two months, he successfully generated equality tests and their proofs. While much of the credit goes to his learning abilities, this experience confirmed that Rocq-Elpi could effectively support research in this area. Nevertheless, the solution implemented by Luc, although correct, lacked modularity. Further improvements led to [Tas19], which describes a schema for generating so-called deep induction principles and demonstrates their use in proving the correctness of equality tests.

Before discussing this work, we need to introduce the "Swiss army knife" of boilerplate code generation: parametricity.

5.1.1 Parametricity

The so called parametricy translation [KL12] is a family of procedures $[\cdot]_i$ indexed by an arity i. We focus on the unary one, i.e. $i = 1$, and we write U for any Rocq universe, i.e. `Prop`. Given *any* Rocq term $t : r$, the unary parametricity translation $[\cdot]_1$ gives both a unary predicate R on r, e.g. $[r]_1 = R : r \to U$ and a proof T that R holds for t, i.e. $[t]_1 = T : R\ t$ or somewhat more confusingly if $t : r$ then $[t]_1 : [r]_1\ t$.

For example the unary parametricity translation of `nat` is `is_nat : nat → U` and the translation of `3 : nat` is a proof that `is_nat 3`.

[1] https://www-sop.inria.fr/members/Enrico.Tassi/chabassier.pdf

```
Inductive is_nat : nat -> U :=
| is_zero : is_nat 0
| is_succ n : is_nat n -> is_nat (S n).

Definition is_nat_3 : is_nat 3 :=
  is_succ 2 (is_succ 1 (is_succ 0 is_zero)).
```

One can see `is_nat` as a first class description of the typing assignment, i.e. each term is equipped with a proof that it has the expected type, i.e. `0` with `is_zero`; `1` with `is_succ 0 is_zero`, and so on.

The translation becomes interesting when the type has parameters such as `A` in `list A`. In that case the translation builds a predicate parametric in `A` and in its translation.

The translation of `list A` is a predicate `is_list : ∀A, (A → U) → (list A → U)`, and the translation of the constructors `nil` and `cons` are:

```
is_nil: ∀A (isA : A → U), is_list A isA nil
is_cons: ∀A (isA : A → U), ∀a, isA a →
  ∀l, is_list A isA l → is_list A isA (a :: l).
```

As with the translation of natural numbers, each term is accompanied by a proof that it has the expected type. When considering a term `a` of type `A`, the type parameter of the container, it is paired with a proof of `isA a`, the predicate associated with `A`. The unary parametricity translation is essential for systematically expressing that a property holds deep within a container.

Parametricity is a meta-theorem in Rocq, that is, a theorem about Rocq itself. Since it cannot be represented by a Gallina term, a meta-language is required for its effective implementation. Cyril Cohen implemented the binary parametricity translation in the fall of 2017, motivated by his interest in applying the translation to the Rocq-EAL project [CDM13].[2] I based the unary version on his code. Each translation consists of about 250 lines of Elpi code.

5.1.2 Deep induction principles

We say that an induction principle is deep if the induction hypothesis is available for subterms that occur deep within the immediate subterms. As an example, consider the data type of rose trees.

```
Inductive rtree A : U :=
| Leaf (a : A)
| Node (l : list (rtree A)).
```

The induction principle generated by Rocq is shallow: `P` is only available on the immediate subterms of type `rtree`, which, in this case, are none.

[2] https://github.com/rocq-community/coqeal

```
Lemma rtree_ind : ∀A (P : rtree A → U),
  (∀a : A, P (Leaf A a)) →
  (∀l : list (rtree A), P (Node A l)) →
  ∀t : rtree A, P t.
```

This principle is weak, since no element of `l` in `Node A l` satisfies `P`. In [Tas19], we study the synthesis of a stronger principle in which `P` holds deep inside `l`.

```
Lemma rtree_induction A is_A (P : rtree A → U) :
  (∀a, is_A a → P (Leaf A a)) →
  (∀l, is_list (rtree A) P l → P (Node A l)) →
     ∀t, is_rtree A is_A t → P t.
```

The induction principle uses the unary parametricity translation of `l` and its type to systematically construct the hypothesis `is_list (rtree A) P l`, which provides access to `P` on all elements of `l`.

This additional assumption comes at a price: the induction principle does not apply to just any tree, but to a tree `t` such that `is_rtree A is_A t` holds.

This extra argument is synthesized automatically. For non-containers like `nat`, we prove `∀n, is_nat n` by induction on `n` (no different from how we proved `is_nat 3` previously). For containers, we prove the statement under the assumption that the predicate for the type parameter is trivially true. For example:

```
Lemma list_is_list A isA : (∀a, isA a) -> ∀l, is_list A isA l.
Lemma rtree_is_rtree A isA : (∀a, isA a) -> ∀l, is_rtree A isA l.
```

While this assumption may seem strong at first, it actually matches two relevant classes of predicates:

- the one we are defining, e.g., `nat_is_nat : ∀n, is_nat n` fits
- any `P : t -> U` we are defining by induction on the whole type `t`, e.g., the `P` of an induction principle

To prove the deep induction principle, we require some scaffolding, again synthesized automatically via Elpi programs. In particular we need a tool to operate under a unary parametricity translation.

```
Definition is_list_functor A P Q :
  (∀a, P a → Q a) → ∀l, is_list A P l → is_list A Q l.
```

The proof of this brick is not particularly interesting, it just amounts at recursing over `is_list` and using the hypothesis on each element of the list.

We can now present the proof of the deep induction principle for rose trees:

```
Definition rtree_induction (A : U) (PA : A → U) (P : rtree A → U)
    (His_Leaf : ∀a, PA a → P (Leaf A a))
    (His_Node : ∀l, is_list (rtree A) P l → P (Node A l))
:=
  fix IH s1 (x : is_rtree A PA s1) {struct x} : P s1 :=
  match x in (is_rtree _ _ s2) return (P s2) with
  | is_Leaf _ _ a Pa =>
      His_Leaf a Pa
  | is_Node _ _ l Pl =>
      His_Node l (is_list_functor (rtree A) (is_rtree A PA) P IH l Pl)
  end.
```

Note the following type assignments for the variables and terms appearing in the last branch of the **match** on x:

```
l  : list (rtree A)
Pl : is_list (rtree A) (is_rtree A PA) l
IH : ∀r, is_rtree A PA r -> P r
is_list_functor (rtree A) (is_rtree A PA) P IH l Pl :
  is_list (rtree A) P l
```

The last term satisfies the premise of His_Node, which in turn gives the user of the induction principle access to the property P on all elements of l. The is_list_functor property is key to apply the induction hypothesis IH deep, on all the elements of Pl.

5.1.3 Natural equality tests

Now that we have a strong induction principle, we can establish the following properties for the recursive program of interest.

```
Definition correct (T : Type) (eqb : T -> T -> bool) :=
  ∀x y : T, reflect (x = y) (eqb x y).

Definition correct_at (T : Type) (eqb : T -> T -> bool) (x : T) :=
  ∀y : T, reflect (x = y) (eqb x y).
```

This is the form of the correctness lemmas we will prove.

```
Lemma list_eq_correct : ∀A eqA (l : list A),
  is_list A (correct_at A eqA) l ->
    correct_at (list A) (list_eq A eqA) l
```

This lemma states: if eqA is a correct test for all elements of l, then list_eq A eqA is a correct test for l.

