

Deriving proved equality tests in Coq-elpi: Stronger induction principles for containers in Coq

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Abstract

We describe a procedure to derive equality tests and their correctness proofs from inductive type declarations in Coq. Programs and proofs are derived compositionally, reusing code and proofs derived previously.

The key steps are two. First, we design appropriate induction principles for data types defined using parametric containers. Second, we develop a technique to work around the modularity limitations imposed by the purely syntactic termination check Coq performs on recursive proofs. The unary parametricity translation of inductive data types turns out to be the key to both steps.

Last but not least, we provide an implementation of the procedure for the Coq proof assistant based on the Elpi [6] extension language.

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1 Introduction

Modern typed programming languages come with the ability of generating boilerplate code automatically. Typically when a data type is declared a substantial amount of code is made available to the programmer at little cost, code such as an equality test, a printing function, generic visitors etc. For example the `derive` directive of Haskell or the `ppx_deriving` OCaml preprocessor provide these features for the respective programming language.

The situation is less than ideal in the Coq proof assistant. It is capable of synthesizing the recursor of a data type, that, following the Curry-Howard isomorphism, implements the induction principle associated to that data type. It supports all data types, containers such as lists included, but generates a quite weak principle when a data type *uses* a container. Take for example the data type `rose tree` (where `U` stands for a universe such as `Prop` or `Type`):

```
Inductive rtree A : U :=
| Leaf (a : A)
| Node (l : list (rtree A)).
```

Its associated induction principle is the following one:

```
Lemma rtree_ind : ∀ A (P : rtree A → U),
  (∀ a : A, P (Leaf A a)) →
  (∀ l : list (rtree A), P (Node A l)) →
  ∀ t : rtree A, P t.
```

Remark that the recursive step, line 3, lacks any induction hypotheses on (the elements of) `l` while one would expect `P` to hold on each and every subtree. Even a very basic recursive program such as an equality test cannot be proved correct using this induction principle. To be honest, the Coq user is not even supposed to write equality tests by hand, nor to prove them correct interactively. Coq provides two facilities to synthesize equality tests and their correctness proofs called `Scheme Equality` and `decide equality`. The former is fully automatic



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but is unfortunately very limited, for example it does not support containers. The latter requires human intervention and generates a single, large, term that mixes code and proofs.

As a consequence, users often need to manually write induction principles, equality tests and their correctness proofs. This situation is very unfortunate because the need for the automatic generation of boilerplate code such as equality tests is higher than ever in the Coq ecosystem. All modern formal libraries structure their contents in a hierarchy of interfaces and some machinery such as Type Classes [18] or Canonical Structures [9] are used to link the abstract library to the concrete instances the user is working on. For example the first interface one is required to implement in order to use the theorems in the Mathematical Components library [10] on a type T is the `eqType` one, requiring a correct equality test on T .

In this paper we use the framework for meta programming based on Elpi [6, 19] developed by the author and we focus on the derivation of equality tests. It turns out that generating equality tests is easy, while their correctness proofs are hard to synthesize, for two reasons. The first problem is that the standard induction principles generated by Coq, as shown before, are too weak. In order to strengthen them one needs quite some extra boilerplate, such as the derivation of the unary parametricity translation of the data types involved. The second reason is that termination checking is purely syntactic in Coq: in order to check that the induction hypothesis is applied to a smaller term, Coq may need to unfold all theorems involved in the proof. This forces proofs to be transparent that, in turn, breaks modularity: A statement is no more a contract, changing its proof may impact users.

In this paper we describe a derivation procedure for equality tests and their correctness proofs where programs and proofs are both derived compositionally, reusing code and proofs derived previously. This procedure also confines the termination check issue, allowing proofs to be mostly opaque. More precisely the contributions of this paper are the following ones:

- A technique to confine the issue stemming from the purely syntactic termination check implemented by Coq out of the main proofs. In this paper we apply it to the correctness proof of equality tests, but the technique is applicable to all proofs that proceed by structural induction.
- A modular and structured process to derive proved equality tests and, en passant, stronger induction principles for inductive types defined using containers.
- An implementation based on the Elpi extension language for the Coq proof assistant.

By installing the `coq-elpi` package¹ and issuing the command `Elpi derive rtree` one gets the following terms synthesized out of the type declaration for `rtree`:

```

Definition eq_axiom T f x := ∀y, reflect (x = y) (f x y).

Definition rtree_eq : ∀A, (A → A → bool) → rtree A → rtree A → bool.

Lemma rtree_eq_OK : ∀A (A_eq : A → A → bool), (∀a, eq_axiom A A_eq a) →
  ∀t, eq_axiom (rtree A) (rtree_eq A A_eq) t.
```

`reflect` is a predicate stating the equivalence between the proposition $(x = y)$ and the boolean test $(f\ x\ y)$; `rtree_eq` is a (transparent) equality test and `rtree_eq_OK` is its (opaque) correctness proof under the assumption that the equality test `A_eq` is correct.

The paper introduces the problem in section 2 by describing the shape of an equality test and of its correctness proof and explaining the modularity problem that stems for the termination checker of Coq. It then presents the main idea behind the modular derivation

¹ See <https://github.com/LPCIC/coq-elpi> for the installation instructions

95 procedure in section 3. Section 4 briefly introduces the Elpi extension language and section 5
96 describes the full derivation.

97 **2 The problem: opaque proofs v.s. syntactic termination checking**

98 Recursors, or induction principles, are not primitive notions in Coq. The language provides
99 constructors for fix point and pattern matching that work on any inductive data the user
100 can declare. For example in order to test two lists `l1` and `l2` for equality one typically takes
101 in input an equality test `A_eq` for the elements of type `A` and then performs the recursion:

```
102 Definition list_eq A (A_eq : A → A → bool) :=
103   fix rec (l1 l2 : list A) {struct l1} : bool :=
104     match l1, l2 with
105     | nil, nil => true
106     | x :: xs, y :: ys => A_eq x y && rec xs ys
107     | _, _ => false
108   end.
```

111 Coq accepts this definition because the recursive call is on `xs` that is a syntactically smaller
112 term of the argument labelled as decreasing by the `{struct l1}` annotation.

