

The Gamma Distribution

The continuous random variable X is said to have a “gamma distribution” with parameters α and β if X has pdf

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \mathbf{I}_{(0,\infty)}(x)$$

for $\alpha > 0$ and $\beta > 0$.

We write $X \sim \Gamma(\alpha, \beta)$

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β is an “inverse scale parameter”

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \mathbf{I}_{(0,\infty)}(x)$$

Important Note:

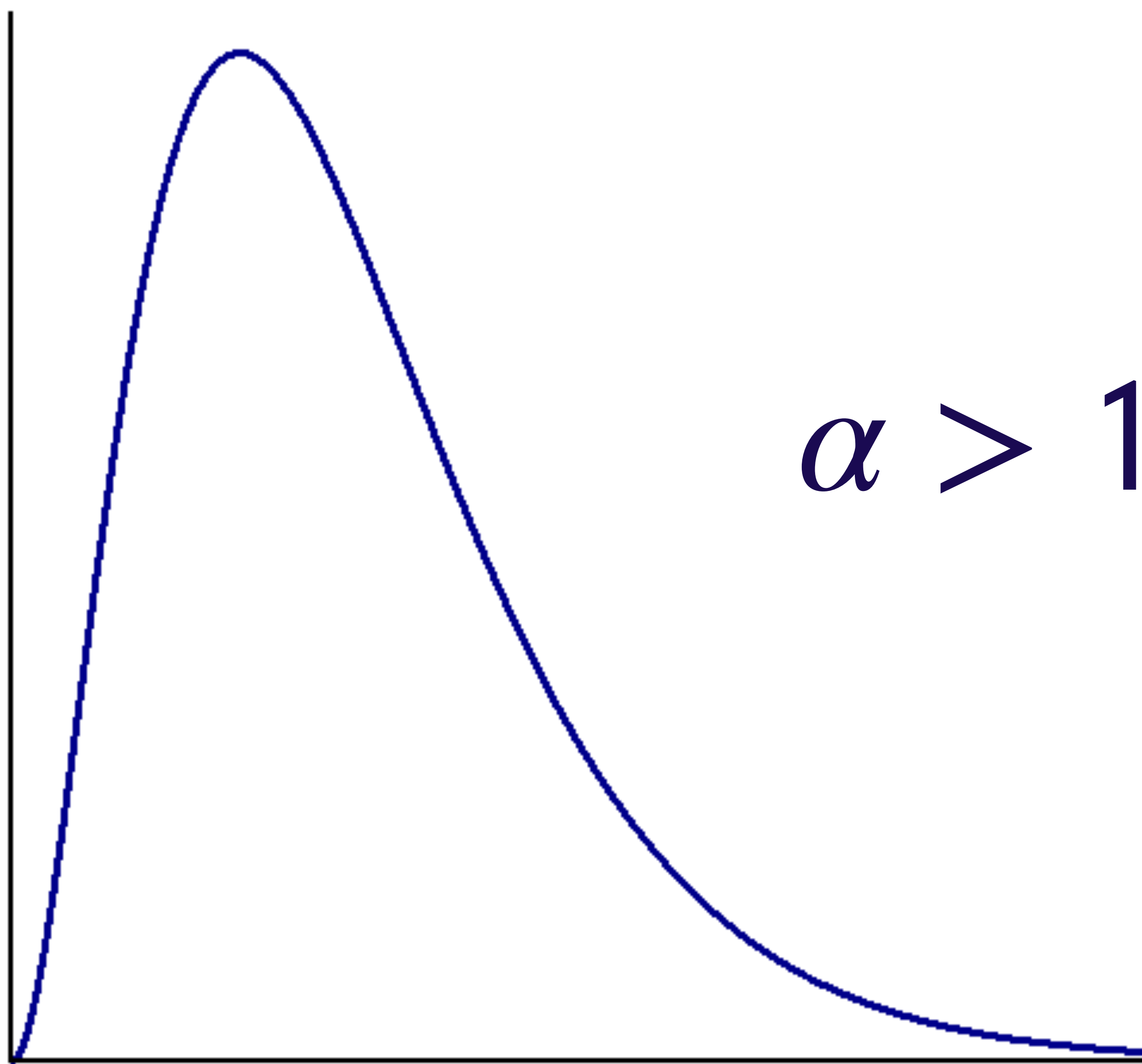
β is a “scale parameter”

Some people use

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \mathbf{I}_{(0,\infty)}(x)$$

