$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

$$\lim_{n\to\infty} \frac{2n^2+3}{3n^3-7} = 0$$

$$\lim_{n\to\infty} \frac{2n^2+3}{3n^2-7} = \frac{2}{3}$$

$$a_n = \frac{n-1}{n} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 1$$

$$a_n = \frac{n-1}{n} \Rightarrow \lim_{n\to\infty} a_n = 1$$

Consider a sequence of random variables: $X_1, X_2, X_3, \ldots$

$$\boxed{\lim_{n\to\infty} X_n = ?}$$

This is meaningless!

Convergence in Probability

The sequence of random variables:

$$X_1, X_2, X_3, \dots$$

converges in probability to a random variable X if, for any ε>0

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$$

We write $X_n \xrightarrow{P} X$.

An Integral Notation:

Rewrite $$\int_0^2 f(x)\,dx = \int_A f(x)\,dx$$

where $A=\{x: 0\leq x\leq 2\}$.

Suppose that X has the exponential distribution with rate λ.

How can we find $P(|\sin(X)|>1/2)$?

How can we find $P(|\sin(X)|>1/2)$?

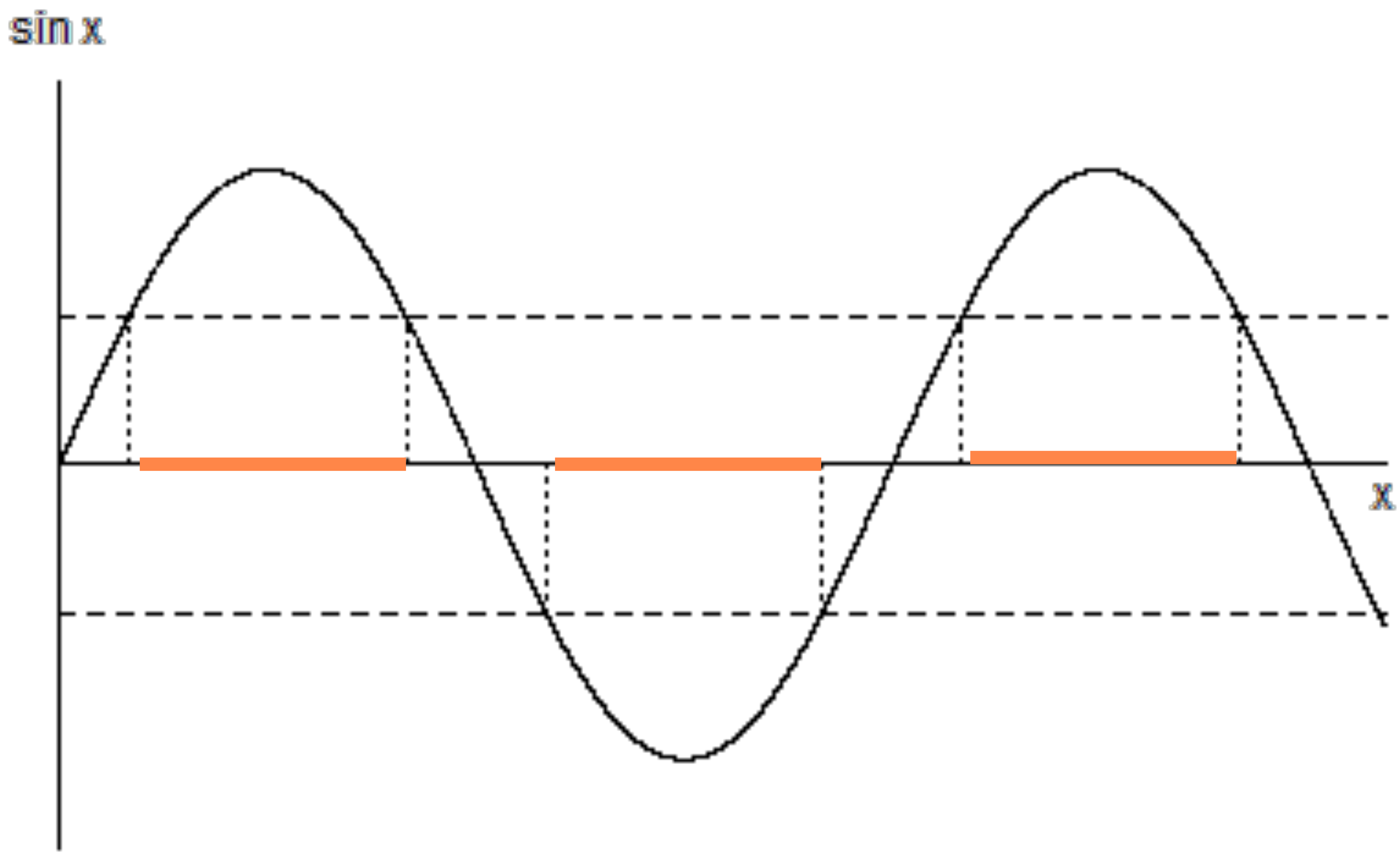
1. We can define a new random variable $Y=|\sin(X)|$, try to find its pdf and then

$$P(|\sin(X)| > 1/2) = P(Y > 1/2)$$

$$= \int_{1/2}^{\infty} f_Y(y)\, dy$$

How can we find $P(|\sin(X)|>1/2)$?