113 We can define the equality test for `rtree` by reusing the equality test for lists:

```
114 Definition rtree_eq B (B_eq : B → B → bool) :=
115   fix rec (t1 t2 : rtree B) {struct t1} : bool :=
116     match t1, t2 with
117     | Leaf x, Leaf y => B_eq x y
118     | Node l1, Node l2 => list_eq (rtree B) rec l1 l2
119     | _, _ => false
120   end.
```

123 Note that `list_eq` is called passing as the `A_eq` argument the fixpoint `rec` itself (line 12). In
124 order to check that the latter definition is sound, Coq looks at the body of `list_eq` to see
125 whether its parameter `A_eq` is applied to a term smaller than `t1`. Since `l1` is a subterm of `t1`
126 and since `x` is a subterm of `l1`, then the recursive call `(rec x y)` at line 5 is legit.

127 The fact that checking `rtree_eq` requires inspecting the body of `list_eq` is not very an-
128 noying: we want both `list_eq` and `rtree_eq` to compute, hence their body matters to us.

129 On the contrary proof terms are typically hidden to the type checker once they have
130 been validated, for both performance and modularity reasons. The desire is to make only
131 the statement of theorems binding, and keep the freedom to clean, refactor, simplify proofs
132 without breaking the rest of the formal development.

133 For example, let's assume that `list_eq_OK` is an opaque proof that `list_eq` is correct.

```
134 Lemma list_eq_OK : ∀ A (A_eq : A → A → bool),
135   (∀ a, eq_axiom A A_eq a) →
136   ∀ l1, eq_axiom (list A) (list_eq A A_eq) l1.
137 Proof. .. Qed. (* proof is opaque, hence hidden *)
```

140 It seems desirable to use this lemma in order to prove the correctness of `rtree_eq`, since it
141 calls `list_eq`.

```
142 Lemma rtree_eq_OK B B_eq (HB: ∀ b, eq_axiom B B_eq b) :
143   ∀ t, eq_axiom (rtree B) (rtree_eq B B_eq) t
144 :=
145   fix IH (t1 t2 : rtree B) {struct t1} :=
146     match t1, t2 with
147     | Node l1, Node l2 => .. list_eq_OK (rtree B) (tree_eq B B_eq) IH l1 l2 ..
148     | Leaf b1, Leaf b2 => .. HB b1 b2 ..
149     | .. => ..
150   end.
```

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Unfortunately this term is rejected: we pass `IH`, the induction hypothesis, as the witness that `(tree_eq B B_eq)` is a correct equality test (the argument at line 10 preceding `IH`) but Coq does not know how `list_eq_OK` uses this argument, since its body is opaque.

The issue seems unfixable without changing Coq in order to use a more modular check for termination, for example based on sized types [1]. We propose a less ambitious but more practical approach here, that consists in putting the transparent terms that the termination checker is going to inspect outside of the main proof bodies so that they can be kept opaque.

The intuition is to “reify” the property the termination checker wants to enforce. It can be phrased as “`x` is a subterm of `t` and has the same type”. More in general we model “`x` is a subterm of `t` with property `P`”.

3 The idea: put unary parametricity translation to good use

Given an inductive type `T` we name `is_T` an inductive predicate describing the type of the inhabitants of `T`. This is the one for natural numbers:

```
Inductive is_nat : nat → U :=
| is_0 : is_nat 0
| is_S n (pn : is_nat n) : is_nat (S n).
```

The one for a container such as `list` is more interesting:

```
Inductive is_list A (is_A : A → U) : list A → U :=
| is_nil : is_list A is_A nil
| is_cons a (pa : is_A a) l (pl : is_list A is_A l) : is_list A is_A (a :: l).
```

Remark that all the elements of the list validate `is_A`.

When a type `T` is defined in terms of another type `C`, typically a container, the `is_C` predicate shows up inside `is_T`. For example:

```
Inductive is_rtree A (is_A : A → U) : rtree A → U :=
| is_Leaf a (pa : is_A a) : is_rtree A is_A (Leaf A a)
| is_Node l (pl : is_list (rtree A) (is_rtree A is_A) l) : is_rtree A is_A (Node A l).
```

Note how line 3 expresses the fact that all elements in the list `l` validate `(is_rtree A is_A)`.

Our intuition is that these predicates reify the notion of being of a certain type, structurally. What we typically write `(t : T)` can now be also phrased as `(is_T t)` as one would do in a framework other than type theory, such as a mono-sorted logic.

It turns out that the inductive predicate `is_T` corresponds to the unary parametricity translation [22] of the type `T`. Keller and Lasson in [8] give us an algorithm to synthesize these predicates automatically. What we look for now is a way to synthesize a reasoning principle for a term `t` when `(is_T t)` holds.

3.1 Stronger induction principles for containers

Let’s have a look at the standard induction principle of lists.

```
Lemma list_ind A (P : list A → U) :
  P nil →
  (∀ a l, P l → P (a :: l)) →
  ∀ l : list A, P l.
```

This principle is parametric on `A`: no knowledge on any term of type `A` such as `a` is ever available. We want to synthesize a more powerful principle that lets us choose an invariant for the subterms of type `A` (the differences are underlined):

```

204
2051 Lemma list_induction A (is_A: A → U) (P: list A → U):
2062   P nil →
2073   (∀ a (pa : is_A a) l, P l → P (a :: l)) →
2084   ∀ l, is_list A is_A l → P l.
209

```

210 Note the extra premise (`is_list A is_A l`): The implementation of this induction principle
 211 goes by recursion on the term of this type and finds as an argument of the `is_cons` constructor
 212 the proof evidence (`pa : is_A a`) it feeds to the second premise (line 3). Intuitively all terms
 213 of type (`list A`) validate the property `P`, while all terms of type `A` validate the property `is_A`.

