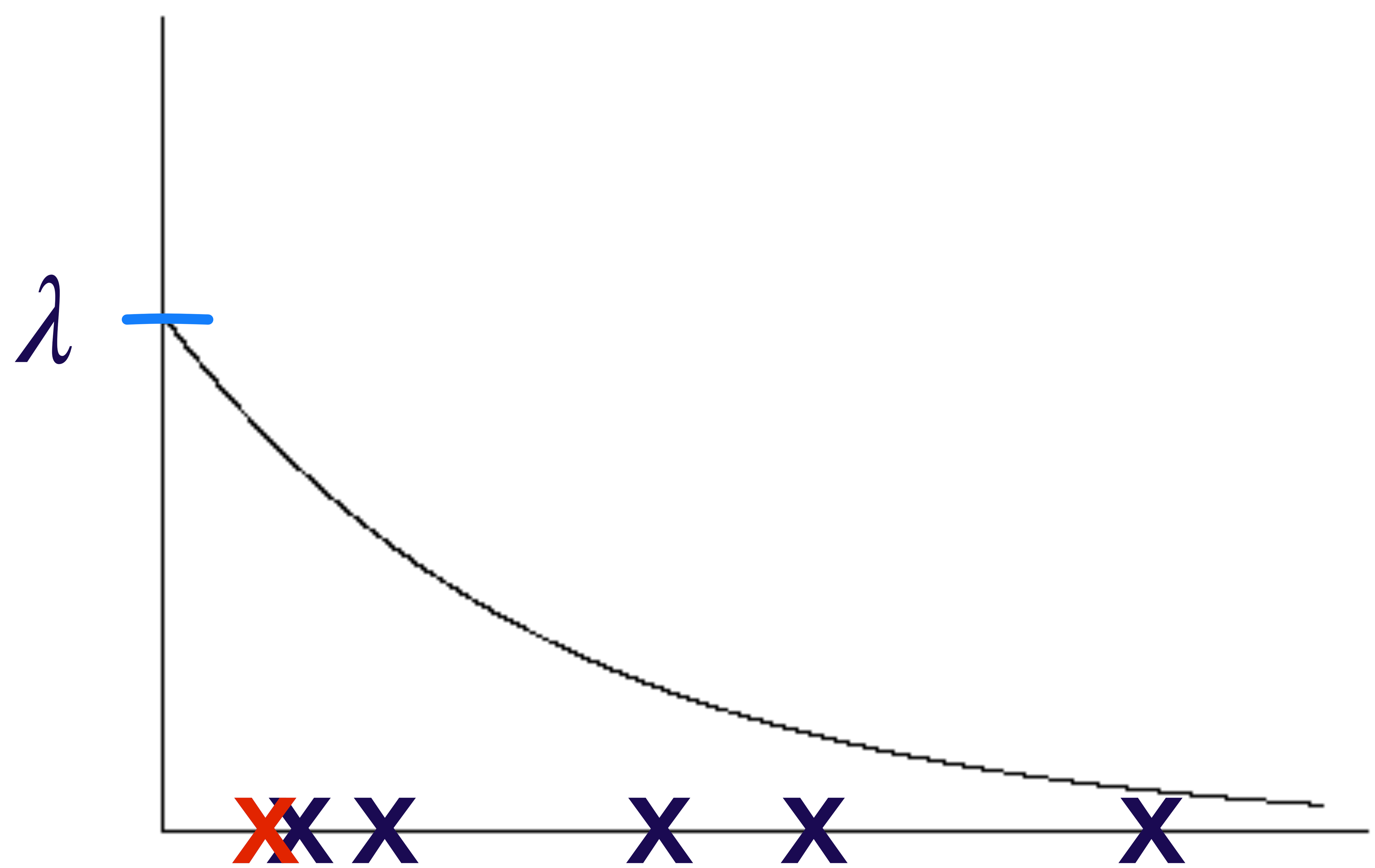
Suppose that $X_1, X_2, ..., X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.