The most interesting case is the equality test for rose trees, which reuses the equality test for lists list_eq, passing the recursive call as the eqA argument.

```
Definition rtree_eq A (A_eq : A → A → bool) :=
  fix rec (t1 t2 : rtree A) {struct t1} : bool :=
    match t1, t2 with
    | Leaf a, Leaf b => A_eq a b
    | Node l, Node s => list_eq (rtree A) rec l s
    | _, _ => false
    end.
```

What is remarkable is that the correctness proof for `rtree_eq` reuses the correctness proof for `list_eq`.

```
Definition rtree_eq_correct (A : Type) (eqA : A -> A -> bool) :
  ∀r, is_rtree A (correct_at A eqA) r ->
        correct_at (rtree A) (rtree_eq A eqA) r
:=
  rtree_induction
    A (correct_at A eqA) (correct_at (rtree A) (rtree_eq A eqA))
    (λ(a0 : A) (Pa : correct_at A eqA a0) =>
       correct_Leaf A eqA a0 Pa)
    (λ(sib : list (rtree A))
         (Psib : is_list (rtree A)
                     (correct_at (rtree A) (rtree_eq A eqA)) sib) =>
       correct_Node A eqA sib
         (list_eq_correct (rtree A) (rtree_eq A eqA) sib Psib)).
```

The proof also relies on two auxiliary lemmas that are not particularly interesting; we refer the reader to [Tas19] for further details.

```
correct_Node : ∀A eqA (l : list (rose A)),
  correct_at (list (rose A)) (list_eq (rose A) (rose_eq A eqA)) l ->
    correct_at (rose A) (rose_eq A eqA) (Node A l)

correct_Leaf : ∀A eqA (a : A),
  correct_at A eqA x ->
    correct_at (rose A) (rose_eq A eqA) (Leaf A x)
```

The final theorem can be obtained by satisfying the extra premise of the deep induction hypothesis, namely that each rose tree is a rose tree.

```
Definition rose_eq_OK A eqA :
  correct A eqA -> correct (rtree A) (rose_eq A eqA)
:=
  λp r =>
    rose_eq_correct A eqA
      r (rtree_is_rtree A (correct_at A eqA) p r).
```

To conclude, both equality tests and their proofs are constructed compositionally. In [Tas19], we also discuss why these proofs can be kept opaque without interfering with the termination checker.

5.1.4 Fast (to check) equality tests

The results in [Tas19] are further refined in [GLT23], where we focus on the size (and thus the type checking time) of the synthesized terms. In particular, the natural equality tests are quadratic: they compare each constructor with all the others. Consequently, the proofs are also quadratic in the number of constructors.

In [GLT23], we devise equality tests that are pseudo-linear in size and prove them correct in a compositional way using the deep induction principles developed in [Tas19].

Here, we outline only the pseudo-linear schema for rose trees. The idea is to assign a tag (a positive number) to each constructor and compare these tags upfront. If the tags differ, the two rose trees are different; if they are equal, we can uniformly unpack the fields and compare them.

We synthesize these three functions upfront.

```
Definition tag A : rtree A -> positive.
Definition fields_t : positive -> Type.
Definition fields A :  ∀r : rtree A, fields_t (tag r).
```

ROCQ

The main code pattern matches over the first `rtree` and pre-computes its tag and fields. The it calls a common piece of code `eqb_body` that will test if the tags agree and proceed if it is the case.

```
Definition rose_eqb A eqA : rtree A -> rtree A -> bool :=
  fix rec (x1 x2 : rose A) {struct x1} : bool :=
    match x1 with
    | Leaf _ a =>
        (* 1 = tag (Leaf _ a); a = fields (Leaf _ a) *)
        eqb_body 1 (eqb_fields A eqA rec) a x2
    | Node _ sib =>
        (* 2 = tag (Node _ sib); sib = fields (Node _ sib) *)
        eqb_body 2 (eqb_fields A eqA rec) sib x2
    end.
```

ROCQ

When the tags match, `eqb_body` extracts the fields of the second rose tree and calls `eqb_fields`. Since the two fields have types `fields_t t1` and `fields_t t2`, respectively, a cast is necessary.

```
Definition eqb_body t1 eqb_fields v1 x2 :=
  let t2 := tag x2 in
  match pos_eq_dec t2 t1 with
  | left heq =>
    let f2 : fields_t t2 := fields x2 in
    eqb_fields t1 f1 (match heq with eq_refl => f2 end)
  | right _ => false
  end.
```

Comparing the fields amounts to calling the equality test for the appropriate type, possibly using the recursive call to `rose_eqb`.

```
Definition eqb_fields A eqA rec t :
  fields_t t -> fields_t t -> bool
:=
  match t as i return (fields_t i -> fields_t i -> bool) with
  | 1%positive => λa b : A => eqA a b
  | 2%positive => λa b : list (rtree A) =>
                   list_eqb (rtree A) rec a b
  | _ => λ_ _ => true (* impossible, only 2 constructors *)
```

As we can see, the two matches for the constructors are not nested, but rather chained. Both `rose_eqb` and `eqb_fields` have a number of branches proportional to the number of constructors.

The main downside is that, unless the equality tests are extracted to OCaml, the type cast that uses the evidence provided by `pos_eq_dec` must be evaluated.

Readers interested in benchmarks can find them in [GLT23]. Benchmarks essentially confirm that the correctness proofs for these tests typecheck fast, in linear proportion to the size[3] of the inductive data type being compared. Proofs as presented in the previous section typecheck in quadratic time.

5.1.5 The role of Rocq-Elpi in derive

The Elpi language proved instrumental in this research, as it enabled the rapid and concise development of derivations. Some derivations, such as the one for deep inductive principles, were previously unknown, and the ease of experimentation was crucial for their discovery.

We believe that the automatic handling of names and holes was essential. Only a few Rocq APIs were truly necessary: reading from and writing to the logical environment, and invoking type inference. Although these APIs require glue code that is far from trivial – since they read and write inductive types in HOAS form – performance was not critical according to the benchmarks in [GLT23]. All bottlenecks were due to inefficiencies in Rocq, particularly in the APIs for managing universe constraints.

[3] The size is the number of constructors and the number of arguments of constructors.

5.2 Hierarchy Builder

The Mathematical Components library is the largest and most widely used mathematical library for Rocq, according to the 2022 Rocq users survey [Alm+23]. It is the foundation built to tackle the mechanization of the Odd Order theorem [Gon+13]. A key ingredient of this foundation is the hierarchy of interfaces around which the contents are organized, and the fine-tuning of the mechanisms that link the interfaces to their instances: the Rocq elaborator is programmed to automatically find a justification when an abstract theory is applicable to an instance.

The Achilles' heel of the library is its steep learning curve. While documentation such as [MT22] is certainly helpful, the complexity of building the hierarchy and programming the elaborator cannot be overcome by explanation alone, just as assembly language cannot be made accessible merely by documenting it. Hierarchy Builder (HB) provides a high- level, declarative language to describe the hierarchy and a "compiler" that translates this language into the foundational language of Rocq, taking care of programming the elaborator to make the hierarchy work.