214 More in general to each type we attach a property. For parameters we let the user choose
 215 (we take another parameter, `is_A` here). For the type being analysed, `list A` here, we take
 216 the usual induction predicate `P`. For terms of other types we use their unary parametricity
 217 translation. Take for example the induction principle for `rtree`.

```

218
2191 Lemma rtree_induction A is_A (P : rtree A → U) :
2202   (∀ a, is_A a → P (Leaf A a)) →
2213   (∀ l, is_list (rtree A) P l → P (Node A l)) →
2224   ∀ t, is_rtree A is_A t → P t.
2234

```

224 Line 3 uses `is_list` to attach a property to `l`, and given that `l` has type (`list (rtree A)`) the
 225 property for the type parameter (`rtree A`) is exactly `P`. Note that this induction principle
 226 gives us access to `P`, the property one is proving, on the subtrees contained in `l`.

227 3.1.1 Synthesizing stronger induction principles

228 We postpone a detailed description of the synthesis to section 5.4, here we just sketch how
 229 to build the type on the induction principle.

230 It turns out that the types of the constructors of `is_T` give us a very good hint on the
 231 type of the induction principle. The type of the first premise

```

232
2333 (∀ a, is_A a → P (Leaf A a)) →

```

235 is exactly the type of the `is_Leaf` constructor

```

236
2373 | is_Leaf a (pa : is_A a) : is_rtree A is_A (Leaf A n)
238

```

239 where (`is_rtree A is_A`) is replaced by `P`. The same holds for the other premise: its type
 240 can be trivially obtained from the type of `is_Node`.

241 Our intuition is that the inductive predicate `is_T` provides the same information that
 242 typing provides. Induction principles give `P` on (smaller) terms of the same type, that would
 243 be terms for which `is_T` holds. Given their inductive nature, `is_T` predicates are able to
 244 propagate the desired property inside parametric containers.

245 3.2 Isolating the syntactic termination check problem

246 As one expects, it is possible to prove that `is_T` holds for terms of type `T`.

```

247
248 Definition nat_is_nat : ∀ n : nat, is_nat n :=
249   fix rec n : is_nat n :=
250     match n as i return (is_nat i) with
251     | 0 => is_0
252     | S p => is_S p (rec p)
253   end.
254

```

255 For containers (`T A`) we can prove (`is_T A is_A`) when `is_A` is trivial.

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```

256
257 Definition list_is_list : ∀ A (is_A : A → U), (∀ a, is_A a) → ∀ l, is_list A is_A l.
258
259 Definition rtree_is_rtree : ∀ A (is_A : A → U), (∀ a, is_A a) → ∀ t, is_rtree A is_A t.
260

```

These facts are then to be used in order to satisfy the premise of our induction principles.

Going back to our goal, we can build correctness proofs of equality tests in two steps. For example, for natural numbers we can generate two lemmas:

```

264
265 Lemma nat_eq_correct : ∀ n, is_nat n → eq_axiom nat nat_eq n :=
266   nat_induction (eq_axiom nat nat_eq) P0 PS.
267
268 Lemma nat_eq_OK n : eq_axiom nat nat_eq n :=
269   nat_eq_correct n (nat_is_nat n).
270

```

where P0 and PS (line 2) stand for the two proof terms corresponding to the base case and the inductive step of the proof. We omit them here for brevity.

For containers such as `(list A)` we can link the pieces in a similar way (at line 3 we omit the proofs for `nil` and `cons` as before).

```

275
276 Lemma list_eq_correct A A_eq : ∀ l, is_list A (eq_axiom A A_eq) l →
277   eq_axiom list A (list_eq A A_eq) l :=
278   list_induction A (eq_axiom A A_eq) (eq_axiom (list A) (list_eq A A_eq)) Pnil Pcons.
279
280 Lemma list_eq_OK A A_eq (HA : ∀ a, eq_axiom A A_eq a) l :
281   eq_axiom (list A) (list_eq A A_eq) l :=
282   list_eq_correct A A_eq l (list_is_list A (eq_axiom A A_eq) HA l).
283

```

It is interesting to look at a data type that uses a container such as `rtree`: the induction hypothesis P1 given by `rtree_induction` perfectly fits the premise of `list_eq_correct` (line 7).

```

286
287 Lemma rtree_eq_correct A A_eq : ∀ t, is_tree A (eq_axiom A A_eq) t →
288   eq_axiom (rtree A) (rtree_eq A A_eq)
289   :=
290   rtree_induction A (eq_axiom A A_eq) (eq_axiom (rtree A) (rtree_eq A A_eq))
291   PLeaf
292   (fun l (P1 : is_list (rtree A) (eq_axiom (rtree A) (rtree_eq A A_eq)) l) =>
293     .. list_eq_correct (rtree A) (rtree_eq A A_eq) l P1 ..).
294
295 Lemma rtree_eq_OK A A_eq (HA : ∀ a, eq_axiom A A_eq a) t :
296   eq_axiom (rtree A) (rtree_eq A A_eq) t :=
297   rtree_eq_correct A A_eq t (rtree_is_rtree A (eq_axiom A A_eq) HA t).
298

```

Type checking the terms above does not require any term to be transparent. Actually they are applicative terms, there is no apparently recursive function involved.

Still there is no magic, we just swept the problem under the rug. In order to type check the proof of `rtree_is_rtree` Coq needs to look at the proof term of `list_is_list`:

```

303
304 Definition rtree_is_rtree A is_A (His_A : ∀ a, is_A a) :=
305   fix IH t {struct t} : is_rtree A is_A t :=
306     match t with
307     | Leaf a => is_Leaf A is_A a (His_A a)
308     | Node l => is_Node A is_A l (list_is_list (rtree A) (is_rtree A) IH l)
309     end.
310

```

As we explained in section 2 Coq would reject this term if the body of `list_is_list` was opaque.

Even if we cannot make the problem disappear (without changing the way Coq checks termination), we claim we confined the termination checking issue to the world of reified type information. The transparent proofs of theorems such as `T_is_T` are separate from the other, more relevant, proofs that can hence remain opaque as desired.

4 Elpi: an extension language for Coq

Elpi [6] is a dialect of λ Prolog [13], a higher order logic programming language. Elpi can be used as an extension language for Coq [19] in order to develop new commands in a programming language that has native support for bound variables.

Coq terms are represented in λ -tree syntax style [12] (sometimes also called Higher Order Abstract Syntax) reusing the binders of the programming language to represent the ones of Coq. For example, the term `(fun x => fact x)` is represented as `(lam (λ x, app["fact",x]))`. We say that `app` and `lam` are object level term constructors standing for iterated (n-ary) application and unary lambda abstraction; `"fact"` is a constant and `x` is a variable bound by λ x, that is the binder of the programming language.²

Programs are organized in clauses that represent both a data base of known facts and a set of rules to derive new facts out of known ones. For example one could use a relation named `eq-db` to link a type to its equality test.