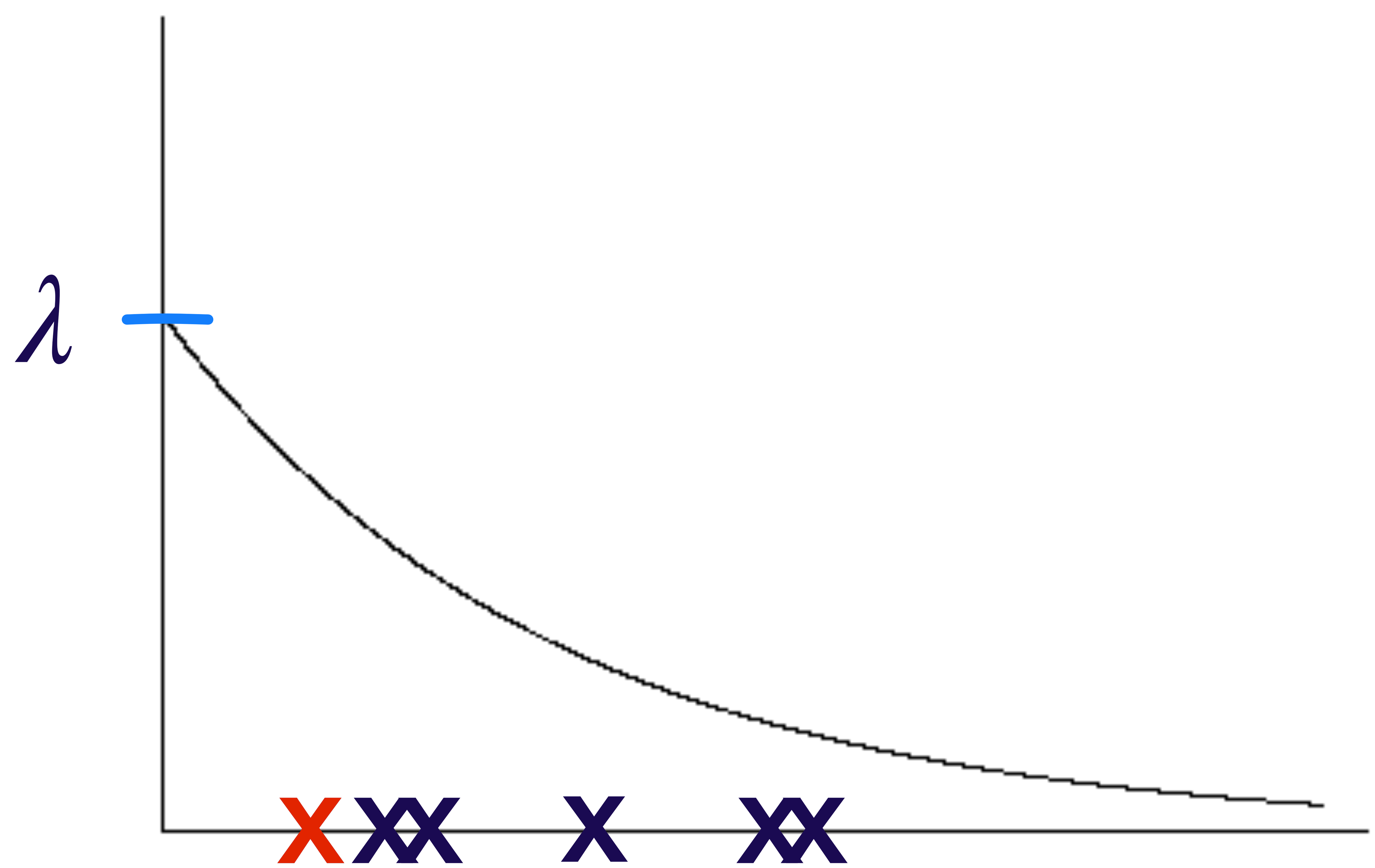
Construct a 95% confidence interval for λ based on the minimum value in the sample.

What is the distribution of the minimum?

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.



Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda>0$.



