The Gamma Distribution

The continuous random variable X is said to have a "gamma distribution" with parameters α and β if X has pdf

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

for $\alpha > 0$ and $\beta > 0$.

We write $X \sim \Gamma(\alpha, \beta)$

$$X \sim \Gamma(\alpha, \beta)$$

β is an "inverse scale parameter"

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

Important Note:

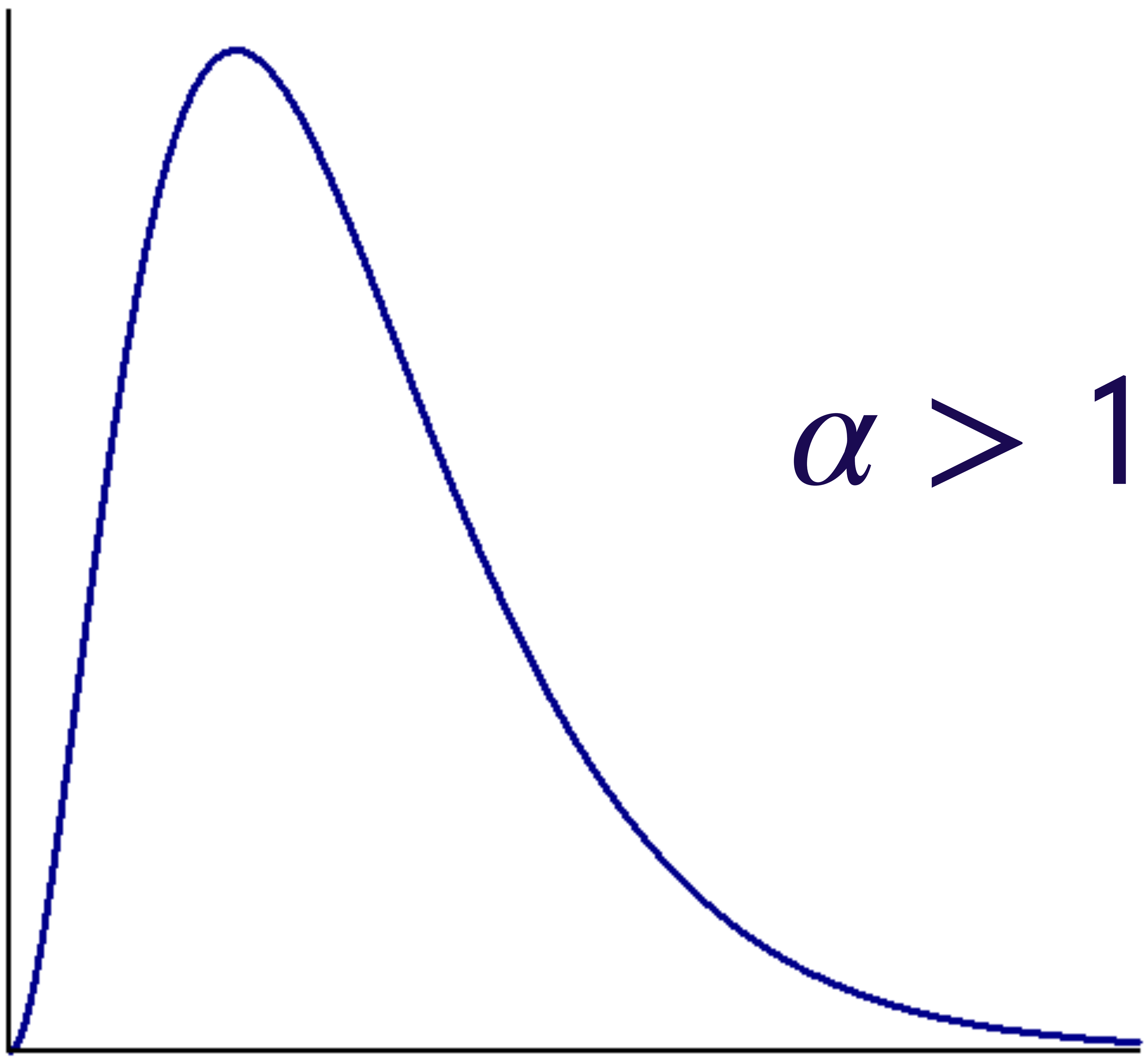
Some people use

β is a "scale parameter"

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} I_{(0,\infty)}(x)$$

α is a "shape parameter"

