Variance:

- Measure of "spread" of a distribution
- Denoted by $\mathrm{Var}[X]$ or σ^2
- Defined as

$$\boxed{\mathrm{Var}[X] = \mathrm{E}[(X - \mu)^2]}$$

where $\mu = \mathrm{E}[X]$

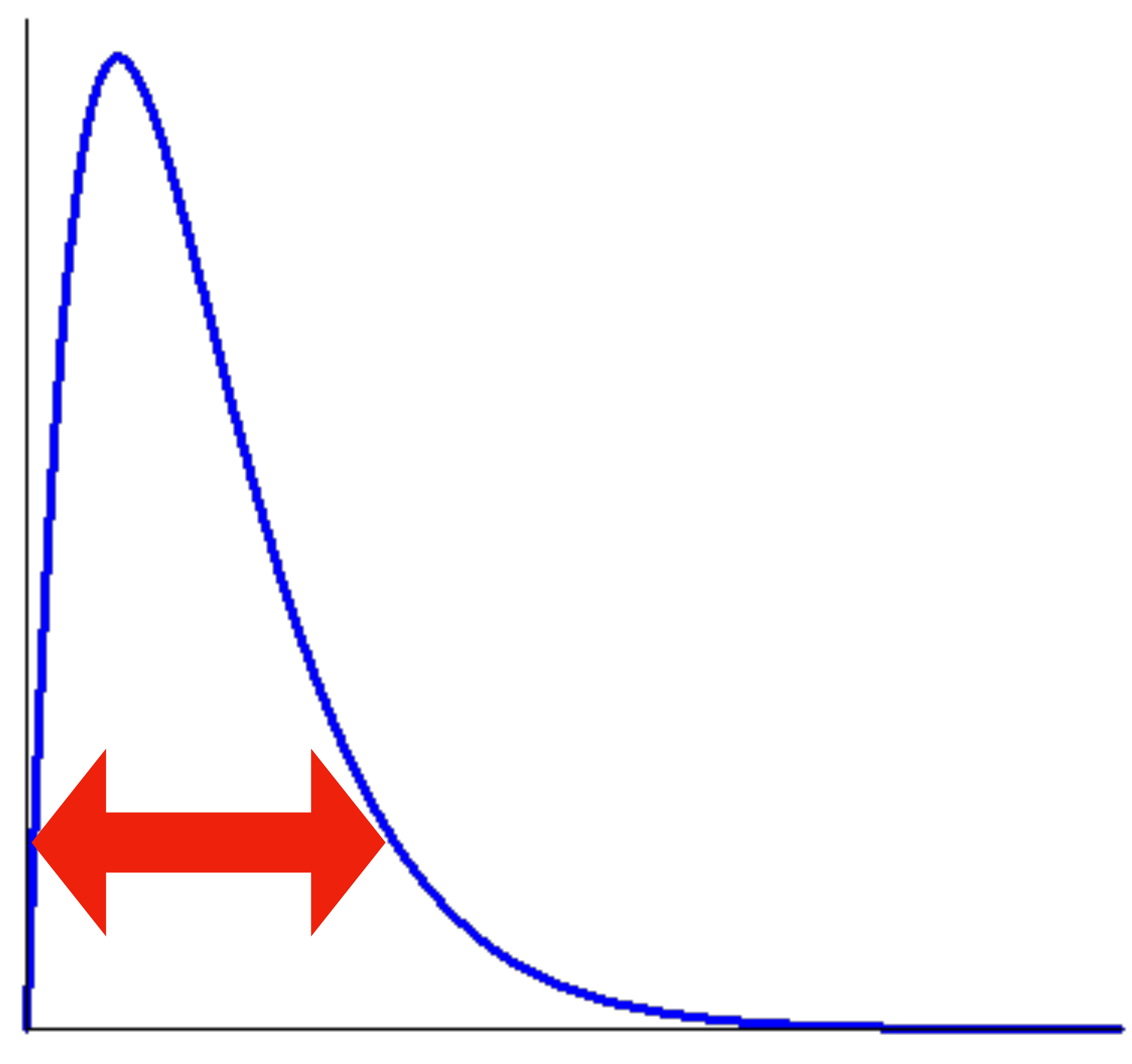
- Why not $\mathrm{E}[X - \mu]$?