```
eq-db "nat" "nat_eq".
eq-db (app["list", B]) (app["list_eq", B, B_eq]) :- eq-db B B_eq.
```

The first clause is a fact stating that `nat_eq` is the equality test for type `nat`. The second clause is an inference one and reads: the equality test for `(list B)` is `(list_eq B B_eq)` if `B_eq` is the equality test for `B`.

The `eq-db` data base can be queried for an equality test for, say, `(list nat)` by writing the goal `(eq-db (app["list", "nat"]) F)` where `F` is a variable to be filled in. By chaining the two clauses Elpi answers `(F = app["list_eq", "nat", "nat_eq"])` that reads back in the Coq syntax as `(list_eq nat nat_eq)`, the desired equality test for `(list nat)`.

It is worth pointing out that in λ Prolog the set of clauses is dynamic: a program is allowed to add clauses inside a specific scope (typically the one of a binder) and the runtime collects them when the scope ends. As we will see, this feature is useful when a derivation takes place under an hypothetical context, e.g. when one assumes a parameter `A` and an equality test `A_eq`. No other feature of the Elpi language is relevant to this paper.

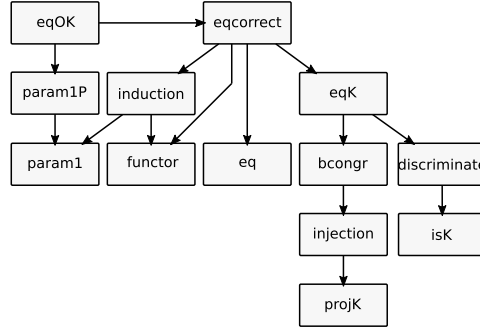
Finally, the integration of Elpi in Coq exposes to the extension language primitives to access the logical environment, e.g. to read an inductive data type declaration; to declare a new inductive type; to define a new constant; etc.

5 Anatomy of the derivation

The structure of the derivation is depicted in the following diagram. Each box represents a component deriving a complete term. An arrow from component A to component B tells that the terms generated by B are used by the terms generated by A. The interfaces between these components are indeed types: one can replace the work done by each component with a few hand written terms, if necessary.

² Here we simplify a little the embedding and use strings to represent named terms, omitting their nodes: For example `nat`, an inductive type, is actually written `(indt "Coq.Init.Datatypes.nat")`, while `fact`, a defined constant, is written `(const "Coq.Arith.Factorial.fact")`.

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355

356 The *eq* component is in charge of synthesizing the program performing the equality test.
357 The correctness proof generated by *eqcorrect* goes by induction on the first term of the two
358 being compared and then goes on in a different branch for each constructor *K*. The property
359 being proved by induction is expressed using *eq_axiom* that, as we will detail in section 5.6
360 is equivalent to a double implication. The *bcongr* component proves that the property is
361 preserved by equal contexts, that is when the two terms are built using the same constructor.
362 When they are not the program must return false and the equality be false as well: this is
363 shown by *eqK*, that performs the case split on the second term. The no confusion property
364 of constructor is key to this contextual reasoning. *projK* and *isK* generate utility functions
365 that are then used by *injection* and *discriminate* to prove that constructors are injective
366 and different. As we sketched in the previous sections the unary parametricity translation
367 plays a key role in expressing the induction principle. The inductive predicate *is_T* for an
368 inductive type *T* is generated by *param1* while *param1P* shows that terms of type *T* validate
369 *is_T*. *functor* shows that *is_T* is a functor when *T* has parameters. This property is both used
370 to synthesize induction principles and also to combine the pieces together in the correctness
371 proof. The *eqOK* component hides the *is_T* relation from the theorems proved by *eqcorrect*
372 by using the lemmas *T_is_T* proved by *param1P*.

373 5.1 Equality test

374 Synthesizing the equality test for a type *T* proceeds as follows. First the test takes in input
375 each type parameter *A* together with an equality test *A_eq*. Then the recursive function takes
376 in input two terms of type *T* and inspects both via pattern matching. Outside the diagonal,
377 where constructors are different, it says *false*. On the diagonal it composes the calls on the
378 arguments of the constructors using boolean conjunction. The code called to compare two
379 arguments depends on their type: If it is *T* then it is a recursive call; if it is a type parameter
380 *A* then we use *A_eq*; if it is another type it uses the corresponding equality test.

381 Let us take for example the equality test for rose trees:

```

382 Definition rtree_eq A (A_eq : A → A → bool) :=
383   fix rec (t1 t2 : rtree A) {struct t1} : bool :=
384     match t1, t2 with
385     | Leaf a, Leaf b => A_eq a b
386     | Node l, Node s => list_eq (rtree A) rec l s
387     | _, _ => false
388   end.
389

```

391 Line 5 calls *list_eq* since the type of *l* and *s* is *(list (rtree A))* and it passes to it *rec* since
392 the type parameter of *list* is *(rtree A)*.

Here is an excerpt of Elpi code used to synthesize the body of the branches:

```

393
394 eq-db "A" "A_eq".
395 eq-db (app["rtree", "A"]) "rec".
396 eq-db (app["list", B]) (app["list_eq", B, B_eq]) :- eq-db B B_eq.
397

```

The first clause says that `A_eq` is the equality test for type `A`, and is used to build the branch at line 4. The third clause, chained with the second one, combines `list_eq` with `rec` building the branch at line 5. The first two clauses are present only during the derivation of the body of the fixpoint, under the context formed by the type parameter `A`, its equality test `A_eq`, and the recursive call `rec` itself. Once the derivation is complete both clauses are removed from the data base and the following one is permanently added.