2. We can integrate the pdf for X over the relevant region.



How can we find P(|sin(X)|>1/2)?

2. We can integrate the pdf for X over the relevant region.

$$P(|\sin(X)| > 1/2)$$

$$= \int_{\sin^{-1}(1/2)}^{\pi-\sin^{-1}(1/2)} f_X(x)\,dx + \int_{\pi+\sin^{-1}(1/2)}^{2\pi-\sin^{-1}(1/2)} f_X(x)\,dx + \cdots$$

Notation

$$= \int_{\{x\,:\,|\sin(x)| > 1/2\}} f_X(x)\,dx$$

An Inequality:

Let X be a random variable. Let g be a non-negative function and let c>0.

Then

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

When $g(x)=|x|$,

this is known as Markov's inequality.

An Inequality:

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_{\{x \,:\, g(x) \geq c\}} g(x) f_X(x)\, dx + \int_{\{x \,:\, g(x) < c\}} g(x) f_X(x)\, dx$$

An Inequality:

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_{\{x\,:\,g(x) \geq c\}} g(x) f_X(x)\, dx + \int_{\{x\,:\,g(x) < c\}} g(x) f_X(x)\, dx$$

$$\geq \int_{\{x\,:\,g(x) \geq c\}} g(x) f_X(x)\, dx \quad \geq \int_{\{x\,:\,g(x) \geq c\}} c\, f_X(x)\, dx$$

An Inequality:

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

$$E[g(X)] \geq \int_{\{x\,:\,g(x) \geq c\}} c\, f_X(x)\, dx$$

$$= c \int_{\{x\,:\,g(x) \geq c\}} f(x)\, dx \quad = c\, P(g(X) \geq c)$$

Chebyshev's Inequality:

Let X be a random variable with mean μ and variance $\sigma^2 < \infty$. Let $k > 0$.

Then

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or, equivalently,

$$P(|X - \mu| < k\sigma) > 1 - \frac{1}{k^2}$$

"the probability that X is within k standard deviations of its mean"

Chebyshev's Inequality:

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Proof:

$$P(|X-\mu| \geq k\sigma) = P(\underbrace{(X-\mu)^2}_{g(x)} \geq \underbrace{k^2\sigma^2}_{c})$$

$$\leq \frac{E[g(X)]}{c} = \frac{E[(X-\mu)^2]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2}$$

$$= 1/k^2 \checkmark$$

Convergence in Probability $(X_n \xrightarrow{P} X)$

The sequence of random variables:

$$X_1, X_2, X_3, \ldots$$

converges in probability to a random variable X if, for any ε>0

$$\lim_{n\to\infty} P(\,|X_n - X| > \varepsilon\,) = 0$$

could be a constant!

Ex: X=3 w.p.1

$$P(X = x) = \begin{cases} 1, & \textbf{if x=3} \\ 0, & \textbf{otherwise} \end{cases}$$

The Weak Law of Large Numbers

Suppose that $X_1, X_2, X_3, \ldots$ is a sequence of iid random variables from any distribution with mean μ and variance $\sigma^2 < \infty$.

Then $$\overline{X} \xrightarrow{P} \mu$$

- $E[\overline{X}] = \mu$
- $Var[\overline{X}] = \sigma^2/n$

The Weak Law of Large Numbers

$$\overline{X} \xrightarrow{P} \mu$$

Proof: Let ε>0.

Chebyshev:
$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Here:
$$P(|\overline{X}-\mu_{\overline{X}}| \geq k\sigma_{\overline{X}}) \leq \frac{1}{k^2}$$

Which is:
$$P(|\overline{X}-\mu| \geq k\sigma/\sqrt{n}) \leq \frac{1}{k^2}$$

Choose k so that this is ε.

The Weak Law of Large Numbers

$$P(|\bar{X} - \mu| \geq \varepsilon) \leq \frac{1}{(\varepsilon\sqrt{n}/\sigma)^2} = \frac{\sigma^2}{\varepsilon^2 n}$$

$$\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{\varepsilon^2 n} = 0$$

$$\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \varepsilon) = 0$$

$$\bar{X} \xrightarrow{P} \mu$$

The Weak Law of Large Numbers

$$\lim_{n \to \infty} P(|\overline{X} - \mu| \geq \varepsilon) = 0$$

$$\Downarrow$$

$$\overline{X} \xrightarrow{P} \mu$$

Example:

$$X_1, X_2, X_3, \ldots \overset{\text{iid}}{\sim} \exp(\text{rate} = \lambda)$$

$$\overline{X} \xrightarrow{P} 1/\lambda$$

$$X_1, X_2, X_3, \ldots \overset{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

$$\overline{X} \xrightarrow{P} \alpha/\beta$$