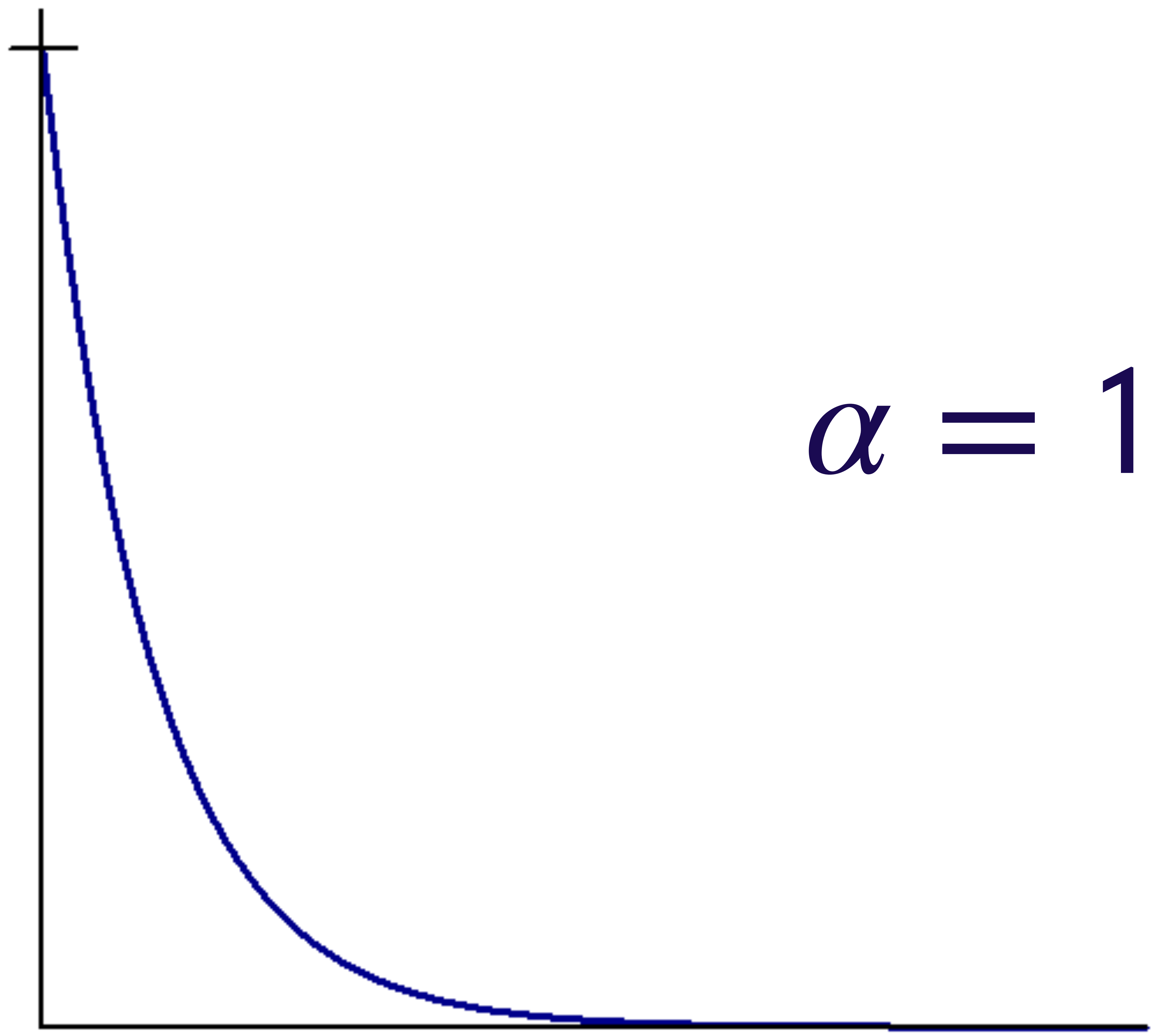
$$x^{\alpha-1}\, e^{-\beta x}$$



$\alpha > 1$

α is a "shape parameter"