Let $Y_n = \min(X_1, X_2, \ldots, X_n)$.

The cdf for each X_i is

$$F(x) = P(X_i \leq x) = 1 - e^{-\lambda x}$$

The cdf for Y_n is

$$F_{Y_n}(y) = P(Y_n \leq y)$$

$$= P(\min(X_1, X_2, \ldots, X_n) \leq y)$$

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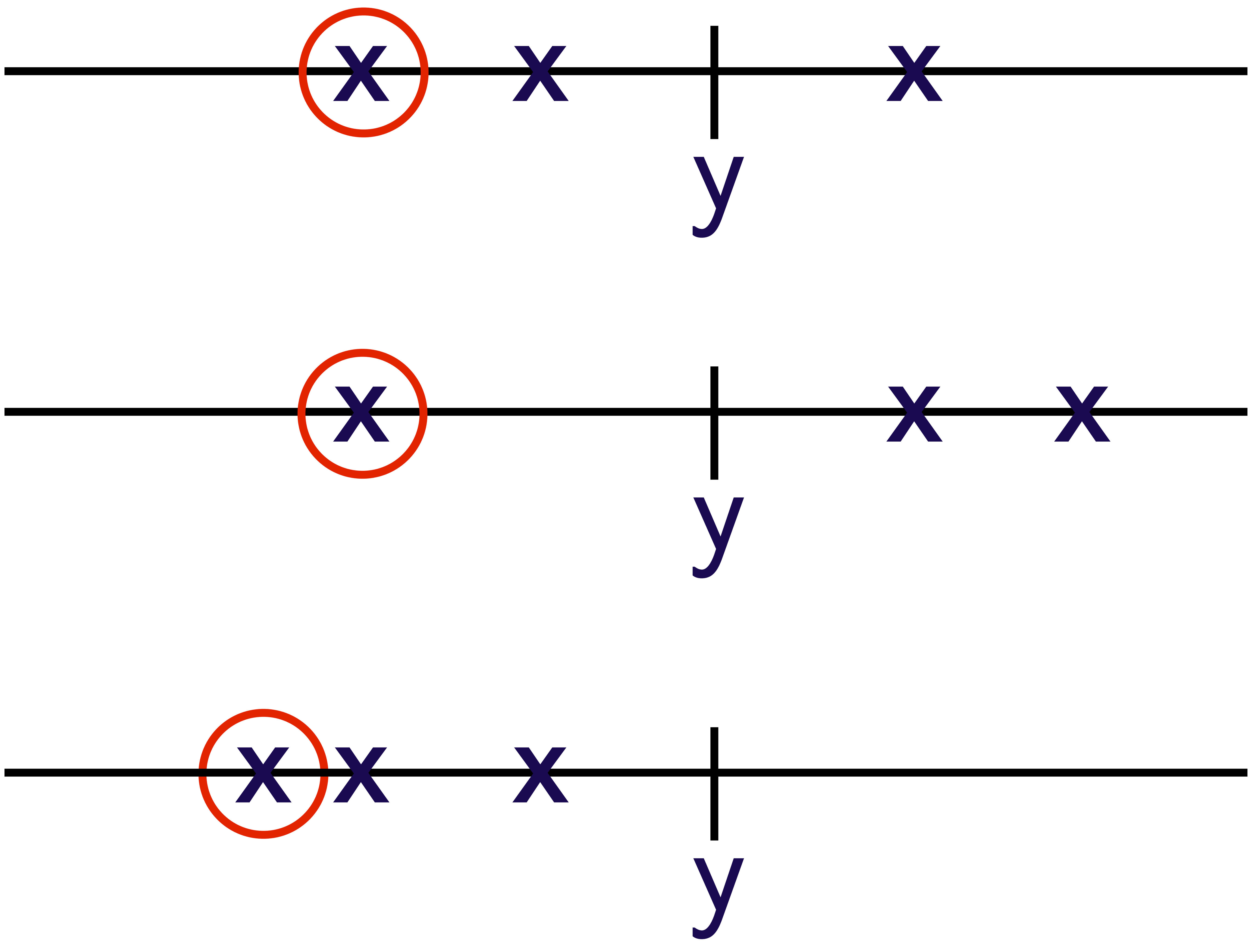
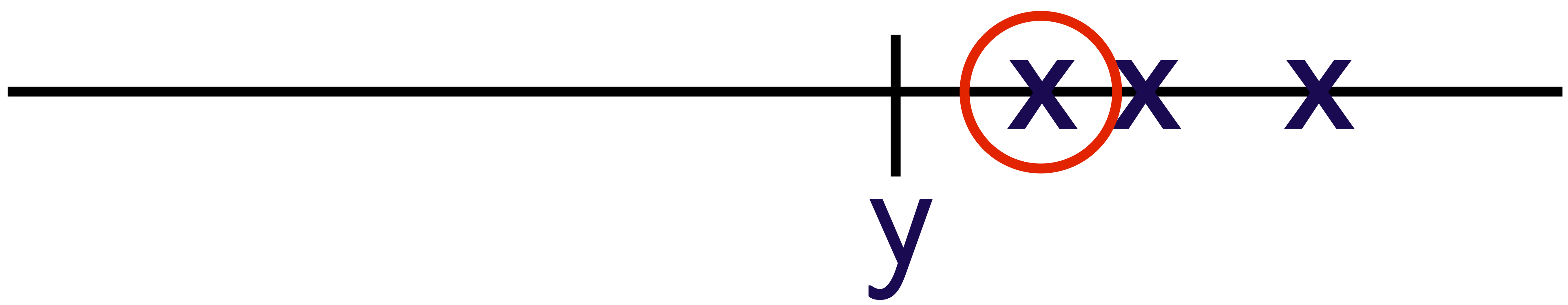