The design of the language was brought to our attention in spring 2019 by Cyril Cohen, as a result of fruitful discussions with other researchers in this domain at a Dagstuhl seminar. The implementation took six months [CST20] and pushed Rocq-Elpi to bind a large set of Rocq APIs. The "boilerplate" code that HB replaces with a metaprogram must declare Rocq modules, sections, notations, implicit arguments, canonical structures, . . . , in addition to records and functions between records. HB also motivated the ability to pass Rocq-Elpi programs arguments that are Rocq declarations, such as those introduced by the `Record` or `Definition` keywords.

5.2.1 Mathematical Components 2.0 and Mathematical Components Analysis

Even though HB was, in principle, capable of describing all the interfaces of the Mathematical Components library, porting the library to the HB language was still a significant effort. We were fortunate that developers and power users of Mathematical Components agreed to work on this during a week-long coding sprint [Aff+21]. The results exceeded our expectations: not only was the library ported, but many satellite projects also migrated to HB.

The ease of defining new interfaces provided by HB unlocked the growth of the Mathematical Components library, particularly its order theory component. The graph in fig. 5.1 shows in green the number of interfaces defined using HB. One can see that since HB was introduced, the number of interfaces has grown steadily. In the last three years, about a hundred were added, almost doubling their total, while in the previous seven years only a handful were added. One can also see that the size of the library shrank as a result of the port, since the HB language hides boilerplate code that, when handwritten, is quadratic in the number of related interfaces.

We want to emphasize the link between the growth of the hierarchy of interfaces and the growth of the actual contents (definitions and theorems) of a formal library. Figure 5.2 shows the steady progress made by the Mathematical Components Analysis library (MCA).

The developers of MCA adopted HB even before MC was ported to it, essentially from the very beginning of the library's development. The graph demonstrates how the growth of that library paralleled the expansion of its hierarchy of interfaces. A similar trend can be observed in fig. 5.1, although before the advent of HB, adding an interface required introducing a large amount of boilerplate code, which somewhat forced the correlation between the two measures. The port to HB resulted in the removal of 5,000 lines of such boilerplate code.

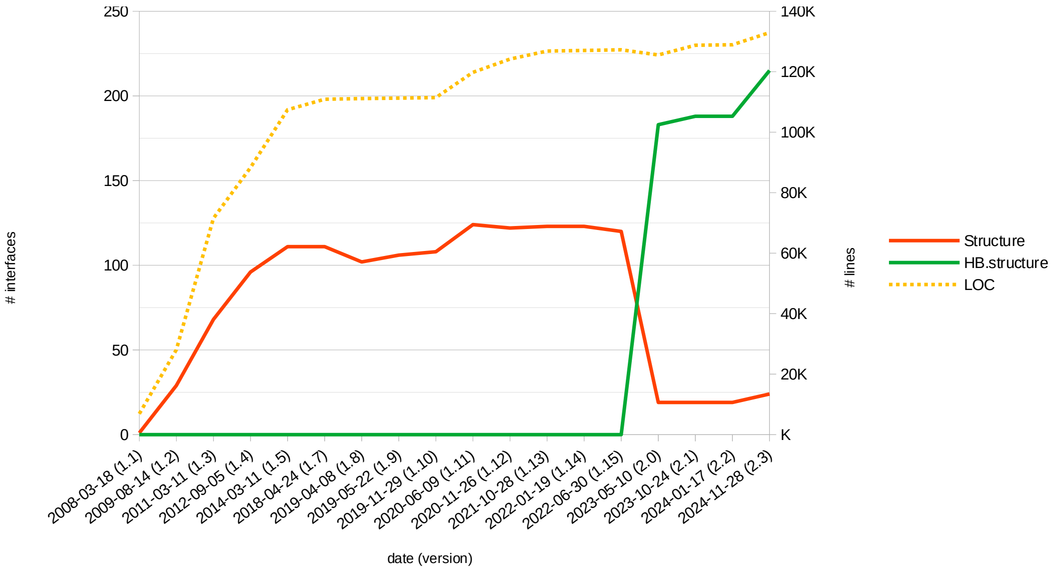


Figure 5.1: Evolution of the Mathematical Components library

5.2.2 The role of Rocq-Elpi in Hierarchy Builder

During the initial implementation of HB (version 1.0), Elpi as a language was truly a technical advantage: its high-level nature allowed us to quickly experiment with different solutions. Once the design was finalized, we thought that rewriting the tool in OCaml, if necessary, would be feasible.

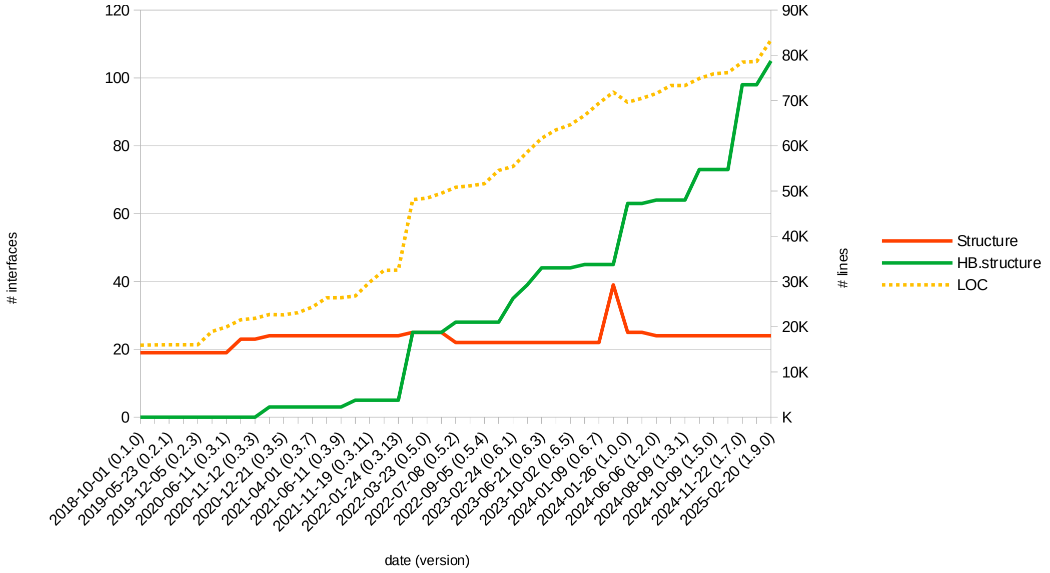


Figure 5.2: Evolution of the Mathematical Components Analysis library

However, HB later had to be substantially revised to accommodate interfaces with parameters, with the goal of fully applying it to the Mathematical Components library. This major rewrite was made possible because Elpi featured a type checker and the change could be modeled as a data type transformation: the description of an interface moved from a flat list to a HOAS structure, describing how formal (bound) parameters distribute over the subinterfaces. In hindsight, we believe we could have performed the same refactoring in an OCaml program, but while it was natural to move to HOAS in Elpi, it is unclear whether the right intuition would have emerged in OCaml, where binding is not necessarily reflected in the types.

The development and production use of HB revealed a few inefficiencies in the implementations of both HB and Elpi. Fortunately, these were mostly due to a few very naive routines in terms of computational complexity.