$= \mathrm{E}[X] - \mu$

$= \mu - \mu$

$= 0$

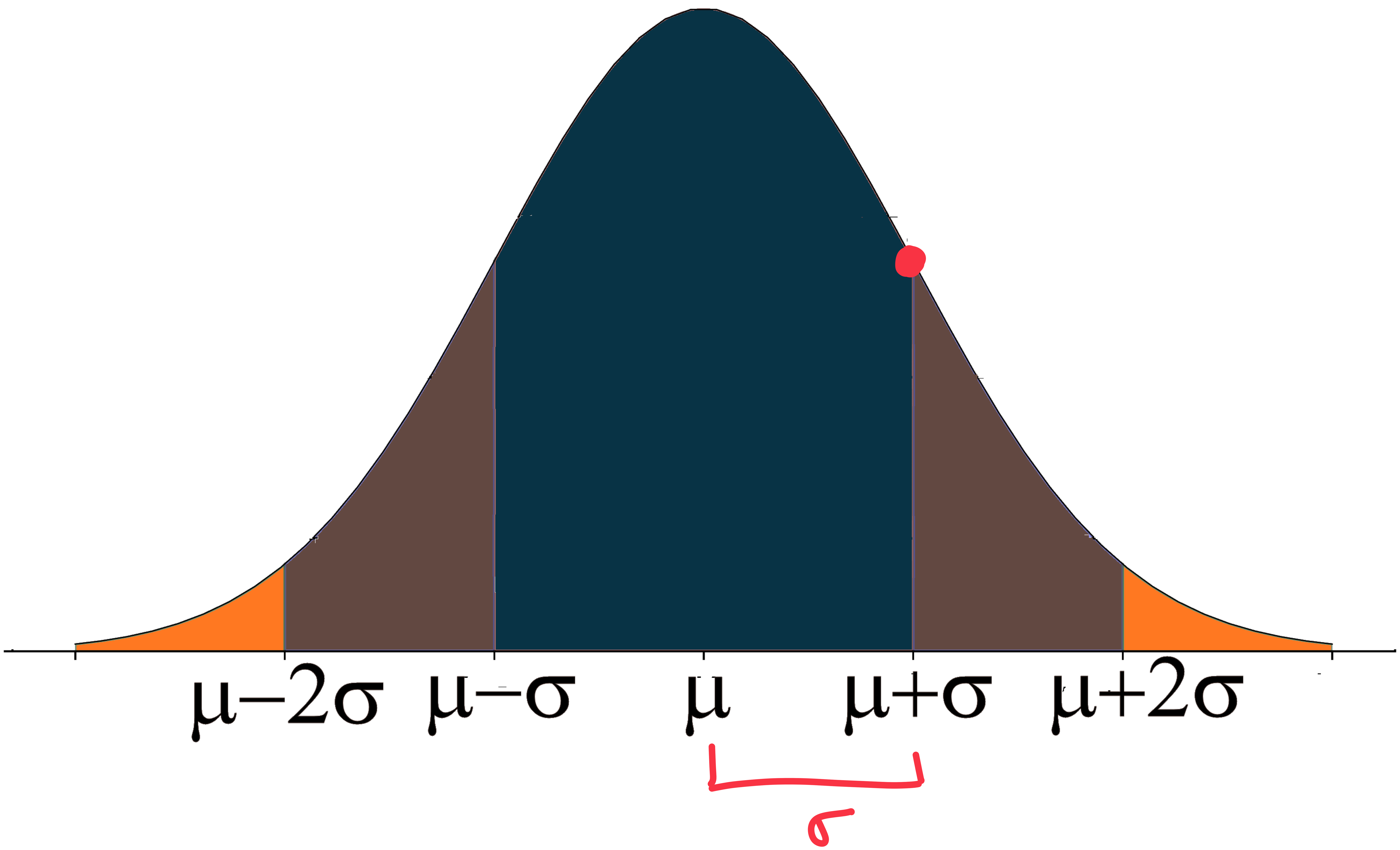
- Why not $\mathrm{E}[\,|X - \mu|\,]$?



Standard Deviation:

$$\sigma = \sqrt{\sigma^2}$$

Ex: $X \sim N(\mu, \sigma^2)$



Another Way to Compute Variance:

$$
\begin{aligned}
\mathrm{Var}[X] &= E[(X-\mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2 \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

Properties of Variance:

$$\mathrm{Var}[aX] = ?$$

Let $Y = aX$.

Then

$$\mu_Y = E[Y] = E[aX] = aE[X] = a\mu_X.$$

$$\begin{aligned}\Rightarrow \quad \mathrm{Var}[aX] = \mathrm{Var}[Y] &= E[(Y - \mu_Y)^2] \\ &= a^2 E[(X - \mu_X)^2] \\ &= a^2\,\mathrm{Var}[X]\end{aligned}$$

Properties of Variance:

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]\,?$$

Not necessarily!