$$F_{Y_n}(y) = P(Y_n \le y)$$

$$= P(\min(X_1, X_2, \ldots, X_n) \le y)$$

$$= 1 - P(\min(X_1, X_2, \ldots, X_n) > y)$$



$$= 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y)$$

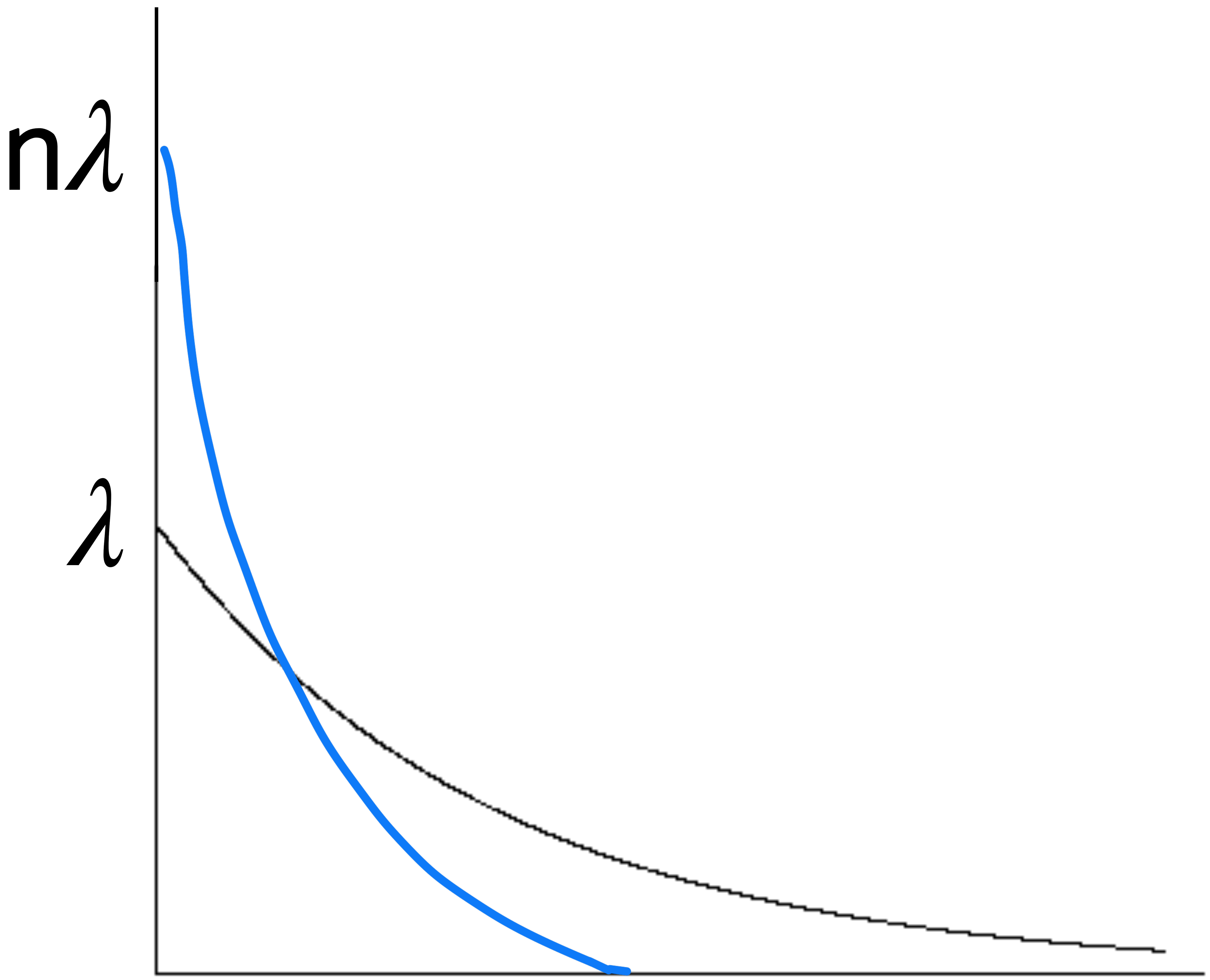
$$\overset{\text{indep}}{=} 1 - P(X_1 > y) \cdot P(X_2 > y) \cdots P(X_n > y)$$

$$\overset{\text{ident}}{=} 1 - [P(X_1 > y)]^n = 1 - [1 - F(y)]^n$$

$$
\begin{aligned}
F_{Y_n}(y) &= P(Y_n \le y) \\
&= 1 - [1 - F(y)]^n \\
&= 1 - [1 - (1 - e^{-\lambda y})]^n \\
&= 1 - [e^{-\lambda y}]^n \\
&= 1 - e^{-n\lambda y}
\end{aligned}
$$

$$
f_{Y_n}(y) = \frac{d}{dy} F_{Y_n}(y) = n\lambda e^{-n\lambda y}
$$

The minimum of n iid exponential with rate λ is exponential with rate nλ !



Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda>0$.

Construct a 95% confidence interval for λ based on the minimum value in the sample.

Step One: Choose a statistic.

$$Y_n = \min(X_1, X_2, \ldots, X_n)$$

Step Two: Find a function of the statistic and the parameter you are trying to estimate whose distribution is known and parameter free.