All the inefficiencies in HB can be attributed to the limited standard library of Elpi. For example, a hierarchy is essentially a Directed Acyclic Graph (DAG) that describes an order relation. Topologically sorting a subpart of it is a frequent operation in HB, and our initial naive implementation was cubic in complexity. Similarly, when we view hierarchy nodes as designated sets and edges as the superset relation, we often need to find all designated sets included in a given one. An algorithm that properly exploits the DAG can discard all supersets of any set not included in the given one, while a naive algorithm can only filter the designated sets linearly.

More surprisingly, the inefficiencies found in Elpi were not in the runtime of the language, but rather in its compiler, and more specifically in its lack of incrementality. As mentioned in section 3.3 and exemplified in section 4.5, HB synthesizes and accumulates rules as it goes to represent its state. In the Mathematical Components library alone, the number of such rules reaches 40,000, and any routine that was not incremental and/or had poor complexity had to be revised. In particular, version 2.0 of Elpi features a fully incremental compiler: adding a rule to an existing program (or database) has logarithmic complexity in the size of the relevant part of the existing program, i.e., the existing rules for the same predicate.

To the best of our knowledge, most of the time spent in HB today is actually consumed by calls to the Rocq API, typically to typecheck and define the terms that are automatically synthesized.

5.3 Other applications

Elpi and Rocq-Elpi have found applications in other projects in which I did not participate directly. Here, I mention some that I am aware of.

5.3.1 Algebra Tactics

Algebra tactics is a port (and improvement) of the **ring**, **field**, and **lra** tactics to the Mathematical Components library [Sak22]. These tactics are reflexive; that is, a pre-processing phase called reification recognizes in a goal the syntax of an algebraic structure and reflects it into a datatype that is then manipulated by a certified Gallina program. Given the hierarchy of algebraic structures, some operations of of a structure may be inherited from a substructure. Since the type theory of Rocq does not feature subtyping, casting a structure to a substructure leaves a trace in the terms of Gallina, making the reification process more delicate.

For example, the addition of a ring actually comes from the additive abelian group it inherits from. The code snippet below must compare the abelian group instance `U` with the ring instance `R` to which the tactic is applied, possibly in the context `C` of a ring morphism.

```
ring C {{ @GRing.opp lp:U lp:In1 }} {{ @ROpp lp:R lp:OutM1 }} Out VM :-
  rocq.unify-eq { rmorphism->zmod C } U ok,
  rmorphism->ring C R, !,
  ring C In1 OutM1 Out1 VM, !,
  build.opp Out1 Out.
```

ELPI

5.3.1.1 The role of Rocq-Elpi in Algebra Tactics

The code makes critical use of the ability to call `rocq.unify-eq` deep within the goal to identify terms up to Rocq's unification. In other words, it leverages the sophisticated machinery of contextual conversions described in section 3.4.2.2.

The algebra tactics project includes a few benchmarks, such as a five-hundred-line polynomial equation in six variables. The **ring** tactic takes about five seconds to solve it.

While the standard, simpler reification written in OCaml is an order of magnitude faster, the reification procedure written in Elpi is not the dominant factor in the five seconds, making it usable in practice even on such an extreme goal (for an interactive theorem prover).

5.3.2 Trakt and TRocq: proof transfer tools

The most frequently asked question regarding the Mathematical Components library is: how do we make it compute?

Almost all definitions in the library are computable, but they are not immediately executable by Rocq for one of two reasons. Either they are executable but inefficient, being optimized for proofs rather than computation, or they are locked behind an opaque module signature to enforce an abstraction barrier, preventing the Rocq type checker from potentially taking an unreasonable amount of time.

The answer to this FAQ is to transfer an expression involving inefficient or locked operations to (equivalent) fast and unlocked ones. While defining such operations and proving their correspondence is feasible – and sometimes already part of the library – the tooling to automatically perform the transfer never really existed in an easy-to-use form.

The Trakt tool [Blo+23] and its successor Trocq [CCM24] fill this gap. Both are implemented in Elpi and are organized as sketched in section 4.5. Trocq is based on a gradualized version of the Univalent Parametricity translation [TTS21]. Unlike the parametricity relation sketched in section 5.1.1, which concerns bare-bone relations, the univalent version focuses on relations that are equivalences. Trocq improves on this by studying relations that are in between, such as implication (i.e., type-theoretic maps). It turns out that in many cases of practical interest, the type theory of Rocq – which does not internalize the principle of univalence – is sufficient to represent the result of the parametricity translation for non-bare-bone relations.

5.3.2.1 The role of Rocq-Elpi in Trocq

In his Ph.D. dissertation [Cra23, Page 115], Enzo Crance highlights how naturally pen-and-paper Trocq rules can be transposed into Elpi. It is worth mentioning that Crance uses Constraints Handling Rules to store axiom requirements when translating an expression, and only when all requirements are known does he compute the minimum set of Rocq axioms needed to represent the parametricity-based translation.

5.3.3 BlueRock's BRiCk

BlueRock Systems[4] is a company developing, among other things, the BRiCk program logic for C++. This consists of a set of Rocq libraries, partially based on Iris [Jun+15; Kre+17], for reasoning about existing C++ code: a `cpp2v` tool translates C++ classes to a formal representation on which specifications can be written.

They employ Rocq-Elpi to automatically synthesize code, extending the derive utility described in section 5.1. In particular, they synthesize type class instances for the `EqDecision`, `Countable`, `Finite`, and `Inhabited` std++ classes, as well as a serialization class called `ToBit`. Finally, they also synthesize Lenses, which are support code for record updates.

We focus here on another use they make of Rocq-Elpi, namely its NES application.

5.3.3.1 N.E.S. – Namespace Emulation System

It is well known that Rocq modules can, to some extent, emulate namespaces. Rocq modules are inspired by OCaml modules and are closed entities: once a module is closed, no new items can be added to it. One can define a new module and include the old one together with some additions, but the resulting module is a new one, with a different name. In contrast, namespaces provide an open-ended notion: new items can always be added to an existing namespace.

The key ingredient for emulating namespaces with modules is the presence in Rocq of a so-called "nametab": an environment of non-ambiguous names that can be extended by importing names from a module, including the names of modules themselves. This allows the illusion that `A.x` and `A.y` are items from the same module `A`, even if the two `A`s have different (long) names, e.g., `X1.A.x` and `X2.A.y`.

[4] https://www.bluerock.io/

NES emulates namespaces using the trick above. It was developed together with Cyril Cohen as a proof of concept, possibly as an addition to Hierarchy Builder. We believe BRiCk uses it to mimic C++ namespaces.

The basic features of NES can be seen in the Rocq code below:

```
(* we place ourselves in a name space *)
NES.Begin This.Is.A.Long.Namespace.
  (* we add some stuff to the name space *)
  Definition stuff := 1.
NES.End This.Is.A.Long.Namespace.

(* we are now outside the name space *)
(* we pick the same name to live dangerously *)
Definition stuff := 2.

(* we place ourselves back in an existing namespace, *)
(* its contents are accessible via their short name *)
NES.Begin This.Is.A.Long.Namespace.
  (* we add some more stuff to the name space *)
  Definition more_stuff := stuff.
NES.End This.Is.A.Long.Namespace.

(* we can access short names (shadow possible collisions) *)
NES.Open This.Is.A.Long.Namespace.
Print stuff. (* = 1 *)
```

It is worth noting how the definition of `more_stuff` uses the contents of the namespace; that is, the second definition of `stuff` does not cause name capture (in another namespace).

```
Eval lazy in more_stuff. (* = 1 *)
```

5.3.3.2 The role of Rocq-Elpi in BRiCk

Although BlueRock has made some of its code public, much of it remains private, so we cannot speculate too much about which features of the Elpi language play which roles. Certainly, the large set of APIs provided by Rocq-Elpi is important here.