- We will see that this is true if X and Y are independent.
- Need concept of "covariance".

Covariance:

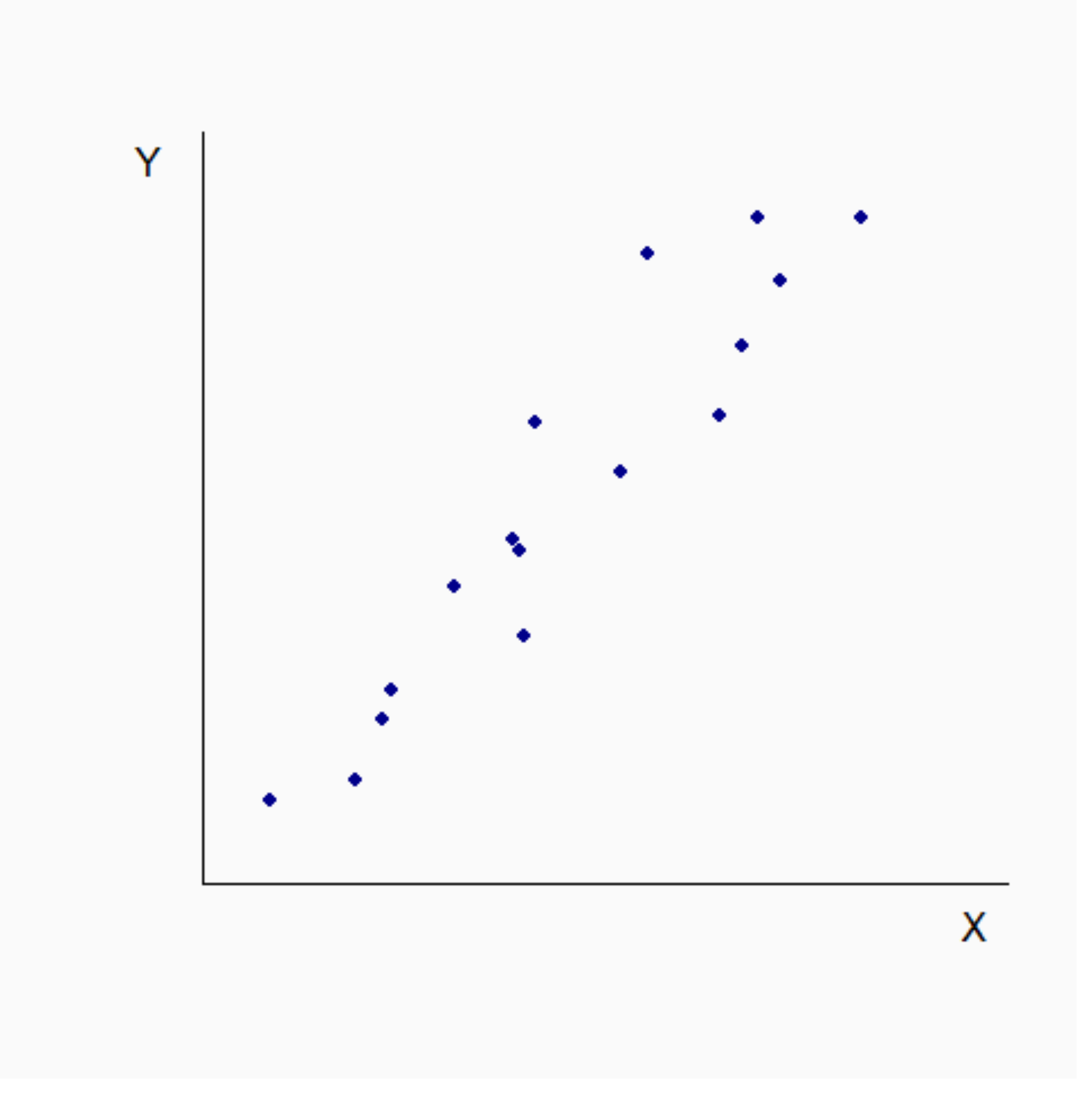
Let X and Y be random variables.