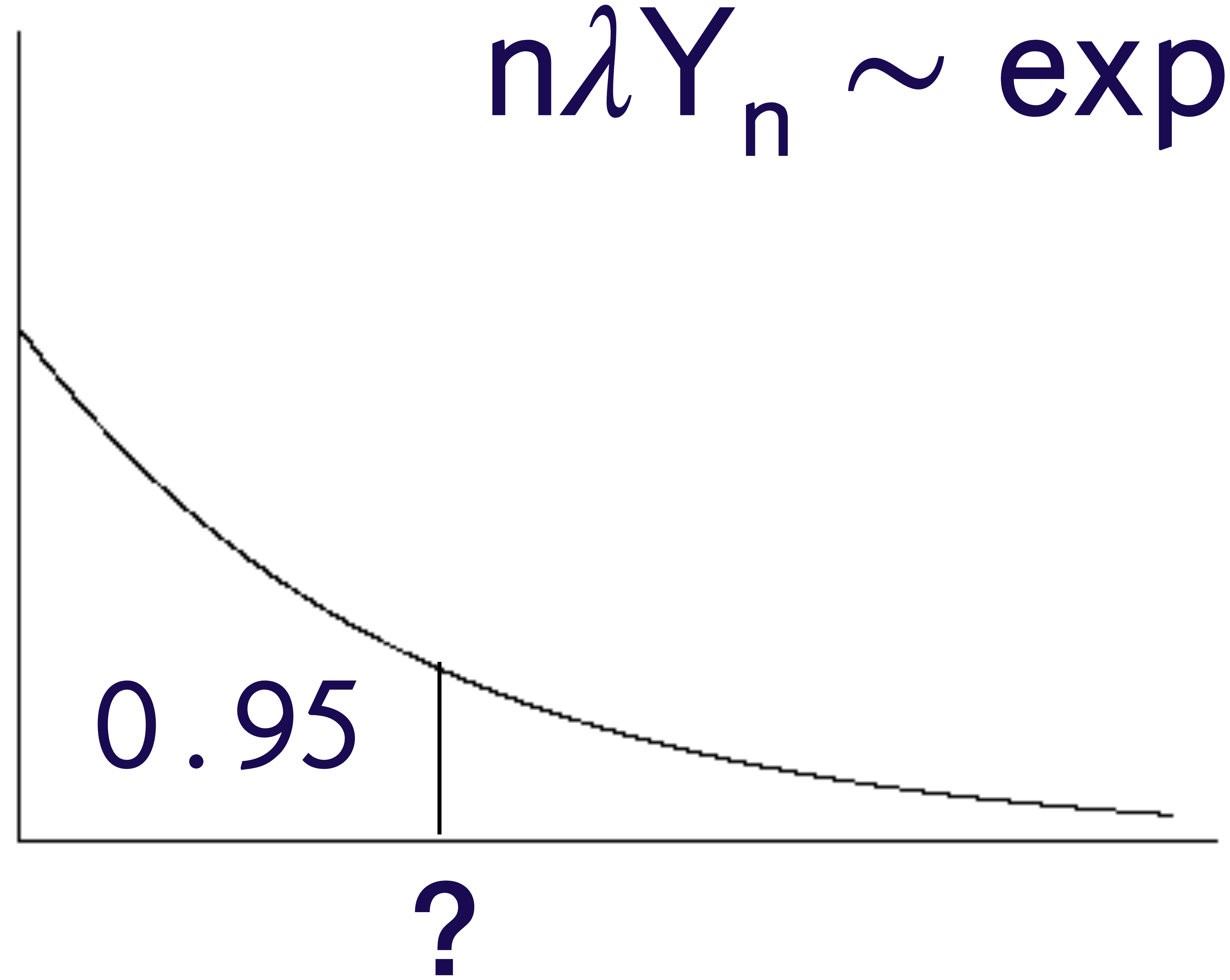
$$Y_n = \min(X_1, X_2, \ldots, X_n) \sim \exp(\text{rate} = n\lambda)$$

$$= \Gamma(1, n\lambda)$$

$$\Rightarrow \quad n\lambda Y_n \sim \Gamma(1, 1) = \exp(\text{rate} = 1)$$

Step Three: Find appropriate critical values.

