More Maximum Likelihood Estimation!

Special Cases in this video:

- The Invariance Property of MLEs!!!

Suppose you have a random sample from a distribution with mean μ and variance σ^2.

- An unbiased estimator of μ is $\overline{X}$.
- If we wanted to estimate μ^2, Should we use $\overline{X}^2$?

Not if we want an unbiased estimator!

$$E[\overline{X}^2] = Var[\overline{X}] + (E[\overline{X}])^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

The Invariance Property of MLEs

$$X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} \exp(\text{rate} = \lambda)$$

Find the MLE of λ^2.

Let $\tau = \lambda^2$.

Reparameterize the pdf:

$$f(x;\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$$

Becomes:

$$f_2(x;\tau) = \sqrt{\tau} e^{-\sqrt{\tau} x} I_{(0,\infty)}(x)$$

The problem is now:

Let $X_1, X_2, \ldots, X_n$ be iid

from a distribution with pdf:

$$f_2(x;\tau) = \sqrt{\tau}e^{-\sqrt{\tau}x}\,I_{(0,\infty)}(x)$$

Now this is a "straightforward" MLE problem!

You will get

$$\hat{\tau} = \frac{1}{\overline{X}^2}$$

The Invariance Property of MLEs

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \lambda)$$

Find the MLE of λ^2.

Let $\tau = \lambda^2$.

The MLE for λ is $\hat{\lambda} = \dfrac{1}{\overline{X}}$

The MLE for τ is $\hat{\tau} = \dfrac{1}{\overline{X}^2} = \hat{\lambda}^2$

This is not a coincidence!

The Invariance Property of MLEs

$$f(x;\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$$

$$f(\vec{x};\lambda) \overset{iid}{=} \prod_{i=1}^{n} f(x_i,\lambda)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} I_{(0,\infty)}(x_i)$$

$$= \boxed{\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}} \prod_{i=1}^{n} I_{(0,\infty)}(x_i)$$

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

If $\tau = \tau(\lambda) = \lambda^2$, we can rewrite this

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} = \underbrace{(\sqrt{\tau})^n e^{-\sqrt{\tau} \sum_{i=1}^{n} x_i}}_{L(\tau)\,?}$$

$$L(\tau) = \tau^n e^{-\tau \sum_{i=1}^{n} x_i}$$

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

If $\tau = \tau(\lambda) = \lambda^2$, we can rewrite this

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} = (\sqrt{\tau})^n e^{-\sqrt{\tau} \sum_{i=1}^{n} x_i}$$

$$= \widetilde{L}(\tau)$$

$$= \widetilde{L}(\tau(\lambda))$$

Proof of invariance when τ is invertible and L has a unique maximum:

$$\mathrm{L}(\theta) = \widetilde{\mathrm{L}}(\tau(\theta))$$

$$\tau \text{ invertible} \Rightarrow \tau'(\theta) > 0 \text{ or } \tau'(\theta) < 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{L}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\widetilde{\mathrm{L}}(\tau(\theta)) = \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathrm{L}}(\tau(\theta)) \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\tau(\theta)$$

$$\frac{d}{d\theta} L(\boldsymbol{\theta}) = \frac{d}{d\tau} \widetilde{L}(\tau(\boldsymbol{\theta})) \cdot \underbrace{\frac{d}{d\theta} \tau(\boldsymbol{\theta})}_{\text{never 0 since } \tau \text{ is invertible}}$$

$\frac{d}{d\theta} L(\boldsymbol{\theta})$ is zero when we plug in $\hat{\boldsymbol{\theta}}$ by definition of the MLE

This implies that we must have

$$\frac{d}{d\tau} \widetilde{L}(\tau(\hat{\boldsymbol{\theta}})) = 0$$

$$\frac{d}{d\tau}\tilde{L}(\tau(\hat{\theta})) = 0$$

However,

$\frac{d}{d\tau}\tilde{L}(\tau(\theta))$ is zero when we plug in $\hat{\tau}(\theta)$ by definition of the MLE for $\tau(\theta)$

So $$\frac{d}{d\tau}\tilde{L}(\tau(\hat{\theta})) = 0 = \frac{d}{d\tau}\tilde{L}(\hat{\tau}(\theta))$$

$$\frac{d}{d\tau}\widetilde{L}(\tau(\widehat{\theta})) = 0 = \frac{d}{d\tau}\widetilde{L}(\widehat{\tau}(\theta))$$

L has a unique maximum

$$\Rightarrow \widetilde{L} \text{ has a unique maximum}$$

$$\Rightarrow \quad \tau(\widehat{\theta}) = \widehat{\tau}(\theta)$$

The MLE of $\tau(\theta)$ is $\tau(\theta)$ with the MLE of θ plugged in!