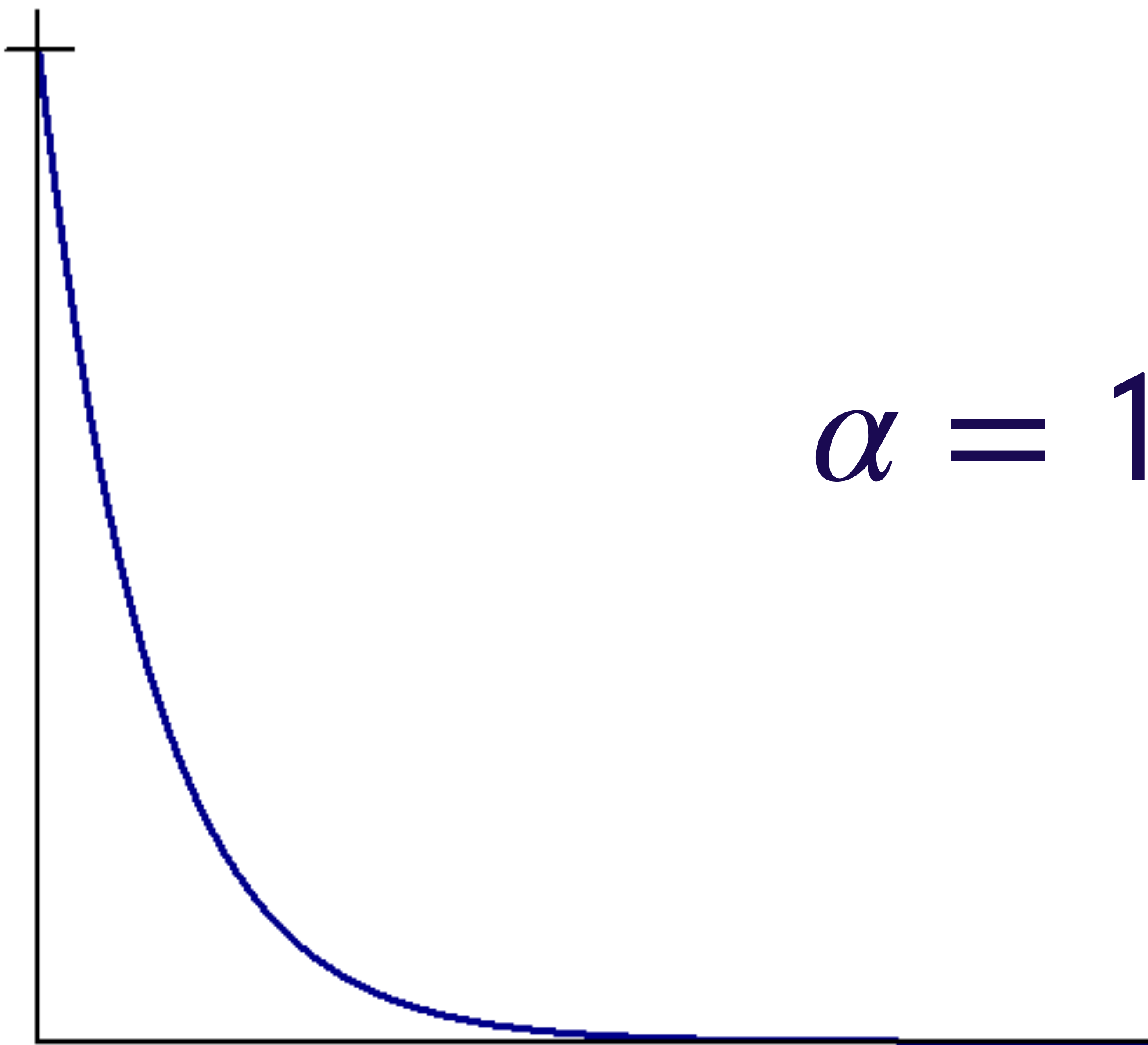
α is a “shape parameter”

