

Suppose that

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

How can we estimate α ?

$$E[X_1] = \frac{\alpha}{\beta}$$

- We could estimate the true mean α/β with the sample mean \bar{X} , but we still can't get at α if we don't know β .
- One possibility is to use
Method of Moments Estimation.

Method of Moments Estimators:

Recall that the “moments” of a distribution are defined as $E[X]$, $E[X^2]$, $E[X^3]$, ...

These are distribution or “population” moments.

- $\mu = E[X]$ is a probability weighted average of the values in the population.
- \bar{X} is the average of the values in the sample.

It was natural for us to think about estimating μ with the average in our sample.

- $E[X^2]$ is a probability weighted average of the squares of the values in the population.

It is intuitively nice to estimate it with the average of the squared values in the sample:

$$\frac{1}{n} \sum_{i=1}^n x_i^2$$

The kth population moments:

$$\mu_k = E[X^k] \quad k = 1, 2, 3, \dots$$

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The k th sample moments:

$$M_k = \frac{1}{n} \sum_{i=1}^n X_i^k \quad k = 1, 2, 3, \dots$$

Method of Moments Estimators (MMEs)

Idea: Equate population and sample moments and solve for the unknown parameters.

Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \exp(\text{rate} = \lambda)$$

First population moment:

$$\mu_1 = \mu = E[X] = \frac{1}{\lambda}$$

First sample moment:

$$M_1 = \frac{1}{n} \sum_{i=1}^n X_i = \bar{X}$$

Equate: $\frac{1}{\lambda} = \bar{X}$



WARNING: This makes no sense! On the left side we have a constant and on the right side we have a random variable!

It is just an intermediate step to facilitate clean algebra.

Equate: $\frac{1}{\lambda} = \bar{X}$

Solve for the unknown parameter...

$$\lambda = \frac{1}{\bar{X}}$$

... and “put a hat on it”!

The MME is

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

We left the “hat” off originally just to facilitate the algebra. Technically, it should have been there from the beginning.

Question:

Is this an unbiased estimator of λ ?

Check:

$$E[\hat{\lambda}] = E\left[\frac{1}{\bar{X}}\right] = E\left[\frac{n}{\sum_{i=1}^n X_i}\right]$$

It is super important to note that:

$$E\left[\frac{1}{\bar{X}}\right] \neq \frac{1}{E[\bar{X}]}$$

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Super Important Notes:

$$E\left[\frac{1}{\bar{X}}\right] \neq \frac{1}{E[\bar{X}]}$$

$$\frac{1}{\sum_{i=1}^n X_i} \neq \sum_{i=1}^n \frac{1}{X_i}$$

$$E\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x) dx \neq \frac{1}{\int_{-\infty}^{\infty} x \cdot f_X(x) dx}$$

$$E[\hat{\lambda}] = E\left[\frac{1}{\bar{X}}\right] = E\left[\frac{n}{\sum_{i=1}^n X_i}\right]$$

$$= n E\left[\frac{1}{Y}\right] \quad \text{where } Y \sim \Gamma(\alpha, \beta)$$

$$= n \int_{-\infty}^{\infty} \frac{1}{y} f_Y(y) dy = n \int_0^{\infty} \frac{1}{y} \cdot \frac{1}{\Gamma(n)} \lambda^n y^{n-1} e^{-\lambda y} dy$$

$$= n \int_0^{\infty} \frac{1}{\Gamma(n)} \lambda^n \underbrace{y^{n-2} e^{-\lambda y}}_{\text{Looks like a } \Gamma(n-1, \lambda) \text{ pdf}} dy$$

$$= n\lambda \frac{\Gamma(n-1)}{\Gamma(n)} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(n-1)} \lambda^{n-1} y^{n-2} e^{-\lambda y} dy}_{=1} = n\lambda \frac{\Gamma(n-1)}{\Gamma(n)}$$

$$E[\hat{\lambda}] = E\left[\frac{1}{\bar{X}}\right] = n\lambda \frac{\Gamma(n-1)}{\Gamma(n)}$$

$$= n\lambda \frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{n}{n-1} \lambda \neq \lambda$$

So, MMEs are not necessarily unbiased estimators.

Can we find an unbiased estimator of λ for the exponential distribution based on \bar{X} ?

$$E\left[\frac{1}{\bar{X}}\right] = \frac{n}{n-1} \lambda \Rightarrow E\left[\frac{n-1}{n} \frac{1}{\bar{X}}\right] = \lambda$$

Unbiased estimator of λ :

$$\hat{\lambda} = \frac{n-1}{\sum_{i=1}^n X_i}$$

Example:

$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

First population moment:

$$\mu_1 = \mu = E[X] = \frac{\alpha}{\beta}$$

First sample moment:

$$M_1 = \frac{1}{n} \sum_{i=1}^n X_i = \bar{X}$$

Equate: $\frac{\alpha}{\beta} = \bar{X}$

One equation, two unknowns. Go for second moments!

Second Population Moment:

$$\begin{aligned}\mu_2 = E[X^2] &= \text{Var}[X] + (E[X])^2 \\ &= \frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta}\right)^2\end{aligned}$$

Second sample moment:

$$M_2 = \frac{1}{n} \sum_{i=1}^n X_i^2$$

Equate:

$$\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n} \sum_{i=1}^n X_i^2$$

$$\frac{\alpha}{\beta} = \overline{X}$$

$$\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n} \sum_{i=1}^n x_i^2$$

$$\hat{\beta} = \frac{\overline{X}}{\frac{1}{n} \sum_{i=1}^n x_i^2 - \overline{X}^2}$$

$$\hat{\alpha} = \frac{\overline{X^2}}{\frac{1}{n} \sum_{i=1}^n x_i^2 - \overline{X}^2}$$