The company has disclosed that they run about 6,500 lines of Elpi code, which is, to our knowledge, the largest Elpi code base in existence, given that Hierarchy Builder, all included, is less than 5,000 lines of code.

In recent years, the formal methods team at BlueRock has contributed valuable patches and early feedback to Elpi and Rocq-Elpi.

5.3.4 eIris

In his master disseration [Maa24], Luko van der Maas reimplemented[5] some parts of the Iris proof mode [Kre+17] in Rocq-Elpi. Later, Krebbers, Maas and myself studied an encoding of inductive

[5] https://github.com/lukovdm/MasterThesisIrisElpi

predicates as fixpoints internal to the Iris logic an implemented, in Rocq-Elpi, a command that takes an indcutive specification, as the example below, and generates the corresponding induction principle [KMT25].

```
Iris Inductive is_del_list : loc → list val → iProp :=
| is_del_list_nil l : l ↦ NIL -∗ is_del_list l []
| is_del_list_cons l l' v vs :
    l ↦ CONS (#l',v) -∗ is_del_list l' vs -∗ is_del_list l (v :: vs)
| is_del_list_del l l' vs :
    l ↦ DEL #l' -∗ is_del_list l' vs -∗ is_del_list l vs.
```

5.3.4.1 The role of Rocq-Elpi in eIris

The example above takes advantage of the capability of Elpi commands to receive Gallina declarations as arguments, also in *raw* form. In particular the inductive declaration above does not make any sense taken as a Gallina term, if only because it uses the linear function space of separation logic, and the `iProp` sort. It is only after the elaboration performed by the Elpi code that the declaration becomes a well typed construction.

5.3.5 ProofCert

Elpi was created, by coincidence, during the ProofCert[6] project led by Miller, which aimed to devise a foundation for integrating a broad spectrum of formal methods and to establish formal properties of computer systems. The project centered around the notion of foundational proof certificates (FPC) [Mar16], which has a reference implementation in λProlog. In his Ph.D. dissertation [Bla17], Blanco studied some applications of FPC and ran their implementation with both Teyjus and Elpi. Blanco is likely the first Elpi user. Later, Manighetti, Blanco, Miller, and Momigliano linked FPC with Rocq by translating proof evidence from one system to the other using Rocq-Elpi [RM20; MMM20].

5.3.5.1 The role of Elpi in ProofCert

While Elpi played a very minor role in Blanco's Ph.D., the glue between λProlog and Rocq provided by Rocq-Elpi was instrumental for the later works.

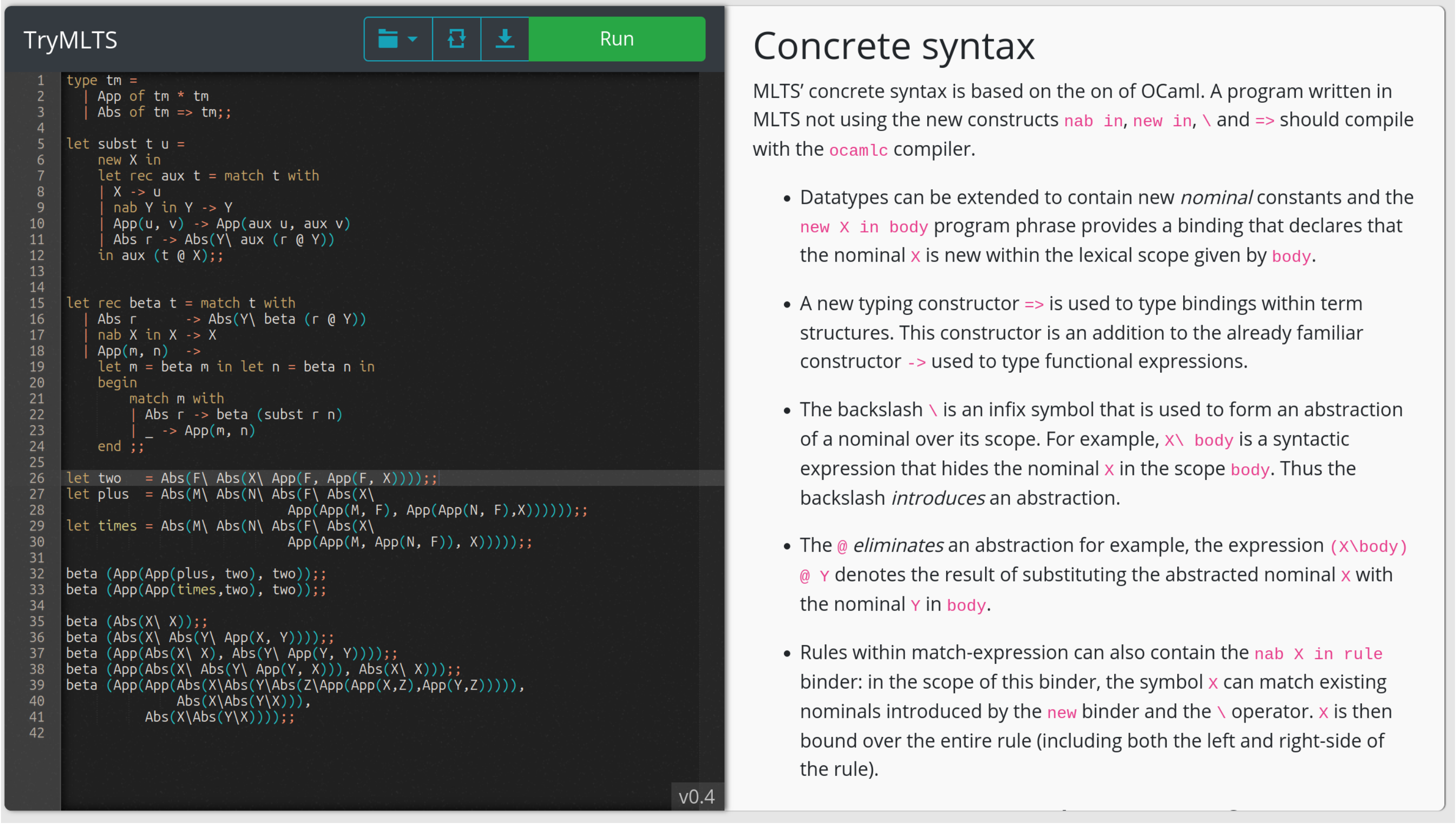
5.3.6 MLTS

As part of his Ph.D., Ulysse Gérard designed and implemented a functional programming language with support for binder mobility, called MLTS [GMS19]. His implementation relies on Elpi for animating the rules defining its semantics, and he made a prototype available in the browser by transpiling the Elpi runtime to JavaScript, see fig. 5.3.

5.3.6.1 The role of Elpi in MLTS

In Gérard's Ph.D., Elpi played only a technological role: being a relatively simple, self-contained interpreter written in OCaml, Elpi could be transpiled to a JavaScript library quite easily.