Example:

$$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$$

How can we estimate the probability that a typical measurement from this data set is greater than zero?

i.e. How can we estimate

$$p = P(X_i > 0) \text{ ?}$$

One Answer:

$$\hat{p} = \frac{\text{\# values in the sample that are >0}}{n}$$

Example:

$$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$$

How can we estimate the probability that a typical measurement from this data set is greater than zero?

Can we do this more formally with a maximum likelihood estimator?

$$P(X_i > 0) = 1 - P(X_i = 0)$$

$$= 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

$\tau(\lambda)$

$$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$$

The pdf is:

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} I_{\{0,1,2,\ldots\}}(x)$$

The joint pdf is:

$$f(\vec{x};\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} I_{\{0,1,2,\ldots\}}(x_i)$$

$$= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\cancel{\prod_{i=1}^{n}(x_i!)}} \cancel{\prod_{i=1}^{n} I_{\{0,1,2,\ldots\}}(x_i)}$$

$$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$$

A likelihood is:

$$L(\lambda) = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}$$

The log-likelihood is:

$$\ell(\lambda) = -n\lambda + (\sum_{i}^{n} x_i) \ln \lambda$$

$$\frac{d}{d\lambda}\ell(\lambda) = 0 \quad \Rightarrow \quad \hat{\lambda} = \overline{X}$$

$$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$$

The MLE for λ is $\widehat{\lambda} = \overline{X}$

By the invariance property of MLEs, the MLE for $p = \tau(\lambda) = 1 - e^{-\lambda}$ is

$$\widehat{\tau}(\lambda) \overset{\text{invar}}{=} \tau(\widehat{\lambda}) = 1 - e^{-\overline{X}}$$

$$\hat{p}_1 = \frac{\text{\# values in the sample that are >0}}{n}$$

$$\hat{p}_2 = 1 - e^{-\overline{X}}$$

Which is better?

$$\hat{p}_1 = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i>0\}}$$

$$E[\hat{p}_1] = \frac{1}{n}\sum_{i=1}^{n} E[I_{\{X_i>0\}}] = E[I_{\{X_1>0\}}]$$

$$E[I_{\{X_1>0\}}] = 0 \cdot P(I_{\{X_i>0\}} = 0) + 1 \cdot P(I_{\{X_i>0\}} = 1)$$

$$= P(I_{\{X_i>0\}} = 1)$$

$$= P(X_i > 0) = p = 1 - e^{-\lambda}$$

$$E[\hat{p}_1] = E[I_{\{X_1>0\}}] = p = 1 - e^{-\lambda}$$

$\hat{p}_1$ **is an unbiased estimator of p.**

$$E[\hat{p}_2] = E[1 - e^{-\overline{X}}]$$

$$= 1 - E[e^{-\overline{X}}]$$

?

- **Method One:**

$$E[e^{-\overline{X}}] = \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} e^{-\frac{1}{n}\sum x_i} \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod (x_i!)}$$

No thanks...

$$E[\hat{p}_2] = 1 - E[e^{-\overline{X}}] = 1 - E[e^{-Y/n}]$$

- **Method Two:**

$$\text{Let } Y = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$$

Use moment generating functions, we can show that $Y \sim \text{Poisson}(n\lambda)$

$$E[e^{-\overline{X}}] = E[e^{-\frac{1}{n}Y}] = \sum_{y=0}^{\infty} e^{-\frac{1}{n}y} \frac{e^{-n\lambda}(n\lambda)^y}{y\,!}$$

No thanks...

$$E[\hat{p}_2] = E[1 - e^{-\overline{X}}] = 1 - E[e^{-\overline{X}}]$$

- **Method Three:**

Find the distribution of $\overline{X}$.

Let $W = \overline{X}$.

$$P(W = w) = P(\overline{X} = w) = P\left(\frac{1}{n}Y = w\right)$$

$$= P(Y = nw) = \frac{e^{-n\lambda}(n\lambda)^{nw}}{(nw)!}$$

$$E[e^{-\overline{X}}] = E[e^{-W}] = \sum_{?} e^{w} \cdot P(W = w)$$

No thanks... **0, 1/n, 2/n, 3/n,...**

$$E[\hat{p}_2] = E[1 - e^{-\overline{X}}] = 1 - E[e^{-\overline{X}}]$$

- **Method Four:**

Let $Y = \sum_{i=1}^{n} X_i$.