Define/denote the covariance as

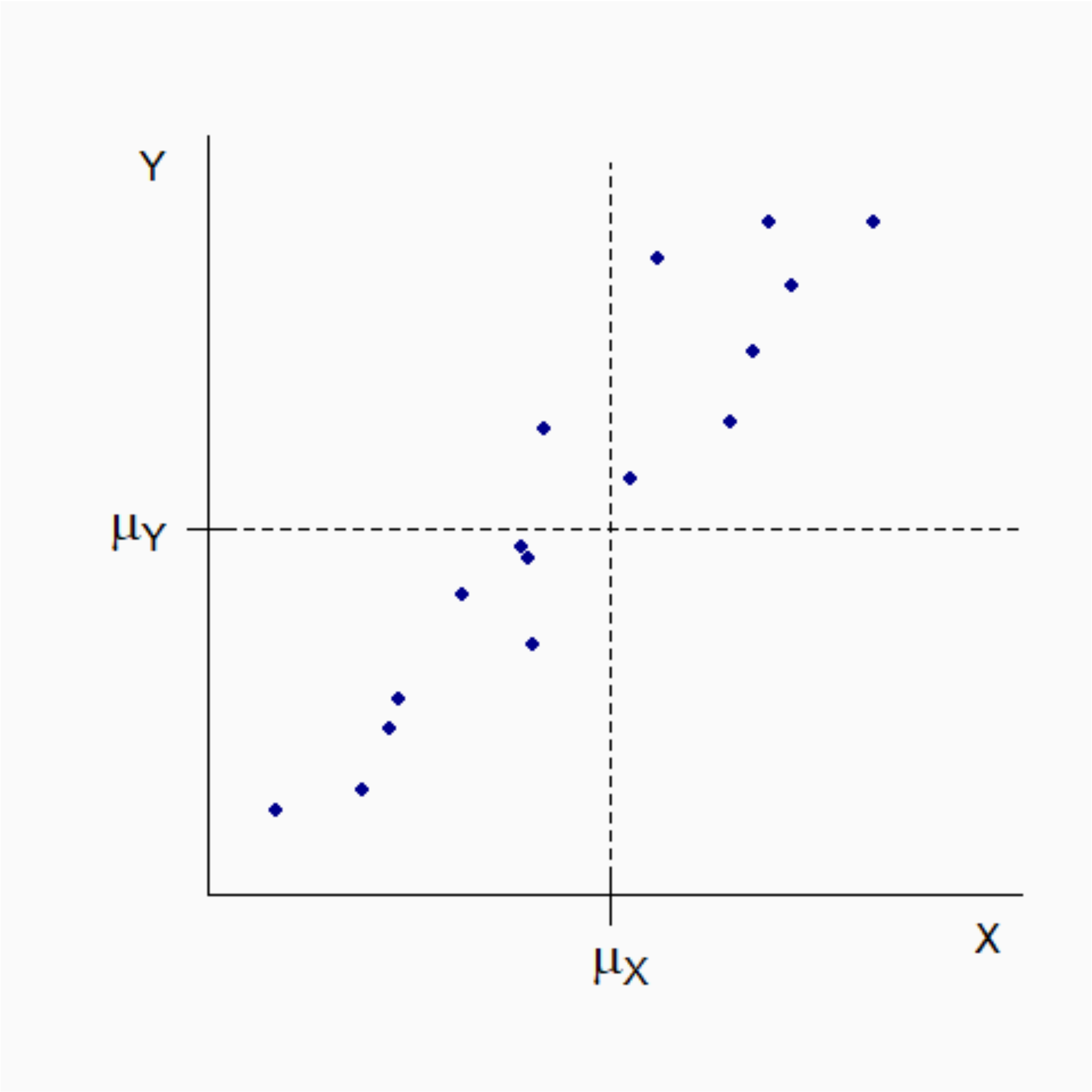
$$\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

