The Cramér-Rao Lower Bound

Let $X_1, X_2, \ldots, X_n$ be a random sample from some distribution with pdf $f(x;\theta)$.

Consider estimating some $\tau(\theta)$.

Suppose that $\widehat{\tau(\theta)}$ is any unbiased estimator of $\tau(\theta)$.

Then
$$Var[\widehat{\tau(\theta)}] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

the CRLB

The Cramér-Rao Lower Bound

$$\mathrm{Var}[\widehat{\tau(\theta)}] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

Fisher information:

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

Notation:

$$\mathrm{Var}[\widehat{\tau(\theta)}] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

The estimator: $\widehat{\tau(\theta)} = T = t(\vec{X})$

so

$$\mathrm{Var}[T] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

The proof of the Cramér-Rao Lower Bound depends on the Cauchy-Schwartz inequality:

$$\left(\int g(x)\,h(x)\,dx\right)^2 \leq \left(\int g^2(x)\,dx\right)\left(\int h^2(x)\,dx\right)$$

<u>Proof:</u> Consider

$$\int\int \big(g(x)h(y) - g(y)h(x)\big)^2\,dx\,dy \geq 0$$

Square the left-hand side out and run the integrals through.

Get

$$0 \le \iint g^2(x)h^2(y)\,dx\,dy$$

$$-2\iint g(x)h(y)g(y)h(x)\,dx\,dy$$

$$+\iint g^2(y)h^2(x)\,dx\,dy$$

these are the same

Get

$$0 \le 2\int\int g^2(x)h^2(y)\,dx\,dy$$

$$-2\boxed{\int\int g(x)h(y)g(y)h(x)\,dx\,dy}$$

$$\int\int g(x)h(y)g(y)h(x)\,dx\,dy$$

$$= \int g(y)h(y)\left(\int g(x)h(x)\,dx\right)dy$$

$$= \left(\int g(x)h(x)\,dx\right)\left(\int g(y)h(y)\,dy\right)$$

$$\boxed{= \left(\int g(x)h(x)\,dx\right)^2}$$

Get

$$0 \le 2\iint g^2(x)h^2(y)\,dx\,dy$$

$$-2\left(\int g(x)h(x)\,dx\right)^2$$

$$\boxed{\left(\int g(x)h(x)\,dx\right)^2} \le \iint g^2(x)h^2(y)\,dx\,dy$$

$$\le \left(\int g^2(x)\,dx\right)\left(\int h^2(y)\,dy\right)$$

$$\boxed{\le \left(\int g^2(x)\,dx\right)\left(\int h^2(x)\,dx\right)}$$

$$\left(\int g(x)\, h(x)\, dx \right)^2 \leq \left(\int g^2(x)\, dx \right) \left(\int h^2(x)\, dx \right)$$

- **This holds with sums. (discrete version)**

- **This holds with dx replaced by f(x)dx where f is a pdf:**

$$(E[g(X)h(X)])^2 \leq E[g^2(X)]\, E[h^2(X)]$$

$$\mathrm{Var}[\widehat{\tau(\theta)}] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

Will be done if we can show that:

$$\tau'(\theta) = \underbrace{\left(T - \tau(\theta)\right)}_{g(\vec{X})}\underbrace{\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)}_{h(\vec{X})}$$

where $T=t(\vec{X})$.

$$\tau'(\theta) = \left(\mathsf{T} - \tau(\theta)\right) \left(\frac{\partial}{\partial\theta} \ln \mathsf{f}(\vec{\mathsf{X}}; \theta)\right)$$

$$\Downarrow$$

$$\mathsf{E}[\tau'(\theta)] = \mathsf{E}\left[\left(\mathsf{T} - \tau(\theta)\right) \left(\frac{\partial}{\partial\theta} \ln \mathsf{f}(\vec{\mathsf{X}}; \theta)\right)\right]$$

$$\Vert$$

$$\tau'(\theta)$$

$$[\tau'(\theta)]^2 = \left(\mathsf{E}\left[\left(\mathsf{T} - \tau(\theta)\right) \left(\frac{\partial}{\partial\theta} \ln \mathsf{f}(\vec{\mathsf{X}}; \theta)\right)\right]\right)^2$$

$$[\tau'(\theta)]^2 = \left(\mathrm{E}\left[\left(\mathrm{T} - \tau(\theta)\right) \left(\frac{\partial}{\partial\theta} \ln \mathrm{f}(\vec{\mathrm{X}};\theta) \right) \right] \right)^2$$

$$\leq \mathrm{E}\left[\left(\mathrm{T} - \tau(\theta)\right)^2 \right] \cdot \mathrm{E}\left[\left(\frac{\partial}{\partial\theta} \ln \mathrm{f}(\vec{\mathrm{X}};\theta) \right)^2 \right]$$

$$= \mathrm{Var}[\mathrm{T}] \cdot \mathrm{I}_n(\theta)$$

$$\Rightarrow \quad \mathrm{Var}[\mathrm{T}] \leq \frac{[\tau'(\theta)]^2}{\mathrm{I}_n(\theta)}$$

$$\tau'(\theta) = \big(\mathsf{T} - \tau(\theta)\big)\left(\frac{\partial}{\partial\theta}\ln \mathsf{f}(\vec{\mathsf{X}};\theta)\right)$$

$$\tau'(\theta) = \frac{\partial}{\partial\theta}\tau(\theta) = \frac{\partial}{\partial\theta}\mathsf{E}[\mathsf{T}]$$

$$= \frac{\partial}{\partial\theta}\int \mathsf{t}(\vec{\mathsf{x}})\,\mathsf{f}(\vec{\mathsf{x}};\theta)\,\mathsf{d}\vec{\mathsf{x}}$$