[6] `https://team.inria.fr/parsifal/proofcert/`



```
type tm =
  | App of tm * tm
  | Abs of tm => tm;;

let subst t u =
    new X in
    let rec aux t = match t with
    | X -> u
    | nab Y in Y -> Y
    | App(u, v) -> App(aux u, aux v)
    | Abs r -> Abs(Y\ aux (r @ Y))
    in aux (t @ X);;


let rec beta t = match t with
  | Abs r      -> Abs(Y\ beta (r @ Y))
  | nab X in X -> X
  | App(m, n)  ->
    let m = beta m in let n = beta n in
    begin
        match m with
        | Abs r -> beta (subst r n)
        | _ -> App(m, n)
    end ;;

let two   = Abs(F\ Abs(X\ App(F, App(F, X))));;
let plus  = Abs(M\ Abs(N\ Abs(F\ Abs(X\
                           App(App(M, F), App(App(N, F),X))))));;
let times = Abs(M\ Abs(N\ Abs(F\ Abs(X\
                           App(App(M, App(N, F)), X)))));;

beta (App(App(plus, two), two));;
beta (App(App(times,two), two));;

beta (Abs(X\ X));;
beta (Abs(X\ Abs(Y\ App(X, Y))));;
beta (App(Abs(X\ X), Abs(Y\ App(Y, Y))));;
beta (App(Abs(X\ Abs(Y\ App(Y, X))), Abs(X\ X)));;
beta (App(App(Abs(X\Abs(Y\Abs(Z\App(App(X,Z),App(Y,Z))))),
             Abs(X\Abs(Y\X))),
          Abs(X\Abs(Y\X))));;
```

Concrete syntax

MLTS' concrete syntax is based on the on of OCaml. A program written in MLTS not using the new constructs `nab in`, `new in`, `\` and `=>` should compile with the `ocamlc` compiler.

- Datatypes can be extended to contain new *nominal* constants and the `new X in body` program phrase provides a binding that declares that the nominal `X` is new within the lexical scope given by `body`.
- A new typing constructor `=>` is used to type bindings within term structures. This constructor is an addition to the already familiar constructor `->` used to type functional expressions.
- The backslash `\` is an infix symbol that is used to form an abstraction of a nominal over its scope. For example, `X\ body` is a syntactic expression that hides the nominal `X` in the scope `body`. Thus the backslash *introduces* an abstraction.
- The `@` *eliminates* an abstraction for example, the expression `(X\body) @ Y` denotes the result of substituting the abstracted nominal `X` with the nominal `Y` in `body`.
- Rules within match-expression can also contain the `nab X in rule` binder: in the scope of this binder, the symbol `X` can match existing nominals introduced by the `new` binder and the `\` operator. `X` is then bound over the entire rule (including both the left and right-side of the rule).

Figure 5.3: Screenshot from https://voodoos.github.io/mlts/

At the same time, Gérard's Ph.D. was quite inspirational to me, since bridging the gap between the logic and functional paradigms, and bringing automatic binder management to the latter, seems within reach thanks to his work.

5.3.7 A proof assistant for local set theory

Nathan Guermond developed a tiny proof assistant based on a flavor of set theory [Joh90] using Elpi.[7]

5.3.7.1 The role of Nathan Guermond in Elpi

During the development of his proof assistant, N. Guermond discovered and helped me fix quite a number of bugs involving η-equivalence in Elpi's unifier. In particular, the unifier was very incomplete when dealing with η-equivalence of non-ground terms. In a way, I feel Guermond's experiments were more instrumental to Elpi than Elpi was to his experiment. Thanks Nathan!

[7] https://github.com/nguermond/local-set-theory

Chapter 6

Conclusion

6.1 Summary

In this document, I have presented the Elpi programming language, its implementation, and its main application as an extension language for Rocq.

6.1.1 Elpi the language

The Elpi programming language is built upon two foundational languages from the 1990s that have been extensively studied: λProlog and Constraint Handling Rules (CHR). λProlog plays the primary role, as most Elpi programs manipulate syntax trees with binders. CHR is essential for scaling the programming comfort of λProlog to syntax trees with holes, without burdening the language with numerous non-logical introspection features.

Throughout this document, I have emphasized that the *key* feature of Elpi is its rule-based nature. I have identified three main advantages:

1. Dynamic rules allow the language to scale effectively to syntax with binders (see section 2.2).

2. Static rules make it easy to write code that is easily extensible (see section 4.4.3).

3. Treating rules as code units greatly facilitates self-manipulation (see section 4.5).

I invite the reader to attempt writing the `to_bool` tool shown in section 4.5 in any other extension language, and to match it in both size and functionality.

I also wish to highlight the importance of the clarity provided by λProlog (and CHR) in the design of Elpi. Building on sound foundations draws a clear distinction between syntactic sugar and "hacks." In particular, the parallel between the execution of λProlog code and proof search in intuitionistic proof theory made it evident that I should avoid some ad-hoc introspection primitives, such as those described in [SC18, Section 8 and later], to implement the type inference algorithm for a form of Standard ML (see section 2.7) or the type checker for the simply typed λ-calculus (see section 2.4). The former requires examining the typing context, i.e., the hypothetical rules of the current goal, akin to a proof rule that inspects the proof tree of a premise. The latter is even more complex, as it finds two (suspended) proof branches for similar goals and, under certain conditions, joins them. Explaining these "proof search" steps at the meta-level at which CHR operates is possible – even formally – as the meta-theory of the proof system.

6.1.2 Elpi the software

What I enjoy most about my work is designing and building tools [Asp+07; GMT15; The25]. To me, Elpi is a tool for building tools more efficiently. Implementing Elpi was great fun; in fact, it is my first fully-fledged programming language. The time required to produce a working implementation would have been prohibitive without an Inria position: free from compulsory teaching duties and with engineering support from the institute.

It is also important to acknowledge that I stood on the shoulders of a giant: Elpi is written in OCaml, which enables both good performance and ease of integration with Rocq. The ability to use mutable data structures to represent unification variables allows Elpi to avoid reifying the heap (the assignment), thereby aligning the notion of garbage in Elpi data with that in OCaml data. This makes it possible to rely on the powerful OCaml garbage collector for Elpi. At the same time, OCaml's rich type system simplified the development of the Foreign Function Interface in a generic way, somewhat analogous to how the signature of printf depends on the format string. GADTs proved to be essential for this purpose.

My main contribution to the implementation of λProlog is the observation that De Bruijn levels enable an efficient implementation of a fragment of the language [Dun+15]. The fragment $\mathcal{L}_\lambda^\beta$ occurs frequently in real code and supports constant-time operations and constant-size data representation as the context of bound variables grows in length.

6.1.3 Rocq-Elpi

At the time of writing, Elpi comprises about 20,000 lines of code: approximately 10,000 for the runtime and 10,000 for the parser, static checker, and compiler. Rocq-Elpi totals 13,000 lines: 5,000 for the Higher Order Abstract Syntax (HOAS) of Rocq terms; 5,000 for the bindings of about 300 Rocq APIs; and 3,000 for the vernacular language used to declare programs. The most complex part is the HOAS, but it is also what makes the language stand out, enabling each Rocq API to require, on average, only 16 lines to be exposed to Elpi (including documentation).

Unsurprisingly, its development was driven by the applications written in it, particularly Hierarchy Builder (section 5.2) and Derive (section 5.1).