```

405 eq-db (app["rtree", B]) (app["rtree_eq", B, B_eq]) :- eq-db B_eq.
406

```

5.2 Parametricity

The *param1* component is able to generate the unary parametricity translation of types and terms following [8]. We already gave many examples in section 3. The *param1P* component synthesizes proofs that terms of type `T` validate `is_T` by a trivial structural recursion: constructor `K` is mapped to `is_K`. When `T` is a container we assume the triviality of the property on the type parameter. For example:

```

414 Definition rtree_is_rtree A (is_A : A → U) : (∀ x, is_A x) → ∀ t, is_rtree A is_A t.
415

```

5.3 Functoriality

The *functor* component implements a double service. For non-indexed containers it synthesizes a simple map:

```

420 Definition list_map A B : (A → B) → list A → list B.
421

```

The derivation becomes more interesting when the container has indexes, e.g. when the container is a `is_T` inductive predicate. On indexed data types the derivation avoids to map the indexes and consequently all type variables occurring in the types of the indexes. For example, mapping the `is_list` inductive predicate gives:

```

427 Lemma is_list_funct A P Q : (∀ a, P a → Q a) → ∀ l, is_list A P l → is_list A Q l.
428

```

This property corresponds to the functoriality of `is_list` over the property about the type parameter. Note that parameters of arity one, such as `P`, are mapped point wise.

As we did for the `eq-db` data base of equality tests, we can store these maps as clauses and use the data base later on in the *induction* and *eqcorrect* derivations. Here is an excerpt of Elpi code for this data base, that we call `funct-db`:

```

435 funct-db (app["is_list", A, P]) (app["is_list", A, Q]) (app["is_list_funct", A, P, Q, F]) :-
436   funct-db P Q F.
437

```

Note that the terms involved are “point free”, i.e. the first two arguments are terms of arity one, while the third term is of arity two. The identity is written as follows:

```

441 funct-db P P (lam (λ a, lam (λ p, p))).
442

```

This means that when one has a term `a` and a term `(p : P a)`, in order to obtain a term `(q : Q a)` he can query `funct-db` by asking Elpi to fill in `M` in `(funct-db "P" "Q" M)`. If the answer is `(M = f)` then the desired term is obtained by passing `a` and `p` to `f`, that is `(f a p : Q a)`.

5.4 Induction

In order to derive the induction principle for type T we first derive its unary parametricity translation is_T . The is_T inductive predicate has one constructor is_K for each constructor K of the type T . The type of is_K relates to the type of K in the following way. For each argument $(a : A)$ of K , is_K takes two arguments: $(a : A)$ and $(pa : \text{is}_A a)$. Finally the type of $(\text{is}_K a_1 pa_1 \dots a_n pa_n)$ is $(\text{is}_T (K a_1 \dots a_n))$.

The induction principle is synthesized by following these steps:

1. take in input each parameter $A_1 \text{ is}_A \dots A_n \text{ is}_A$ of is_T .
2. take in input a predicate $(P : T A_1 \dots A_n \rightarrow U)$.
3. for each constructor is_K of type $(\forall A_1 \text{ is}_A \dots A_n \text{ is}_A, \forall a_1 pa_1 \dots a_n pa_n, \text{is}_T A_1 \text{ is}_A \dots A_n \text{ is}_A (K a_1 \dots a_n))$ take in input an assumption HK of type $(\forall a_1 pa_1 \dots a_n pa_n, P (K a_1 \dots a_n))$.
4. take in input $(t : T A_1 \dots A_n)$.
5. take in input $(x : \text{is}_T A_1 \text{ is}_A \dots A_n \text{ is}_A t)$.
6. perform recursion on x and a case split. Then in each branch
 - a. bind all arguments of is_K , namely $(a_1 : A_1) (pa_1 : \text{is}_A a_1) \dots (a_n : A_n) (pa_n : \text{is}_A a_n)$
 - b. obtain qai by *mapping* the corresponding pai (as in `funct-db`, see below).
 - c. return $(\text{HK } a_1 qai \dots a_n qan)$

Lets take for example the induction principle for rose trees:

```

Definition rtree_induction A is_A P
  (HLeaf : ∀ a, is_A a → P (Leaf A a))
  (HNode : ∀ l, is_list (rtree A) P l → P (Node A l)) :
  ∀ t, is_rtree A is_A t → P t
:=
fix IH (t: rtree A) (x: is_rtree A is_A t) {struct x}: P t :=
  match x with
  | is_Leaf a pa => HLeaf a pa
  | is_Node l pl => (* pl: is_list (rtree A) (is_rtree A is_A) l *)
    HNode l (is_list_funct (rtree A) (is_rtree A is_A) P IH l pl)
end.

```

Note how, intuitively, the type of `HLeaf` can be obtained from the type of `is_Leaf` by replacing $(\text{is_rtree } A \text{ is}_A)$ with P .

Finally let us see how the second argument to `HNode` is synthesized. We take advantage of the fact that Elpi is a logic programming language and we query the data base `funct-db` as follows. First we temporarily register the fact that `IH` maps $(\text{is_rtree } A \text{ is}_A)$ to P obtaining, among others, the following clauses.

```

funct-db (app["is_rtree", "A", "is_A"]) "P" "IH".
funct-db (app["is_list", A, P]) (app["is_list", A, Q]) (app["is_list_funct", A, P, Q, F]) :-
  funct-db P Q F.

```

Then we query `funct-db` as follows:

```

funct-db (app["is_list", app["rtree", "A"], app["is_rtree", "A", "is_A"]])
  (app["is_list", app["rtree", "A"], "P"])
  Q.

```

The answer $(Q = \text{app}["\text{is_list_funct}", \text{app}["\text{rtree}", "A"], \text{app}["\text{is_rtree}", "A", "is_A"], "P", "IH"])$ is exactly the second term we need to pass to `HNode` (once applied to `l` and `pl`, line 10 above).

It is worth pointing out that, for the term to be accepted by the termination checker the map over `is_list` must be transparent.

501 To sum up the unary parametricity translation gives us the type of the induction principle,
 502 up to a trivial substitution. The functoriality property of the inductive predicates obtained
 503 by parametricity gives us a way to prove the branches.

504 5.5 No confusion property

505 In order to prove that an equality test is correct one has to show the so called “no confusion”
 506 property, that is that constructors are injective and disjoint (see for example [11]).

507 The simplest form of the property of being disjoint is expressed on `bool`:

508 **Lemma** `bool_discr` : `true = false` $\rightarrow \forall T : \mathbf{U}, T$.
 509

511 This lemma is proved by hand once and for all. What the *isK* component synthesizes is a
 512 per-constructor test to be used in order to reduce a discrimination problem on type `T` to a
 513 discrimination problem on `bool`. For the rose tree data type *isK* generates:

514 **Definition** `is_Node` A (t : `rtree` A) := `match t with Node _ => true | _ => false end`.
 515 **Definition** `is_Leaf` A (t : `rtree` A) := `match t with Leaf _ => true | _ => false end`.
 516
 517

518 The *discriminate* components uses one more trivial fact, `eq_f`³, in order to assemble these
 519 tests together with `bool_discr`.

520 **Lemma** `eq_f` T1 T2 (f : T1 \rightarrow T2) : $\forall a\ b, a = b \rightarrow f\ a = f\ b$.
 521
 522

523 From a term `H` of type `(Node l = Leaf a)` the *discriminate* procedure synthesizes:

524 `(bool_discr (eq_f (rtree A) (rtree A) (is_Node A) H)) : $\forall T : \mathbf{U}, T$`
 525
 526

527 Note that the type of the term `(eq_f .. H)` is `(is_Node A (Node l) = is_Node A (Leaf a))` that
 528 is convertible to `(true = false)`, the premise of `bool_discr`.

529
 530 In order to prove the injectivity of constructors the *projK* component synthesizes a
 531 projector for each argument of each constructor. For the `cons` constructor of `list` we get:

532 **Definition** `get_cons1` A (d1 : A) (d2 : `list` A) (l : `list` A) : A :=
 533 `match l with nil => d1 | x :: _ => x end`.
 534
 535 **Definition** `get_cons2` A (d1 : A) (d2 : `list` A) (l : `list` A) : `list` A :=
 536 `match l with nil => d2 | _ :: xs => xs end`.
 537
 538

539 Each projector takes in input default values for each and every argument of the constructor.
 540 It is designed to be used by the *injection* procedure as follows. Given a term `H` of type
 541 `(x :: xs = y :: ys)` it synthesizes:

542 `(eq_f (list A) A (get_cons1 A x xs) (x :: xs) (y :: ys) H) : x = y`
 543 `(eq_f (list A) (list A) (get_cons2 A x xs) (x :: xs) (y :: ys) H) : xs = ys`
 544
 545

546 These terms are easy to build given that the type of `H` contains the default values to be
 547 passed to the projectors. Note that the type of the second term is actually:

548 `get_cons2 A x xs (x :: xs) = get_cons2 A x xs (y :: ys)`
 549
 550

551 that is convertible to the desired type `(xs = ys)`.

³ `eq_f` is called `f_equal` in the Coq standard library.

5.6 Congruence

In the definition of `eq_axiom` we use the `reflect` predicate [10]. It is a sort of if-and-only-if specialized to link a proposition and a boolean test. It is defined as follows:

```

555 Inductive reflect (P : U) : bool → U :=
556 | ReflectT (p : P) : reflect P true
557 | ReflectF (np : P → False) : reflect P false.
558

```

In our case the shape of `P` is always an equation between two terms of an inductive type, i.e. constructors. When the same constructor occurs in both sides, as in $(k\ x1.. x_n = k\ y1.. y_n)$, the equality test discards `k` and proceeds on each $(x_i = y_i)$. The *bcongr* component synthesizes lemmas helping to prove the correctness of this step. For example:

```

564 Lemma list_bcongr_cons A :
565   ∀ (x y : A) b, reflect (x = y) b →
566   ∀ (xs ys : list A) c, reflect (xs = ys) c →
567   reflect (x :: xs = y :: ys) (b && c)
568
569 Lemma rtree_bcongr_Leaf A (x y : A) b :
570   reflect (x = y) b → reflect (Leaf A x = Leaf A y) b
571
572 Lemma rtree_bcongr_Node A (l1 l2 : list (rtree A)) b :
573   reflect (l1 = l2) b → reflect (Node A l1 = Node A l2) b
574

```

Note that these lemmas are not related to the equality test specific to the inductive type. Indeed they deal with the `reflect` predicate, but not with the `eq_axiom` predicate that we use every time we talk about equality tests.

The derivation goes as follows: if any of the premises is false, then the result is proved by `ReflectF` and the injectivity of constructors. If all premises are `ReflectT` their argument, an equation, can be used to rewrite the conclusion.

```

582 Lemma list_bcongr_cons A
583   (x y : A) b (hb : reflect (x = y) b)
584   (xs ys : list A) c (hc : reflect (xs = ys) c) :
585   reflect (x :: xs = y :: ys) (b && c) :=
586   match hb, hc with
587   | ReflectT eq_refl, ReflectT eq_refl => ReflectT eq_refl
588   | ReflectF (e : x = y → False), _ =>
589     ReflectF (fun H : x :: xs = y :: ys =>
590       e (eq_f (list A) A (get_cons1 A x xs) (x :: xs) (y :: ys) H))
591   | _, ReflectF e =>
592     ReflectF .. (e (eq_f .. (get_cons2 ..) ..) ..) ..
593   end.
594

```

The elimination of `hb` and `hc` substitutes `b` and `c` by either `true` or `false`. In the branch at line 6 the boolean expression is hence $(\text{true} \ \&\& \ \text{true})$ while the proposition is $(x :: xs = x :: xs)$ given that the two equations $(x = y)$ and $(xs = ys)$ were eliminated as well.

The argument of `e` at line 9 is the term generated by the *injection* component. The branch at line 11, covering the case where the heads are equal but the tails different, is very close to lines 8 and 9 but for the fact that the projector for the second argument of `cons` is used, instead of the projection for the first one.

There are other ways one could have expressed these lemmas, for example by not mentioning the `cons` constructor explicitly but rather an abstract function `k` known to be injective on the first and second argument. Even if we find this presentation more appealing on paper, in practice we found no advantage and we hence opted for the current approach.

bcongr gives us lemmas to propagate equality and inequality only under the same constructor. *eqK* complements this work by proving `eq_axiom` also when the constructors differ.

Recall that the induction principle does a case split on one term, the first one of the two being compared. *eqK* generates a lemma for each constructor, to be used in the corresponding branch of the induction, that performs the case split on the second term being compared. This is the lemma generated for *Node*:

```

613 Lemma rtree_eq_axiom_Node A (A_eq : A → A → bool) l1 :
614   eq_axiom (list (rtree A)) (list_eq (rtree A) (rtree_eq A A_eq)) l1 →
615   eq_axiom (rtree A) (rtree_eq A A_eq) (Node A l1)
616 :=
617   fun H (t2 : rtree A) =>
618     match t2 with
619     | Leaf n =>
620       ReflectF (fun abs : Node A l1 = Leaf A n =>
621         bool_discr (eq_f (rtree A) bool (is_Node A) (Node A l1) (Leaf A n) abs) False)
622     | Node l2 =>
623       rtree_bcongr_Node A l1 l2 (list_eq (rtree A) (rtree_eq A A_eq) l1 l2) (H l2)
624 end.
625
626

```

Note that the code for the first branch is what *discriminate* synthesizes; while the code in the second branch is what *bcongr* generates.

5.7 Correctness

The *eqcorrect* component combines the induction principle generated by *induction* with the case split on the second term provided by *eqK*.

Let's recall the type of the correctness lemma for *list_eq*, of the induction principle and then let's analyse the proof of *rtree_eq_correct*:

```

634 Lemma list_eq_correct A (fa : A → A → bool) l,
635   is_list A (eq_axiom A fa) l →
636   eq_axiom (list A) (list_eq A fa) l.
637
638 Definition rtree_induction A is_A P
639   (HLeaf : ∀ y, is_A y → P (Leaf A y))
640   (HNode : ∀ l, is_list (rtree A) P l → P (Node A l)) :
641   ∀ t, is_rtree A is_A t → P t.
642
643 Lemma rtree_eq_axiom_Node A (f : A → A → bool) l1 :
644   eq_axiom (list (rtree A)) (list_eq (rtree A) (rtree_eq A f)) l1 →
645   eq_axiom (rtree A) (rtree_eq A f) (Node A l1).
646
647

```

The proof is a rather straightforward application of the induction principle to the property

```

648 eq_axiom (rtree A) (rtree_eq A fa)
649
650
651

```

Each branch is then proved by the corresponding lemma generated by *eqK* with only one caveat: one may need to adapt the induction hypothesis, *P1* here, in order to make it fit the premise of the lemma generated by *eqK*. In this specific case the "adaptor" is *list_eq_correct*.

```

652 Lemma rtree_eq_correct A (fa : A → A → bool) :=
653   rtree_induction A (eq_axiom A fa)
654   (*P*) (eq_axiom (rtree A) (rtree_eq A fa))
655   (*HLeaf*) (rtree_eq_axiom_Leaf A fa)
656   (*HNode*) (fun l (P1 : is_list (rtree A) (eq_axiom (rtree A) (rtree_eq A fa)) l) =>
657     rtree_eq_axiom_Node A fa l (list_eq_correct (rtree A) (rtree_eq A fa) l P1)).
658
659
660

```

Logic programming provides a natural way to synthesize the adaptor. We load in the data base all the correctness proofs synthesized so far, as follows:

```

661 funct-db (app["is_list", A, is_A])
662   (app["eq_axiom", app["list", A], app["list_eq", A, A_eq]]) R :-
663   R = (app["list_eq_correct", A, A_eq]),
664   funct-db is_A (app["eq_axiom", A, A_eq]).
665
666

```

6:14 Deriving proved equality tests in Coq-elpi

This clause simply gives an operational reading to the type of `list_eq_correct`: the conclusion is true if the premise is. The only cleverness is to separate the premise in two parts, being a `(list A)` with property `is_A` and have `is_A` be a sufficient condition to prove that `A_eq` is correct. In this way clauses compose better: Search peels off just one type constructor at a time. Indeed we extend the `funct-db` predicate, instead of building a new one just for correctness lemmas, because functoriality lemmas are sometimes needed in addition to the correctness ones. Take for example this simple data type of a histogram.

```

678 Inductive histogram := Columns (bars : list nat).
679
680
681 Lemma histogram_induction (P : histogram → Type) :
682   (∀ l, is_list nat is_nat l → P (Columns l)) →
683   ∀ h, is_histogram h → P h.

```

Now look at the lemma synthesized by *eqK* for the `Columns` constructor.

```

686 Lemma histogram_eq_axiom_Columns l :
687   eq_axiom (list nat) (list_eq nat nat_eq) l →
688   ∀ h, eq_axiom_at histogram histogram_eq (Columns l) h.
689
691
692 Lemma histogram_eq_correct h : eq_axiom histogram histogram_eq h :=
693   histogram_induction
694     (eq_axiom histogram histogram_eq)
695     (fun l (P1 : is_list nat is_nat l) =>
696       histogram_eq_axiom_Columns
697         l (list_eq_correct nat nat_eq
698           l (is_list_funct nat is_nat (eq_axiom nat nat_eq) nat_eq_correct l P1))).
699

```

Note that the type of `P1` is `(is_list nat is_nat)` and that it needs to be adapted to match `(is_list nat (eq_axiom nat nat_eq))`. The correctness lemma for `nat_eq`, namely `nat_eq_correct` of type `(∀ n, is_nat n → eq_axiom nat nat_eq n)`, cannot be used directly but must undergo the `is_list_funct` functor.

5.8 eqOK

The last derivation hides the `is_T` predicate to the final user by combining the output of *eqcorrect* and *param1P*.