- **We know that** $Y \sim \text{Poisson}(n\lambda)$.
- **So, we know the mgf for Y is**

$$M_Y(t) = \exp[n\lambda(e^t - 1)]$$

Now

$$E[e^{-\overline{X}}] = E[e^{-\frac{1}{n}Y}] = M_Y\left(-\frac{1}{n}\right)$$

$$E[e^{-\overline{X}}] = E[-\frac{1}{n}Y] = M_Y\left(-\frac{1}{n}\right)$$

$$= \exp[n\lambda(e^{-1/n} - 1)]$$

So

$$E[\hat{p}_2] = E[1 - e^{-\overline{X}}]$$

$$= 1 - \exp[n\lambda(e^{-1/n} - 1)]$$

$$E[e^{-\overline{X}}] = E[-\frac{1}{n}Y] = M_Y\left(-\frac{1}{n}\right)$$

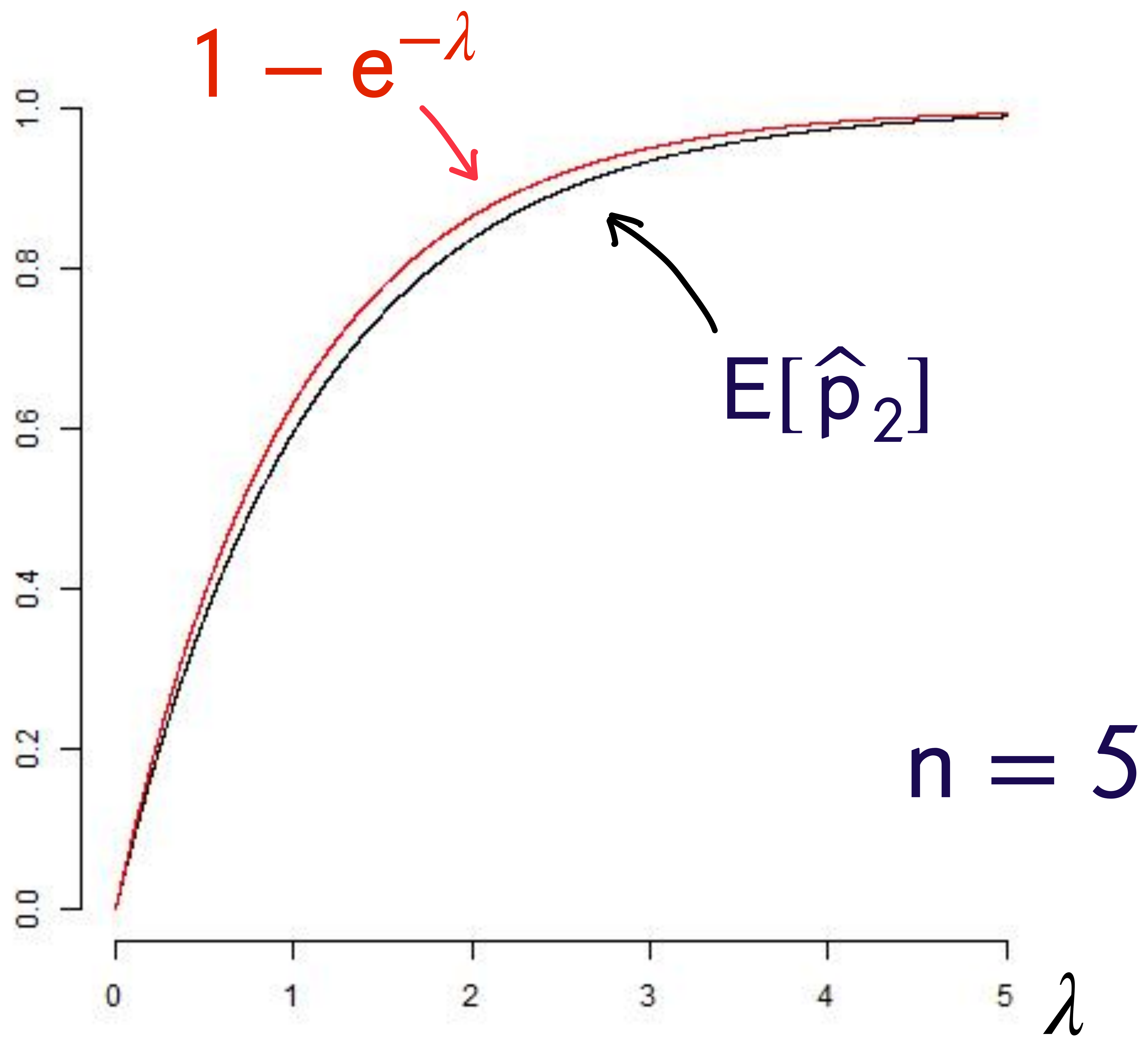
$$= \exp[n\lambda(e^{-1/n} - 1)]$$

So

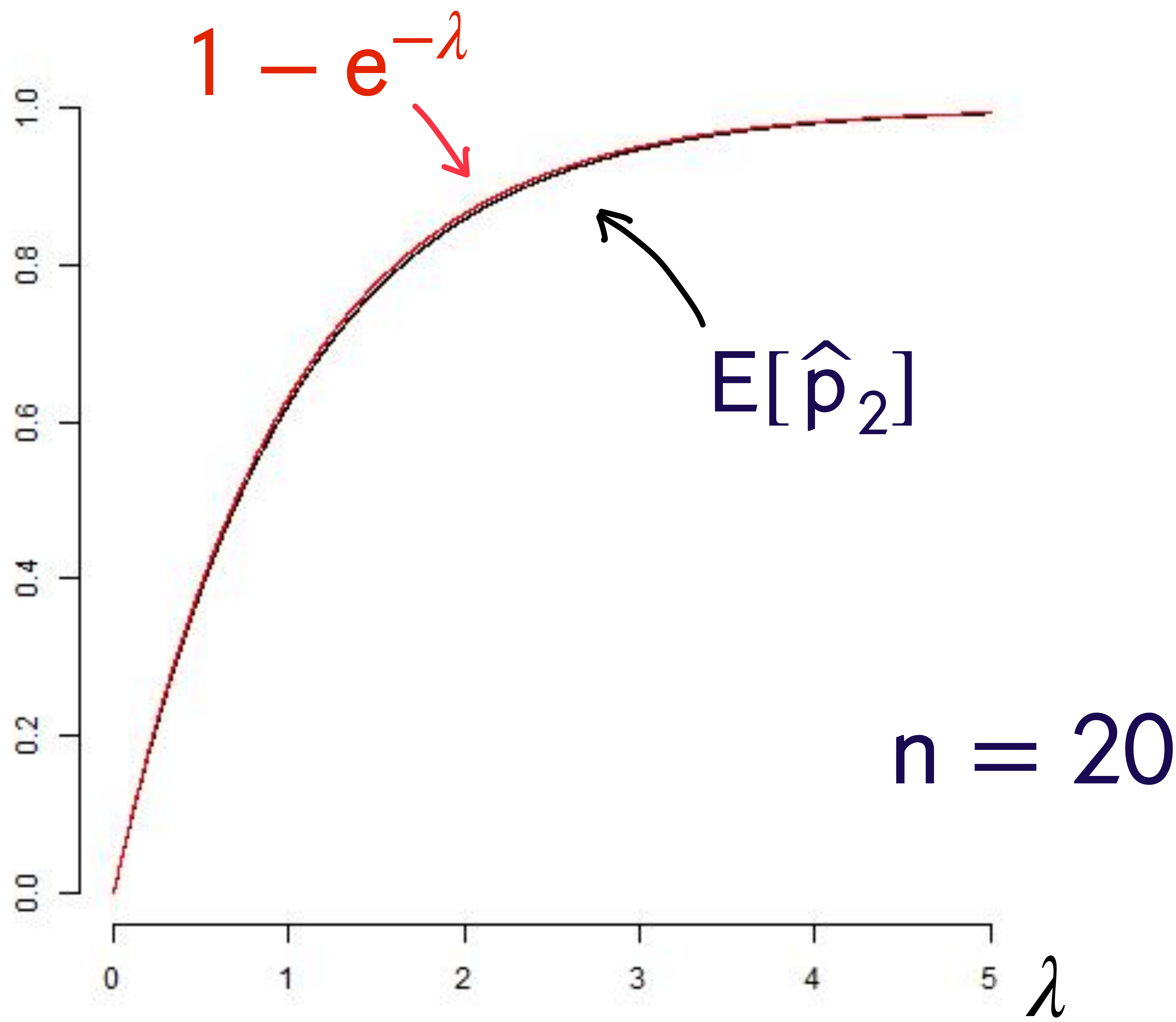
$$E[\hat{p}_2] = E[1 - e^{-\overline{X}}]$$

$$= 1 - \exp[n\lambda(e^{-1/n} - 1)]$$

We want this to be 1-$e^{-\lambda}$.