- Notation: $\sigma_{X,Y}$
- Note that $\mathrm{Cov}(X, X) = \mathrm{Var}[X]$

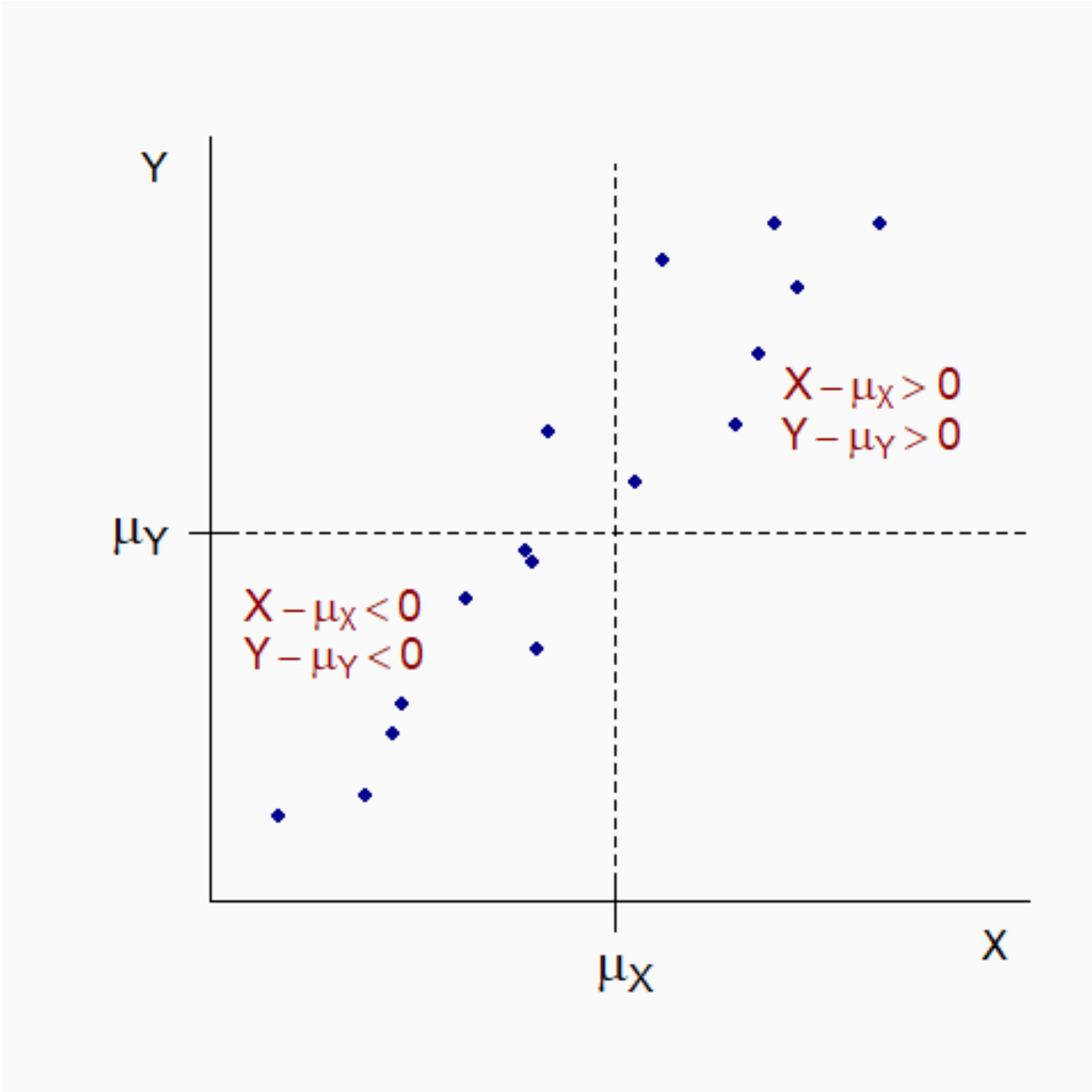
Example:



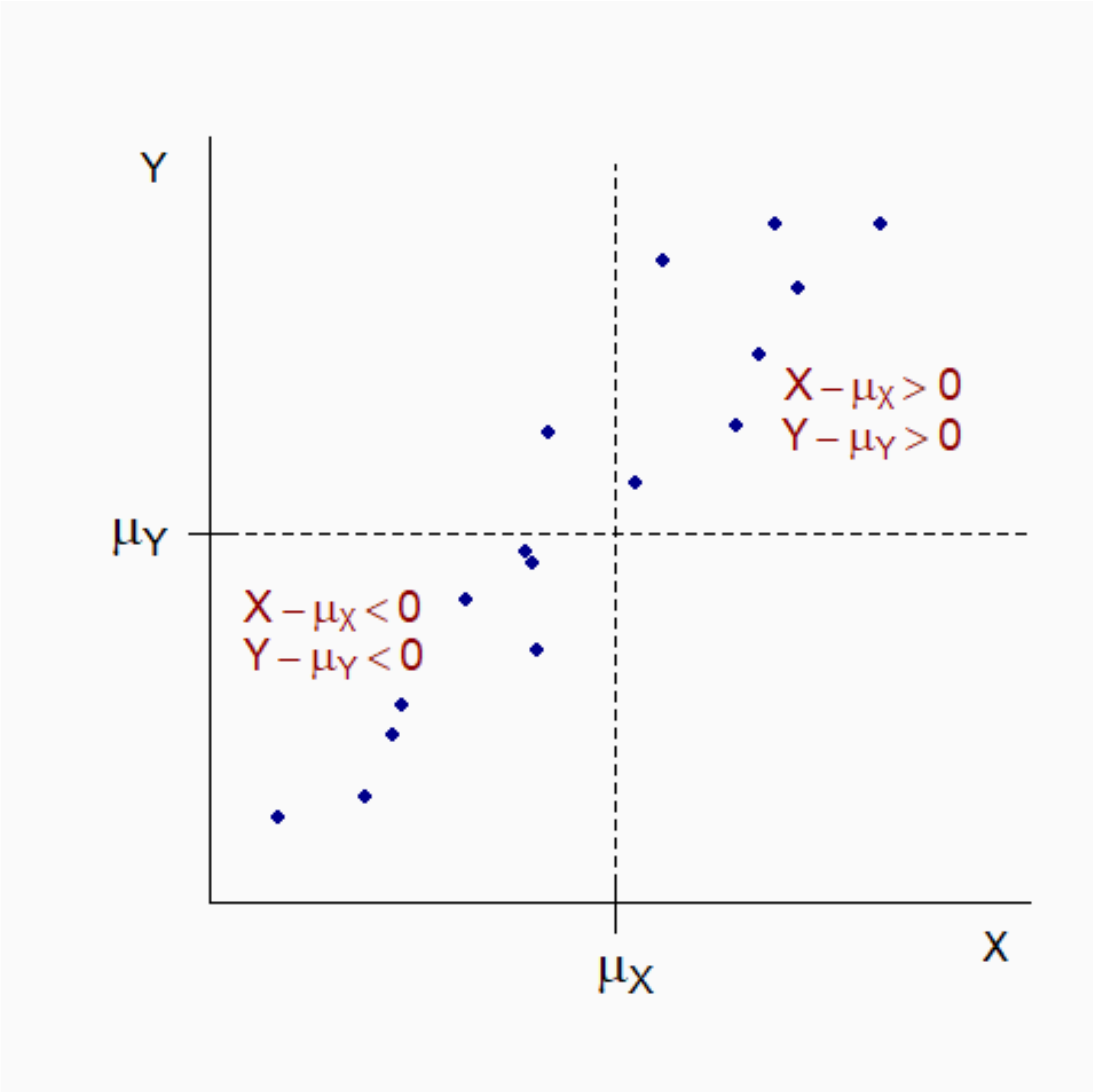
Example:



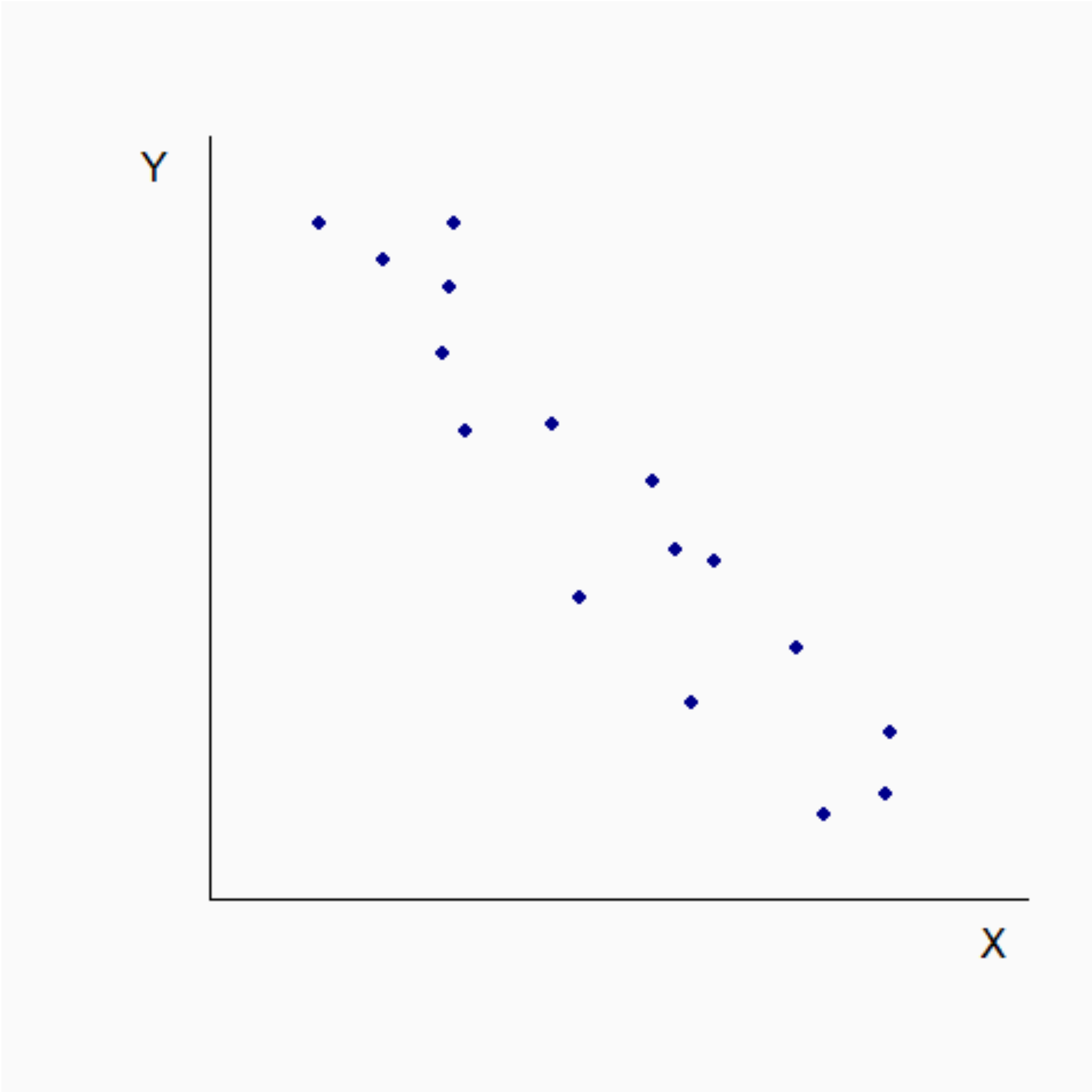
Example:



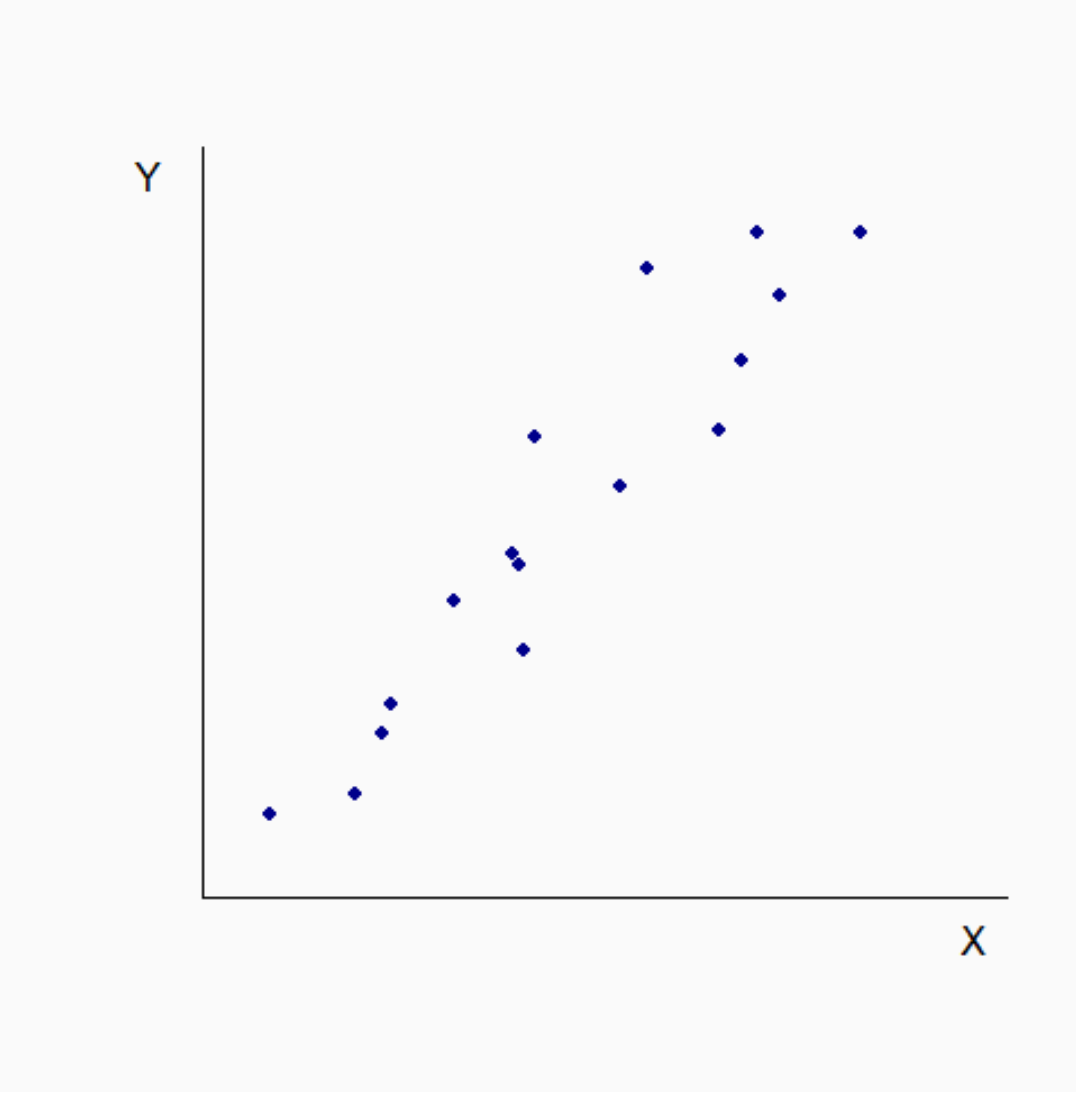
Example: Positive Covariance