$$x^{\alpha-1} e^{-\beta x}$$



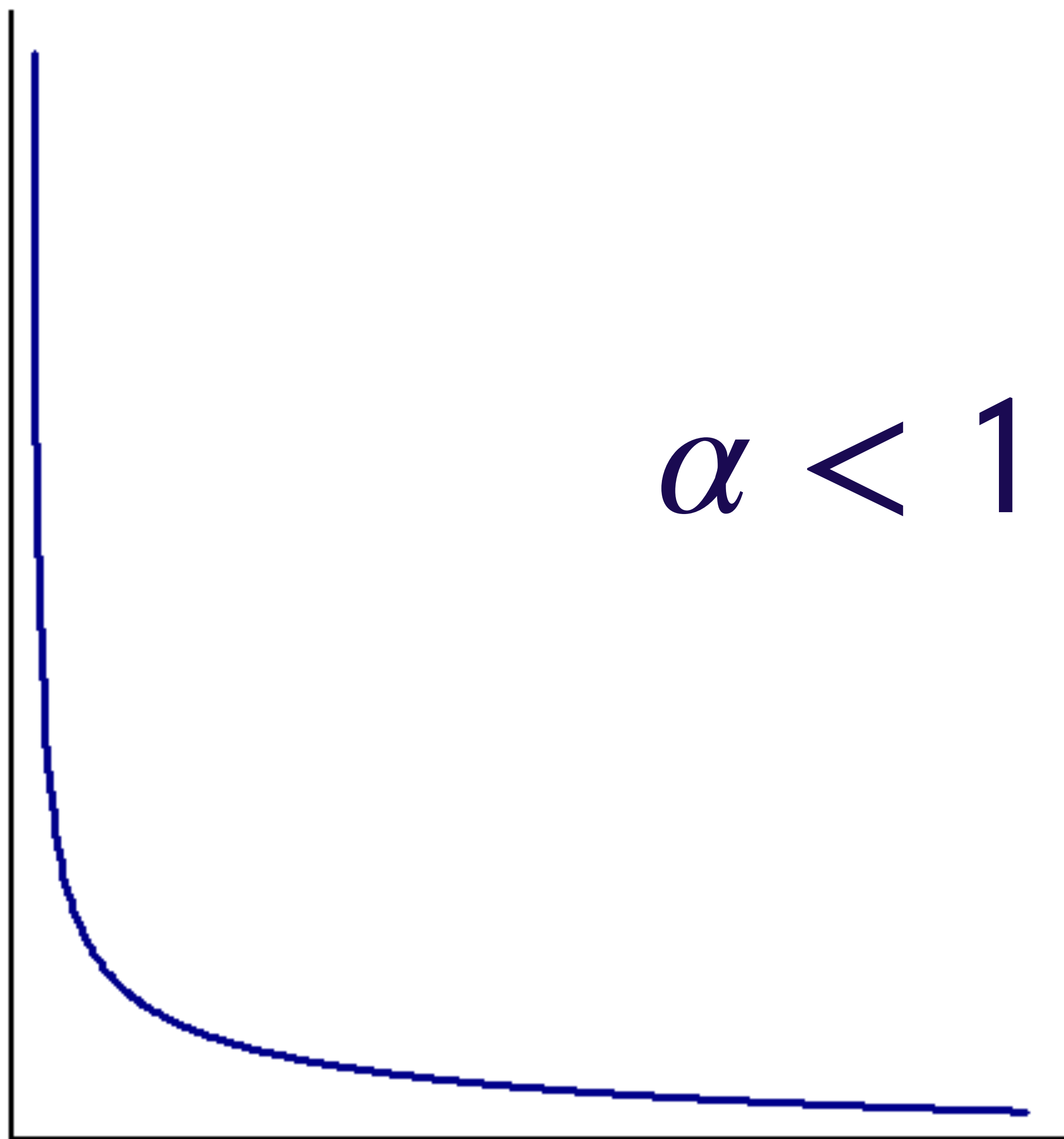
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The Gamma Function

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \mathbf{I}_{(0,\infty)}(x)$$

Different from $\Gamma(\alpha, \beta)$!

The Gamma Function

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \mathbf{I}_{(0,\infty)}(x)$$

Defined as:

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x} dx \\ &= \int_0^\infty \beta^\alpha x^{\alpha-1} e^{-\beta x} dx \\ &\quad (u = \beta x) \end{aligned}$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

- $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = 1$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

- $\Gamma(\alpha) = (\alpha - 1) \Gamma(\alpha - 1)$

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Integration
by parts:

$$\int u dv = uv - \int u dv$$

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$$\Gamma(\alpha) = \int_0^{\infty} \underbrace{x^{\alpha-1}}_u e^{-x} dx$$

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$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \underbrace{e^{-x} dx}_{dv}$$

Integration
by parts:

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Properties of the Gamma Function

$$\int u \, dv = uv - \int v \, du$$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$$

$$= -x^{\alpha-1} e^{-x} \Big|_0^{\infty} + (\alpha - 1) \int_0^{\infty} x^{\alpha-2} e^{-x} \, dx$$

$$= 0 + (\alpha - 1) \Gamma(\alpha - 1)$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

- $\Gamma(n) = (n-1)!$

$$\begin{aligned}\Gamma(n) &= (n-1) \Gamma(n-1) \\ &= (n-1) (n-2) \Gamma(n-2) \\ &\quad \vdots \\ &= (n-1) (n-2) \cdots 2 \cdot \Gamma(1)\end{aligned}$$

Fun Fact:

If you have $X \sim \Gamma(\alpha, \beta)$

and a constant $c > 0$

and define a new random variable

$$Y = cX$$

then

$$Y \sim \Gamma(\alpha, \beta/c)$$

A New Continuous Distribution

Suppose that

$$X \sim \Gamma(n/2, 1/2)$$

We say that X has a

“chi-squared distribution”

with n “degrees of freedom”.

We write

$$X \sim \chi^2(n)$$

Turning a gamma into a chi-squared:

Suppose that

$$X \sim \Gamma(n, \beta)$$

Then

$$2\beta X \sim \Gamma(n, 1/2)$$

$$= \Gamma(2n/2, 1/2)$$

$$= \chi^2(2n)$$