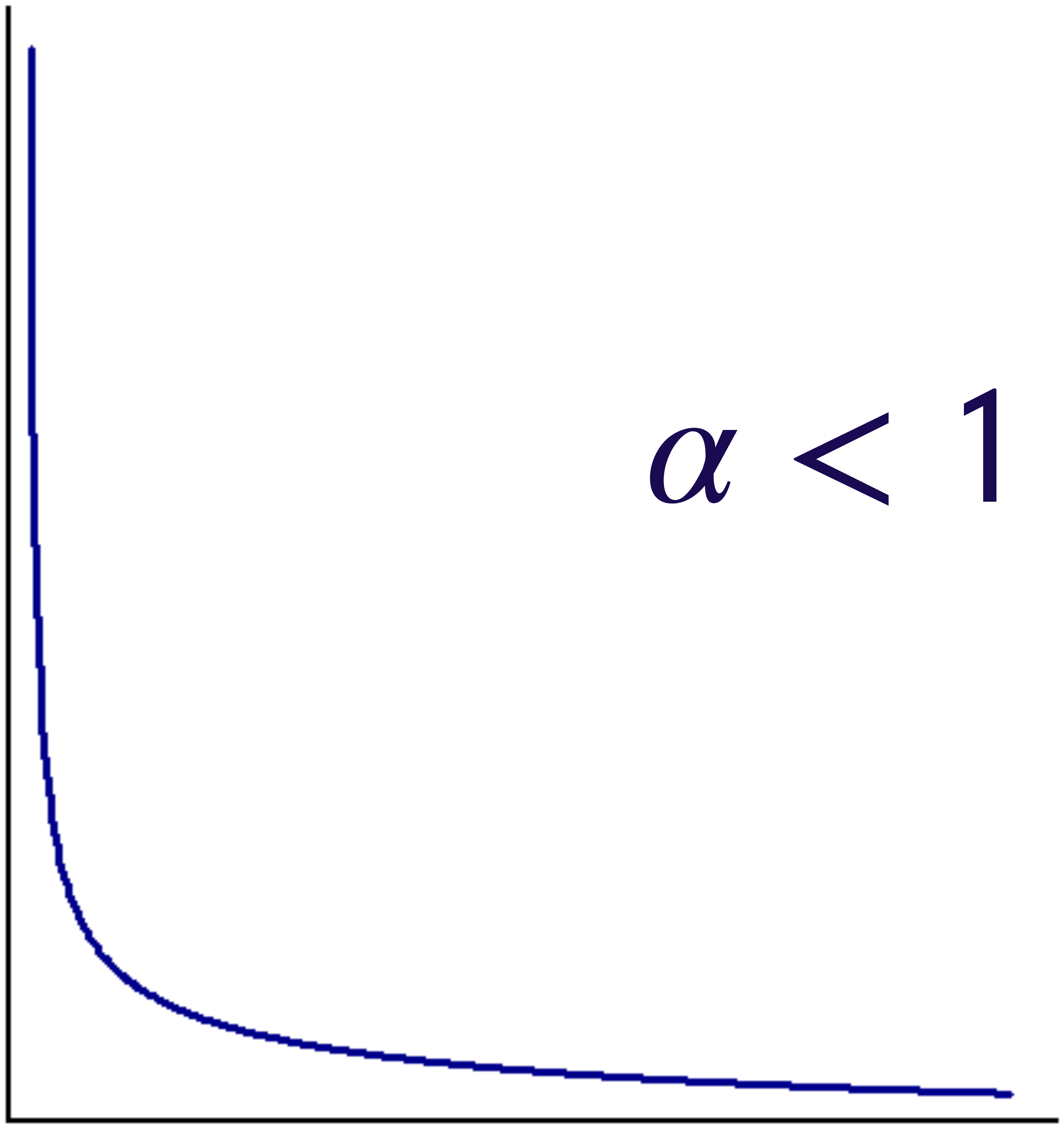
$$x^{\alpha-1}\, e^{-\beta x}$$



$\alpha = 1$

α is a "shape parameter"

$$x^{\alpha-1}\, e^{-\beta x}$$



$$\alpha < 1$$

The Gamma Function

$$f(x) = \frac{1}{\boxed{\Gamma(\alpha)}} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} \, I_{(0,\infty)}(x)$$

Different from $\Gamma(\alpha, \beta)$!

The Gamma Function

$$f(x) = \frac{1}{\boxed{\Gamma(\alpha)}} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

Defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

$$= \int_0^\infty \beta^{\alpha} x^{\alpha-1} e^{-\beta x} \, dx$$

$$(u = \beta x)$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx$$

- $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx$$

- $\Gamma(\alpha) = (\alpha - 1)\, \Gamma(\alpha - 1)$

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Integration by parts:

$$\int u\, dv = uv - \int u\, dv$$

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dv

Integration by parts:

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Properties of the Gamma Function

$$\int u\,dv = uv - \int u\,dv$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\,e^{-x}\,dx$$

$$= -x^{\alpha-1}\,e^{-x}\Big|_0^\infty + (\alpha - 1)\int_0^\infty x^{\alpha-2}\,e^{-x}\,dx$$

$$= 0 + (\alpha - 1)\,\Gamma(\alpha - 1)$$

Properties of the Gamma Function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$$

- $\Gamma(n) = (n-1)!$

$$\begin{aligned}\Gamma(n) &= (n-1)\,\Gamma(n-1) \\ &= (n-1)\,(n-2)\,\Gamma(n-2) \\ &\vdots \\ &= (n-1)\,(n-2)\,2 \cdot \Gamma(1)\end{aligned}$$

$\boxed{\text{Fun Fact:}}$

If you have $X \sim \Gamma(\alpha, \beta)$

and a constant $c > 0$

and define a new random variable

$$Y = cX$$

then

$$Y \sim \Gamma(\alpha, \beta/c)$$

A New Continuous Distribution

Suppose that

$$X \sim \Gamma(n/2, 1/2)$$

We say that X has a

"chi-squared distribution"

with n "degrees of freedom".

We write $$X \sim \chi^2(n)$$

Turning a gamma into a chi-squared:

Suppose that

$$X \sim \Gamma(n, \beta)$$

Then

$$\begin{aligned} 2\beta X &\sim \Gamma(n, 1/2) \\ &= \Gamma(2n/2, 1/2) \\ &= \chi^2(2n) \end{aligned}$$