$$n\lambda Y_n \sim \exp(\text{rate} = 1)$$



$$\int_0^{?} e^{-x}\,dx = 0.95$$

$$\int_0^{?} e^{-x}\,dx = 0.95$$

$$\Downarrow$$

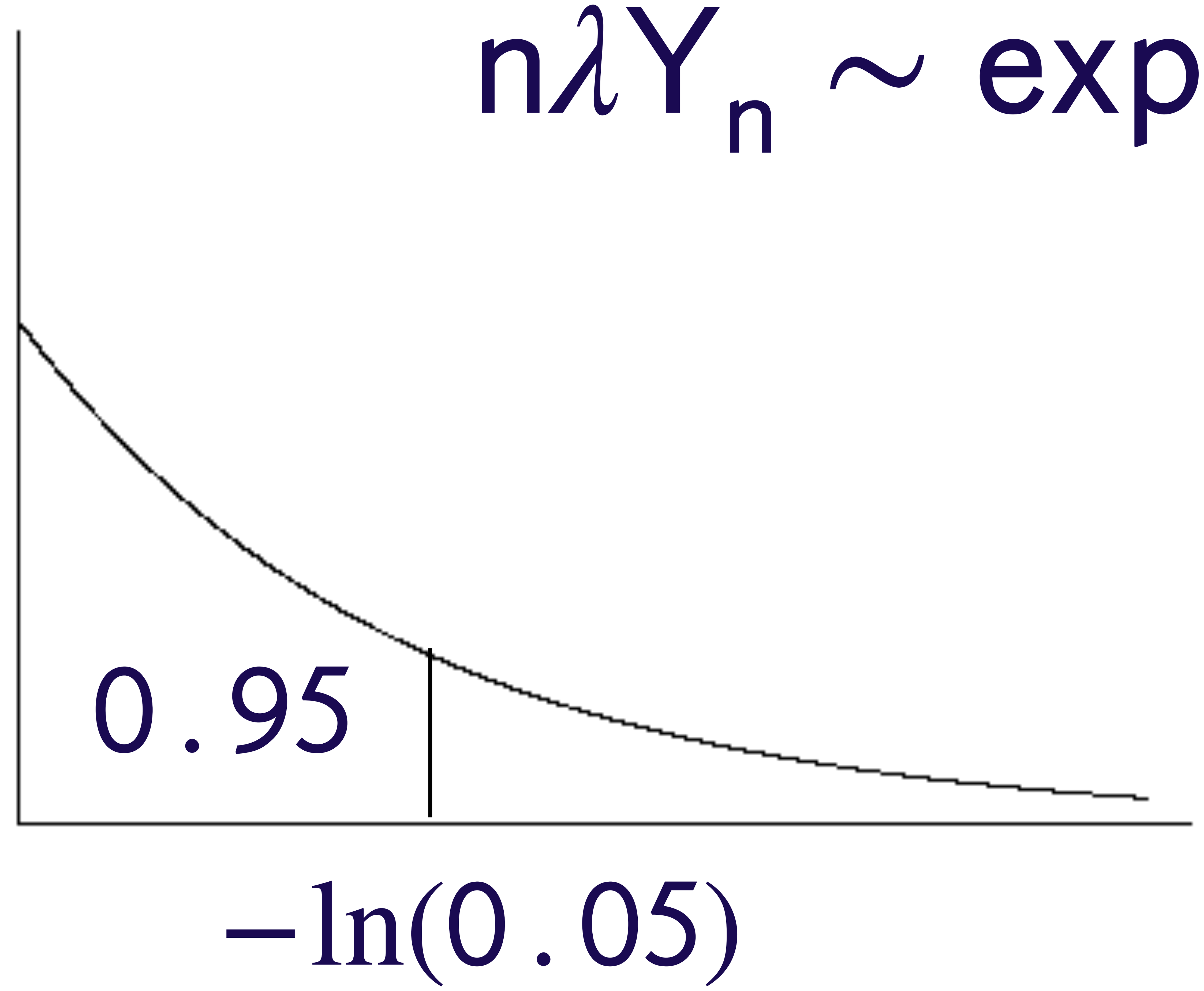
$$1 - e^{-?} = 0.95$$

$$\Downarrow$$

$$? = -\ln(0.05)$$

Step Three: Find appropriate critical values.

$$n\lambda Y_n \sim \exp(\text{rate} = 1)$$



Step Four: Put your statistic from Step Two between the critical values and solve for the unknown parameter "in the middle".

$$0 < n\lambda Y_n < -\ln(0.05)$$

$$\Downarrow$$

$$\left(0, \frac{-\ln(0.05)}{nY_n}\right)$$

where $Y_n = \min(X_1, X_2, \ldots, X_n)$.