```

707 Lemma list_eq_correct A A_eq :
708   ∀ l, is_list A (eq_axiom A A_eq) l → eq_axiom (list A) (list_eq A A_eq) l.
709
710
711 Lemma list_eq_OK A A_eq A_eq_OK l : eq_axiom (list A) (list_eq A A_eq) l :=
712   list_eq_correct A A_eq l (list_is_list A (eq_axiom A A_eq) A_eq_OK).
713

```

Both lemmas are needed. The former composes well and is needed if one defines a type using lists as a container. The latter is what the user needs in order to work with lists.

5.9 Assessment

The code is quite compact thanks to the fact that the programming language is very high level and that its programming paradigm is a good fit for this application.

On the average each components is about 200 lines of code. Simpler derivations like *projK*, *isK* or even *param1P* are under 100 lines.

Debugging this kind of code did not pose particular difficulties. The typical error results in the generated term being ill-typed. In that case the Coq type checker could be used to identify the culprit. Given how small the derivations are, it was simple to identify the lines generating the offending subterm.

The time required to design and develop the entire procedure amounts to approximately six months, but spanned over more than one and a half year: most of the time has been spent improving the integration of Elpi in Coq in response to the experience gathered on this work. At the time of writing the Elpi integration in Coq does not support mutual inductive types, universe polymorphic definitions and primitive projections.

All derivations support polynomial types. Some derivations also support index data, e.g. *eq* is able to synthesize an equality test for vectors. Most of the derivations for contextual reasoning, such as *eqK* and *bcongr* do not support indexes.

6 Related work

Systems similar to Coq [20], e.g. Matita [2], Lean [5] and Isabelle [14] all generate induction principles automatically, with the exception of Agda [15], and some of them also the no confusion properties.

To our knowledge Isabelle is the only system that generates sensible induction principles and proved equality tests when containers are involved. As described in [4] the (co)datatype package is built on top of Bounded Natural Functors [21], a notion that makes the construction of (co)datatypes in Higher Order Logic compositional. Our starting point is very different since Coq, and type theory in general, internalizes the definitional mechanism for (co)datatypes. As a consequence a package like the one described in this paper cannot change it but only work around its eventual limitations. In particular the way Coq checks recursive functions for termination is a fixed, syntactic, non modular, criteria for which some alternatives have been studied (see for example [3, 16]) but never implemented. The non modular criteria applies to induction principles as well, since they are proved using recursion. It is a strength of the construction described in this paper to recover some modularity and hence be able to synthesize mechanically most of what [4] is able to synthesize.

Most Interactive Theorem Provers come with simple forms of Prolog-like automation, usually in the form of Type Classes. The user typically resorts to that in order to perform some of the inductive reasoning one needs in order to synthesize code in a type directed way. To our knowledge no ready-to-use package to synthesize equality tests and their proofs was written this way.

Some systems, notably Lean, come with a whole round meta programming framework. Still, to our knowledge, the primary application is the development of proof commands, not program/proof synthesis, in spite of the stunning similarity.

Coq provides two mechanisms strictly related to this work. The `Scheme Equality` command generates for a type τ the code for the equality test (`T_eqb`) and a proof that equality is decidable on τ . The proof internally uses the equality test, but its type does not:

```
T_eq_dec : ∀x y : T, {x = y} + {x <> y}
```

By unfolding the proof term, that is transparent, it should be possible to recover the fact that `T_eqb` is a correct equality test. Data types defined using containers are not supported. The `decide equality` tactic requires the user to start a lemma with a statement as the one depicted above. The tactic only performs one (case split) step and has to be iterated by hand. It does not remember which equalities were proved decidable before, it is up to the user to eventually share code. The proof term generated is, in a type theoretic sense, a program even if its code mixes the comparison test with its correctness proof. This proof is fully transparent, and inlines all the contextual reasoning steps such as injection and discrimination. As a result the term is very large and computationally heavy when run within Coq.

773 In the programming language world derivation is much more developed. The dominant
 774 approach is to provide some meta programming facilities, e.g. by providing a syntax to the
 775 declaration of types and then use the programming language itself to write derivations [17]
 776 that run at compile time as compiler plugins. Our approach is similar in a sense, since we
 777 work at the meta level on the syntax of types (and terms), but it is also very different since
 778 we pick a different programming language for meta programming. In particular we choose a
 779 very high level one that makes our derivations very concise and hides uninteresting details
 780 such as the representation of bound variables. The derivation described in the paper is the
 781 result of many failed attempts and we believe that the high level nature of the programming
 782 language we chose played an important role in the exploratory phase.

783 The link between the unary parametricity translation, also called predicate lifting, and
 784 induction principles was independently remarked by Kaposi and Kovács in [7].

785 7 Conclusion

786 We described a technique to derive stronger induction principles for Coq data types built
 787 using containers. We use the unary parametricity translation of a data type in order to
 788 fuel its induction principle, to thread an invariant on the contained when used as a con-
 789 tainer and finally to confine the modularity problems stemming from the termination check
 790 implemented in Coq. Finally we provide a Coq package deriving correct equality tests for
 791 polynomial inductive data types.

792 It is work in progress to extend the derivation to inductive types with decidable indexes.
 793 Preliminary work hints that indexes of base types such as `nat` pose no problem. On the
 794 contrary when indexes mention containers, that admit a decidable equality only if their
 795 contained does, the *param1P* component gets substantially more complex. In particular
 796 some notions of Homotopy Type Theory come in to play. For example the notion of being
 797 provable on the entire domain such as $(\forall a : A, P a) \rightarrow (\forall t : T A, is_T A P t)$ seems to
 798 require to be strengthened using the notion of contractibility (that is, the property should
 799 hold and its proof be unique), in order for the construction to compose well.

800 We also look forward to let the user tune the derivation process by annotating the type
 801 declarations. For example the user may want to skip certain arguments when generating
 802 the equality test, such as the integer describing the length of a sub vector in the `cons`
 803 constructor. The resulting equality test surely requires some user intervention in order to
 804 be proved correct, but it features a better computational complexity.

805 Finally, adding other derivations to the package seems appealing. For example the
 806 interface next to `eqType` in the hierarchy used in the Mathematical Component library is the
 807 one of countable types, i.e. types in bijection with natural numbers. The interface requires,
 808 roughly, a serialization function to another countable type, a tedious task that could be
 809 made automatic.

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