

Suppose that

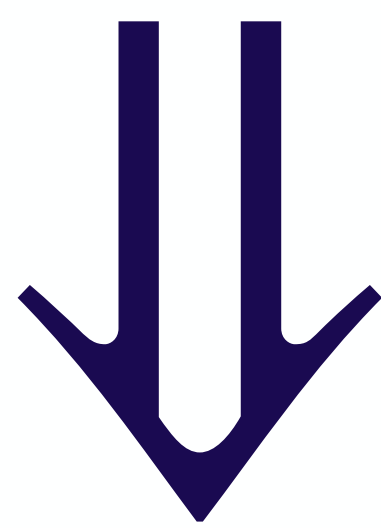
$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Question:

What is the distribution of  $Y = \sum_{i=1}^n X_i$  ?

- Each  $X_i$  takes on the value 1 (“success”) with probability  $p$  and 0 (“failure”) with probability  $1-p$ .
- Summing them up will give the total number of 1’s which is the total number of “S”s in  $n$  trials.

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$$



$$Y = \sum_{i=1}^n X_i \sim \text{bin}(n, p)$$

Not all random variables are so easily interpreted...

## Moment Generating Functions:

Let  $X$  be a random variable.

It's **moment generating function** (mgf) is denoted and defined as

$$M_X(t) = E[e^{tX}]$$

It can be used to produce “moments” for  $X$ :  $E[X]$ ,  $E[X^2]$ ,  $E[X^3]$ , ...

Moment generating functions also uniquely identify distributions.

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

---

Example:  $X \sim \text{Bernoulli}(p)$

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_x e^{tx} f_X(x) \\ &= \sum_x e^{tx} P(X = x) \end{aligned}$$

$$\begin{aligned} M_X(t) &= \sum_x e^{tx} P(X = x) \\ &= e^{t \cdot 0} P(X = 0) + e^{t \cdot 1} P(X = 1) \\ &= 1 \cdot (1 - p) + e^t \cdot p \\ &= 1 - p + pe^t \end{aligned}$$

---

Example:  $X \sim \text{bin}(n, p)$

$$\begin{aligned} M_X(t) &= \sum_x e^{tx} P(X = x) \\ &= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1 - p)^{n-x} \end{aligned}$$

Example:  $X \sim \text{bin}(n, p)$

$$\begin{aligned} M_X(t) &= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} \end{aligned}$$

---

Binomial Theorem:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

---

$$\Rightarrow M_X(t) = (pe^t + 1 - p)^n$$

Suppose that  $X_1, X_2, \dots, X_n$  is a random sample from some distribution with mgf  $M_X(t)$ .

$$\text{Let } Y = \sum_{i=1}^n X_i$$

The mgf for  $Y$  is

$$M_Y(t) = E[e^{tY}] = E\left[e^{t\sum_{i=1}^n X_i}\right]$$

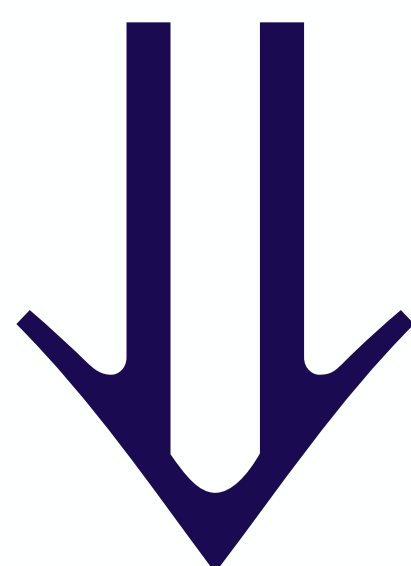
$$= E\left[\prod_{i=1}^n e^{tX_i}\right] = \prod_{i=1}^n E[e^{tX_i}] = \left(M_{X_1}(t)\right)^n$$

*by independence*

*all the same*

$X_1, X_2, \dots, X_n$  iid

$$Y = \sum_{i=1}^n X_i$$



$$M_Y(t) = [M_{X_1}(t)]^n$$



Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Find the distribution of  $Y = \sum_{i=1}^n X_i$ .

$$M_{X_1}(t) = 1 - p + pe^t$$

$$M_Y(t) \stackrel{\text{iid}}{=} [M_{X_1}(t)]^n = [1 - p + pe^t]^n$$

but this is the mgf of the bin(n,p)  
distribution...

and this is enough to say that

$$Y \sim \text{bin}(n, p)$$

Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{exp}(\text{rate} = \lambda)$$

Find the mgf for

$$Y = \sum_{i=1}^n X_i$$

$$M_{X_1}(t) = E[e^{tX_1}] = \int_{-\infty}^{\infty} e^{tx} f_{X_1}(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \int_0^{\infty} \underbrace{\lambda e^{-(\lambda-t)x}}_{\text{like an exp(rate}=\lambda-t)\text{ pdf}} dx$$

$$= \frac{\lambda}{\lambda - t} \underbrace{\int_0^{\infty} (\lambda - t) e^{-(\lambda-t)x} dx}_{= 1} = \frac{\lambda}{\lambda - t} \quad \text{for } \lambda < t$$

## Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

$$M_{X_1}(t) = E[e^{tX_1}] = \int_{-\infty}^{\infty} e^{tx} f_{X_1}(x) dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{1}{\Gamma(\alpha)} \beta^{\alpha} \int_0^{\infty} \underbrace{x^{\alpha-1} e^{-(\beta-t)x}}_{\text{like a } \Gamma(\alpha, \beta-t) \text{ pdf}} dx$$

$$= \frac{\beta^{\alpha}}{(\beta-t)^{\alpha}} \int_0^{\infty} \frac{1}{\Gamma(\alpha)} (\beta-t)^{\alpha} x^{\alpha-1} e^{-(\beta-t)x} dx$$

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

$$\Rightarrow M_{X_1}(t) = \left( \frac{\beta}{\beta - t} \right)^\alpha$$

## Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \exp(\text{rate} = \lambda)$$

Find the distribution of  $Y = \sum_{i=1}^n X_i$ .

$$M_{X_1}(t) = \frac{\lambda}{\lambda - t}$$

$$M_Y(t) \stackrel{\text{iid}}{=} [M_{X_1}(t)]^n = \left( \frac{\lambda}{\lambda - t} \right)^n$$

This is the mgf for the  $\Gamma(n, \lambda)$  distribution.

The sum of  $n$  iid exponential rate  $\lambda$  random variables has a gamma distribution with parameters  $n$  and  $\lambda$ .

- sum of  $n$  iid Bernoulli( $p$ ) random variables is  $\text{bin}(n, p)$
- sum of  $n$  iid  $\text{exp}(\text{rate}=\lambda)$  random variables is  $\Gamma(n, \lambda)$
- sum of  $m$  iid  $\text{bin}(n, p)$  is  $\text{bin}(nm, p)$
- sum of  $n$  iid  $\Gamma(\alpha, \beta)$  is  $\Gamma(n\alpha, \beta)$

- sum of  $n$  iid  $N(\mu, \sigma^2)$  is  $N(n\mu, n\sigma^2)$
- sum of  $n$  independent normal random variable with  $X_i \sim N(\mu_i, \sigma_i^2)$  is  $N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
- $X_1, X_2, \dots, X_n$  independent with  $X_i \sim N(\mu_i, \sigma_i^2)$ .

$$Y = \sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$