$$= \frac{\partial}{\partial\theta}\int \mathsf{t}(\vec{\mathsf{x}})\,\mathsf{f}(\vec{\mathsf{x}};\theta)\,\mathsf{d}\vec{\mathsf{x}} - \tau(\theta)\underbrace{\frac{\mathsf{d}}{\mathsf{d}\theta}\overbrace{\int \mathsf{f}(\vec{\mathsf{x}};\theta)\,\mathsf{d}\vec{\mathsf{x}}}^{1}}_{0}$$

$$= \int \big(\mathsf{t}(\vec{\mathsf{x}}) - \tau(\theta)\big)\frac{\partial}{\partial\theta}\mathsf{f}(\vec{\mathsf{x}};\theta)\,\mathsf{d}\vec{\mathsf{x}}$$

$$\tau'(\theta) = \int \left(t(\vec{x}) - \tau(\theta)\right) \frac{\partial}{\partial \theta} f(\vec{x};\theta)\, d\vec{x}$$

Want to see

- $$E\left[\left(t(\vec{X}) - \tau(\theta)\right)\left(\frac{\partial}{\partial \theta} \ln f(\vec{X};\theta)\right)\right]$$
- Note that

$$\frac{\partial}{\partial \theta} f(\vec{x};\theta) = \frac{\partial}{\partial \theta} \ln f(\vec{x};\theta)\, f(\vec{x};\theta)$$

The CRLB is valid if

- $$\frac{\partial}{\partial \theta} \int \mathrm{f}(\vec{\mathrm{x}}; \theta)\, \mathrm{dx} = \int \frac{\partial}{\partial \theta} \mathrm{f}(\vec{\mathrm{x}}; \theta)\, \mathrm{dx}$$
- $\frac{\partial}{\partial \theta} \ln \mathrm{f}(\vec{\mathrm{x}}; \theta)$ **exists**
- $$0 < \mathrm{E}\left[\left(\frac{\partial}{\partial \theta} \ln \mathrm{f}(\vec{\mathrm{X}}; \theta)\right)^2\right] < \infty$$

The CRLB is valid if

- $$\frac{\partial}{\partial\theta}\int \mathrm{f}(\vec{\mathrm{x}};\theta)\,\mathrm{dx} = \int \frac{\partial}{\partial\theta}\mathrm{f}(\vec{\mathrm{x}};\theta)\,\mathrm{dx}$$

This one doesn't hold whenever the parameter is in the indicator or support of the distribution.

Example:

The CRLB doesn't hold for the unif(0,θ) distribution!

$$\mathrm{Var}[\widehat{\tau(\theta)}] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

Example:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{Bernoulli}(p)$$

Find the Cramér-Rao lower bound of the variance of all unbiased estimators of p.

Here, $\theta=p$ and $\tau(p)=p$.

The Fisher Information:

$$I_n(p) := E\left[\left(\frac{\partial}{\partial p}\ln f(\vec{X};p)\right)^2\right]$$

pdf: $f(x;p) = p^x(1-p)^{1-x} I_{\{0,1\}}(x)$

joint pdf: $f(\vec{x};p)$

$$= p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i} \boxed{\prod_{i=1}^n I_{\{0,1\}}(x_i)}$$

Take the log:

$$\ln f(\vec{x}; p)$$

$$= \left(\sum_{i=1}^{n} x_i\right) \ln p + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1 - p)$$

Take the derivative:

$$\frac{\partial}{\partial p} \ln f(\vec{x}; p) = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1 - p}$$

Simplify:

$$\frac{\partial}{\partial p} \ln f(\vec{x}; p)$$

$$= \frac{(1-p)\sum_{i=1}^{n} x_i - p\left(n - \sum_{i=1}^{n} x_u\right)}{p(1-p)}$$

$$= \frac{\sum_{i=1}^{n} x_i - np}{p(1-p)}$$

Put the random variables in, square, and take the expectation.

Note that $Y = \sum_{i=1}^{n} X_i \sim \text{binomial}(n, p)$

$$
\begin{aligned}
I_n(p) &= E\left[\left(\frac{\partial}{\partial p}\ln f(\vec{X}; p)\right)^2\right] \\
&= E\left[\left(\frac{Y - np}{p(1-p)}\right)^2\right] \\
&= \frac{1}{p^2(1-p)^2}\,\underbrace{E\left[\left(Y - np\right)^2\right]}_{\text{variance of binomial}}
\end{aligned}
$$

Note that $Y = \sum_{i=1}^{n} X_i \sim \text{binomial}(n, p)$

$$I_n(p) = \frac{1}{p^2(1-p)^2} E\left[\left(Y - np\right)^2\right]$$

$$= \frac{1}{p^2(1-p)^2} \mathrm{Var}[Y]$$

$$= \frac{np(1-p)}{p^2(1-p)^2} = \frac{n}{p(1-p)}$$

Example:

$$X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

$$\begin{aligned}
\text{Var}[\hat{p}] &= \frac{[\tau'(p)]^2}{I_n(p)} \\
&= \frac{1^2}{n/[p(1-p)]} \\
&= \frac{p(1-p)}{n}
\end{aligned}$$

Example:

$$X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

mean: p

variance: $p(1-p)$

$$E[\overline{X}] = E[X_1] = p$$

$$\text{Var}[\overline{X}] = \frac{\text{Var}[X_1]}{n} = \frac{p(1-p)}{n}$$

So $\overline{X}$ is an unbiased estimator of p with the smallest possible variance! Woot!

This estimator for p for the Bernoulli distribution is actually a

Uniformly
Minimum
Variance
Unbiased
Estimator.