Rocq-Elpi was developed as an open source project from the very beginning, keeping the Rocq development team informed of the effort. Their commitment to providing patches to Rocq-Elpi whenever a Rocq API evolved made its development sustainable over more than seven years. Contributions from the Rocq community were also very important, especially from power users such as BlueRock Systems, who contributed a substantial amount of code.

6.1.4 Applications written in Elpi

A significant number of tools for Rocq have been written in Elpi, either by myself or by Rocq power users.

Among these, the most impactful for the Rocq community is Hierarchy Builder, as it unblocked the development of the Mathematical Components library – a widely used piece of Rocq code. Its `ssreflect` component is the most widely depended-upon Rocq package, which in turn makes Hierarchy Builder (and Rocq-Elpi) very important parts of the Rocq ecosystem.

It is also worth mentioning the existence of a substantial amount of closed-source Elpi code used internally by BlueRock Systems. I wish them success in their industrial applications.

6.2 Current and Future Work

If only I had a century – or perhaps two.

There are two main axes to consider: Elpi as a language in its own right, and Rocq-Elpi as a framework for extending Rocq. On the Elpi side, the most valuable improvement would be to increase its appeal to programmers already experienced in functional languages. Most newcomers to Elpi are familiar with functional programming and are encountering logic programming for the first time. Improvements related to syntax – a delicate, though not deeply intellectual, aspect of any language – are important. Another significant area is static analysis, which is so prevalent in modern functional languages that programmers often take it for granted (for example, an untyped language no longer feels familiar, even if it is functional). On the Rocq side, it seems reasonable to pursue deeper integration with the system, making Elpi an integral part of Rocq. Power users frequently request a reduction in the inevitable friction caused by a language integrated as a plugin, which is limited by Rocq's public extension points and extensible syntax.

Below, I discuss several concrete aspects of these two directions.

6.2.1 Static Analysis and Functional Programming Languages

Fissore integrated a determinacy checker into Elpi 3.0 [FT25], along with a new keyword, `func`, to be used in place of `pred` in predicate signatures. This static check essentially guarantees that each call to a functional predicate `p` is operationally equivalent to a call to `once p`:

```
pred once i:pred.
once P :- P, !.
```

As expected, `once` can be flagged as `func` since it never leaves any choice points behind. The signature below uses an arrow to separate inputs from outputs, indicating that `once` behaves as a function operationally, i.e., it leaves no choice points even if its argument does.

```
func once (pred) -> .
```

An interesting example comes from the functional programming repertoire:

```
func map (func A -> B), list A -> list B.
map F [] [].
map F [X|XS] [Y|YS] :- F X Y, map F XS YS.
```

The signature states that `map` is a function under the condition that its higher-order argument `F` is a function. If the precondition is not met, `map` is considered miscalled but still runs fine, even if it may leave choice points. Miscalled functions are tracked by the static analyzer and treated as relations for analyzing the surrounding code. In other words, the signature of `map` is allowed to degrade to the following one, which requires nothing of its input but also guarantees nothing about the choice points it may leave:

```
pred map (pred A -> B), list A -> list B.
```

Since the syntax for `func` requires input arguments first, it paves the way for a syntax for functional predicate bodies that is closer to functional languages, where expressions and statements are intermingled. Spilling, Section 2.6.3, already moves in this direction, but one must explicitly invoke it by using curly braces instead of parentheses (which can be omitted when not needed by the parser). One wonders if this default could be reversed for functional predicates, enabling the previous program to be unambiguously written as follows:

```
func map (func A -> B), list A -> list B.
map F [] R :- R = [].
map F [X|XS] R :- R = [F X|map F XS].
```

Currently, one has to write:

```
map F [X|XS] R :- R = [{F X}|{map F XS}].
```

In turn, the `R :- R` part could be made implicit, yielding the familiar code below:

```
func map (func A -> B), list A -> list B.
map F [] = [].
map F [X|XS] = [F X|map F XS].
```

In this way, I wonder whether some syntactic sugar, driven by types and determinacy, could recover some of the benefits of functional languages while retaining *all* the advantages of λProlog. MLTS [GMS19] goes in this direction, maintaining automatic binder management but dropping the rule-based nature, and thus loosing automatic context management and easy self-extensibility.

6.2.2 Certified Runtime

Over the years, I have gained confidence that the runtime is mostly correct, but I would very much like to machine-check some parts.

In the past ten years, I have found and fixed four bugs that I consider relevant (i.e., they could be triggered in legitimate code) and somewhat embarrassing. Two were related to beta reduction: a routine used throughout the codebase had two code paths that could sometimes be incorrect. The root cause was a misnamed variable: `extraargs` did not actually hold *all* the extra arguments (those for which there is no lambda), which misled the programmer into discarding some arguments. The other two bugs consisted of bad optimizations, fortunately not very relevant in practice, in the higher-order unification algorithm.

As Fissore and I are mechanizing [FT25], we have begun to take the first steps in this direction, but I believe it would take a full Ph.D. to complete the mechanization of a proof of correctness of the runtime.

6.2.3 Program Logic for Elpi

This is the area where even a hundred years would not suffice for me, but there are smarter people who could tackle it. Abella [Gac08] is a good starting point, as it comes with a program logic for plain λProlog. This program logic does not consider the cut operator.

The big-step semantics in section 2.5.1 does consider cut, but it is not clear that it would provide a good foundation for a program logic. In the ongoing mechanization of [FT25], we were forced to define an equivalent semantics with a much more explicit description of the ongoing computation in order to reason about the scope of a cut. While this facilitated proofs by making invariants easier to express, it also departs from what the symbolic evaluation of a logic program looks like. In particular, the computation state is much more complex than the list of alternatives used in section 2.5.1. Proving the two semantics equivalent in Rocq is ongoing work.

From a practical perspective, it is unclear to me if programs written in Rocq-Elpi would truly benefit from a program logic. Rocq features five extension languages to our knowledge (Ltac1, Ltac2, MTac, Meta-Rocq, and Elpi), and only Meta-Rocq has one: it uses Gallina itself as a programming language, so, at least in principle, one could use Rocq to prove properties of meta-programs in Rocq itself. While I find it very intriguing intellectually, this possibility has found, at the time of writing, almost no application. So I'm skeptical that providing a program logic for the full Elpi language would make it significantly more practical. Hence, it will not be a priority of mine, but I would love to read such a program logic written on paper by someone else.

6.2.4 Elpi as a Foundational Language

I have always seen, and still see, Elpi as a programming language that puts *practicality* before *foundationality*, i.e., disregarding the possibility of looking at Elpi programs, or their execution, as a formal specification or formal proof.

One of the reasons is surely that I felt in the marmitte[1] of the Calculus of Inductive Constructions very early in my career, possibly because the French and Italian schools are quite close in that respect. So, maybe unconsciously, the foundational language has always been Rocq to me, leaving to Elpi the role of being practical.

At the same time, my voyage in the neighborhood of computational logic gave me the opportunity to see other ways to look at programming languages with support for binders, i.e., as frameworks to formally describe the object of interest and possibly reason about it. Even if I did not choose to follow the footsteps of Twelf [PS99], Beluga [PC15; Pie10], or MMT [RK13] when it comes to being a foundational language, the literature that describes these systems has been very influential to me, and I think it is fair to say that Elpi would not have existed without it.