$1 - e^{-\lambda}$
$E[\hat{p}_2]$
$n = 5$
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
λ



$1 - e^{-\lambda}$
$E[\hat{p}_2]$
$n = 20$
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
λ

In summary, our estimates of $p = 1 - e^{-\lambda}$

- $$\hat{p}_1 = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i > 0\}}$$

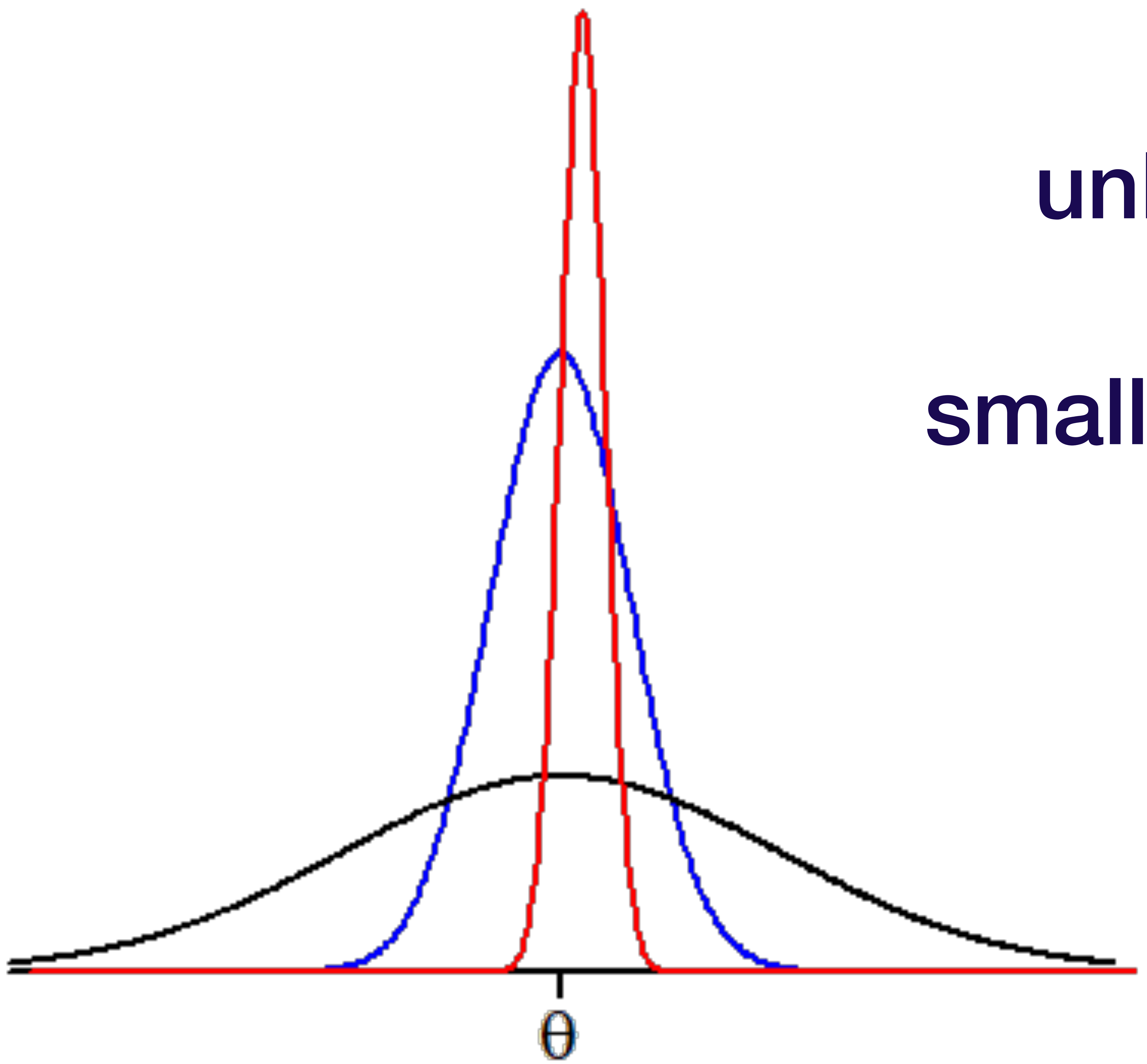
  is an unbiased estimator of p

- $$\hat{p}_2 = 1 - e^{-\overline{X}}$$

  is expected to be a little below p

  (negative "bias")

Should we compare the variances of $\hat{p}_1$ and $\hat{p}_2$?



unbiased
vs
small variance