6.2.5 Deeper Integration in Rocq

There are plenty of Prolog implementations in the Rocq codebase, most of which are disguised as other features. Some tactics to automatically prove tautologies essentially explore the search space by backtracking. Type class resolution essentially amounts to executing a logic program [SO08]. Canonical Structure resolution is yet another, deterministic, logic program [MT13]. The entire proof engine sits atop a logic programming monad à la LogicT [Kis+05].

All these parts could benefit from Elpi, or at least from some of its implementation techniques, but it is unclear how realistic it would be to replace them with Elpi-based alternatives in a production system such as Rocq.

As part of his Ph.D., Fissore has implemented an alternative type class solver. It is based on an Elpi program that compiles type class instances into rules, which are then queried each time Rocq needs to resolve a type class instance [FT23; FT24]. While the approach seems promising,

[1] As Obelix felt in the pot of magic potion as a child, from the French comic Asterix.

the chances of completely replacing the existing engine (with all its quirks) are low. Fissore also found it difficult to find real and computationally demanding examples to justify the introduction of caching mechanisms such as tabling in Elpi [SUM20; Pie05b]. He (or we) will probably implement a simple version of tabling in the future, but focusing on user experience (e.g., preventing loops) rather than performance.

Pierre Roux, Quentin Vermande, and Cyril Cohen have conducted substantial experiments leveraging Elpi in the elaboration process of Rocq, particularly to obtain well-typed terms from intuitively correct but ill-typed ones, both in the context of arithmetical and set-theoretic expressions. Some examples are quite convincing, and deeper integration of Elpi in Rocq would enable the generation of more informative error messages when these elaboration mechanisms fail.

Finally, based on my experience implementing the first Elpi proof of concept (section 3.1), the Rocq proof engine could benefit from the trail mechanism as implemented in Elpi. It turns out that, for a typical Elpi workload, mutable terms and a trail (a log of mutable cells to unset upon backtracking) provide much better performance than immutable terms paired with a substitution implemented as a functional map (a reified heap, essentially). This is partly because the OCaml garbage collector can do its job in the former case, while the substitution grows unchecked in the latter. Reworking the proof engine to validate this efficiency claim would require substantial engineering effort, but could also benefit long running automation.

6.2.6 Rocq as a Logical Framework

The history of logical frameworks and interactive provers is deeply intertwined. Perhaps the most well-known prover rooted in this idea is Isabelle, which hosts various foundational languages, ranging from the set theory Isabelle/ZF to first-order classical logic Isabelle/FOL and the widely recognized higher-order logic Isabelle/HOL. However, a significant drawback in its design is that the logical framework is so weak, from a proof-theoretic perspective, that it cannot effectively address these formal systems. For instance, one cannot relate a result in Isabelle/ZF with one in Isabelle/HOL, nor reuse the automatic proof search tools that make Isabelle/HOL so practical in Isabelle/ZF. Supporting this observation, Paulson implemented Isabelle/ZF and later mechanized ZF in Isabelle/HOL [Pau22].

In essence, each Isabelle/X is a distinct, incompatible prover with Isabelle/Y. The economy achieved by sharing a common implementation infrastructure seems minimal compared to the potential economy of sharing mechanized theorems. One approach to address this issue is to reconcile different theories via external translation, such as those provided by Dedukti [Bla+23]. However, this method makes it challenging to reason about the translation itself.

An emerging approach is to use Rocq's logic as the logical framework. While the Calculus of Inductive Constructions was not explicitly designed for this purpose – for example, the function space is not ideal for an HOAS encoding of terms with binders, and it is unrestricted with respect to duplication – it still offers a programming language, Gallina, that can encode other formal systems. Proof-theoretically, CIC is robust enough to contain, reason about, and share theorems across these systems. A prominent example of this approach is Iris [Kre+17], a separation logic proved sound in Rocq. Thanks to its proof mode [KTB17], Iris can be used to reason about imperative programs within Rocq. Moreover, pointers and heap locations are often related to mathematical (immutable) objects like lists made of cons and nil via inductive predicates. This enables imperative algorithms to be connected to their functional specifications, allowing theorems to be shared at this level possibly by a proved-correct transfer using tools like Trocq [CCM24].

Other examples of frameworks include VST [App11], which implements a program logic for C, and the FOL [FKW20] library, which provides results on its metatheory and enables proof writing within it. Even the Mathematical Component library [MT22] follows this approach, albeit with a thinner framework compared to the ones above. The extensive use of booleans to encode decidable propositions in the Mathematical Component library makes reasoning by excluded middle appear native. This approach paves the way for proof irrelevance, effectively creating a comfortable shell for classical reasoning within CIC without adding axioms, thereby providing a reusable base of theorems for other frameworks. For example, [Kel+23] reuses linear algebra from Mathematical Components to verify in VST the correctness of a C function that multiplies sparse matrices over IEEE-754 conforming floating-point numbers.

If this trend continues, Elpi could play two significant roles. First, it could continue to serve as an extension language. These frameworks require encoding, and Elpi can assist in their implementation or abstract them for the user. For instance, [KMT25] provides users with the illusion of having inductive predicates in the Iris logic. Second, and perhaps more intriguingly, the Rocq developers could attempt to retrofit a logical framework in CIC, inspired by [Pie+21]. If this happens, the techniques used to implement Elpi's runtime could offer a solid foundation for the logical framework infrastructure required by the new type theory.

Index of concepts

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Elpi: rule-based extension language

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Résumé

Ce document présente Elpi, un langage par règles conçu pour étendre des applications telles que les assistants de preuve interactifs (notamment Rocq). Elpi est un mélange de λProlog, un langage de programmation logique d'ordre supérieur, et de Constraint Handling Rules, permettant une manipulation facile des arbres de syntaxe avec des lieurs et des trous.

Le manuscrit décrit Elpi comme un langage de programmation, en le comparant à Prolog et λProlog. Il détaille ensuite son implémentation, en soulignant les clés de son efficacité et de sa facilité d'intégration dans les applications hôtes. Il poursuit avec la description de l'intégration d'Elpi dans Rocq et conclut par une enquête qui recense les applications développées avec Elpi, en identifiant les fonctionnalités du langage importantes pour chacune.

Deux études de cas illustrent la puissance d'Elpi. La première est une version de l'algorithme d'inférence de types Hindley-Milner. La seconde est un outil de transfert de preuves pour Rocq.

Mots-clés : λProlog, CHR, Elpi, OCaml, Rocq.

Abstract

This document presents Elpi, a rule-based language designed to extend applications such as interactive theorem provers, notably Rocq. Elpi combines λProlog, a higher-order logic programming language, with Constraint Handling Rules, enabling concise and expressive manipulation of syntax trees that include binders and holes.

The manuscript presents Elpi as a programming language and contrasts it with Prolog and λProlog. It then describes the implementation, highlighting the factors that contribute to its efficiency and the ease with which it can be integrated into host applications. The text continues with details on Elpi's integration with Rocq and concludes with a survey of applications built on Elpi, identifying the language features most relevant to each.

Two case studies illustrate Elpi's capabilities. The first presents a version of the Hindley-Milner type inference algorithm. The second describes a proof-transfer tool for Rocq.

Keywords: λProlog, CHR, Elpi, OCaml, Rocq.