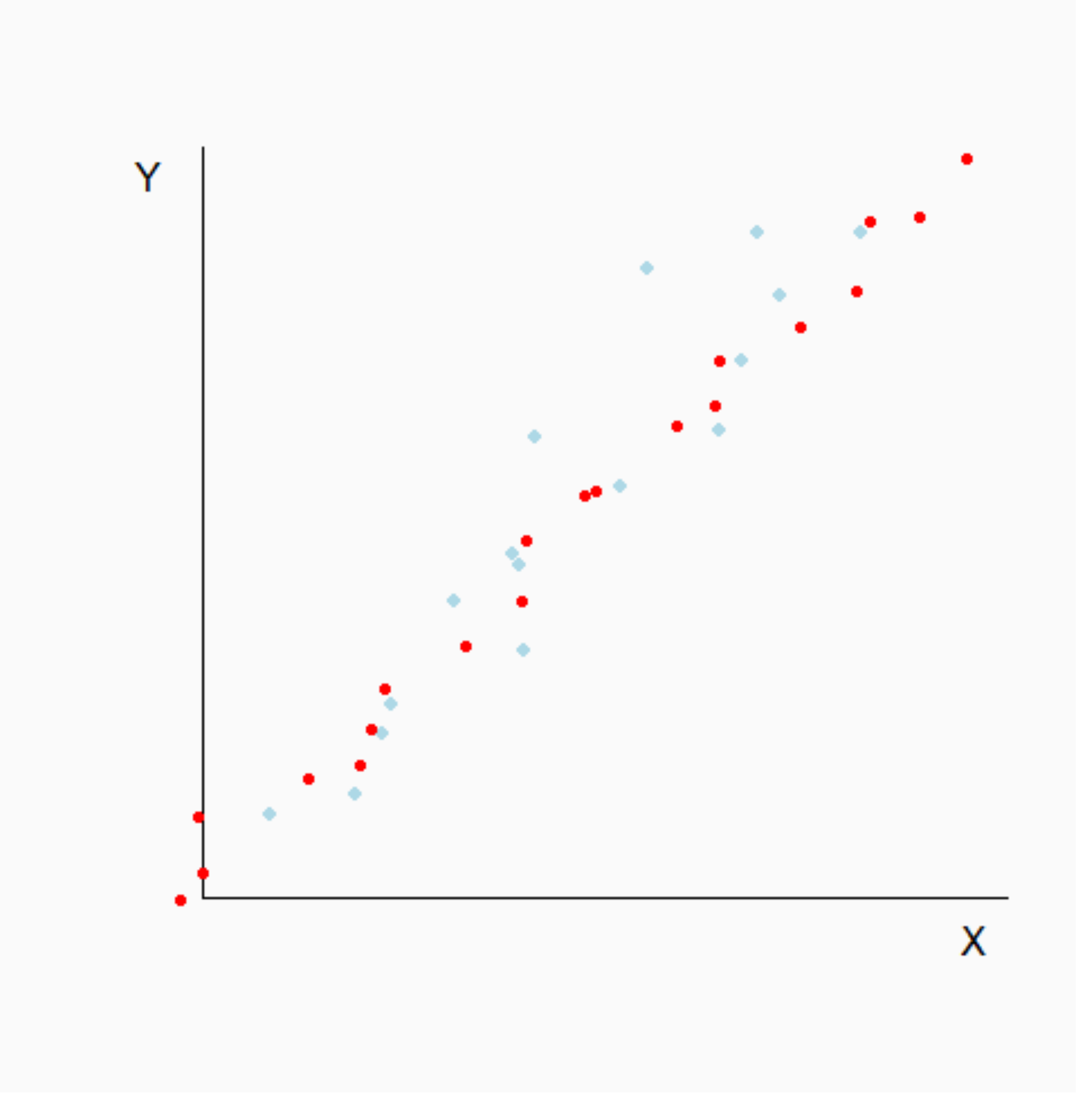
Example: Negative Covariance



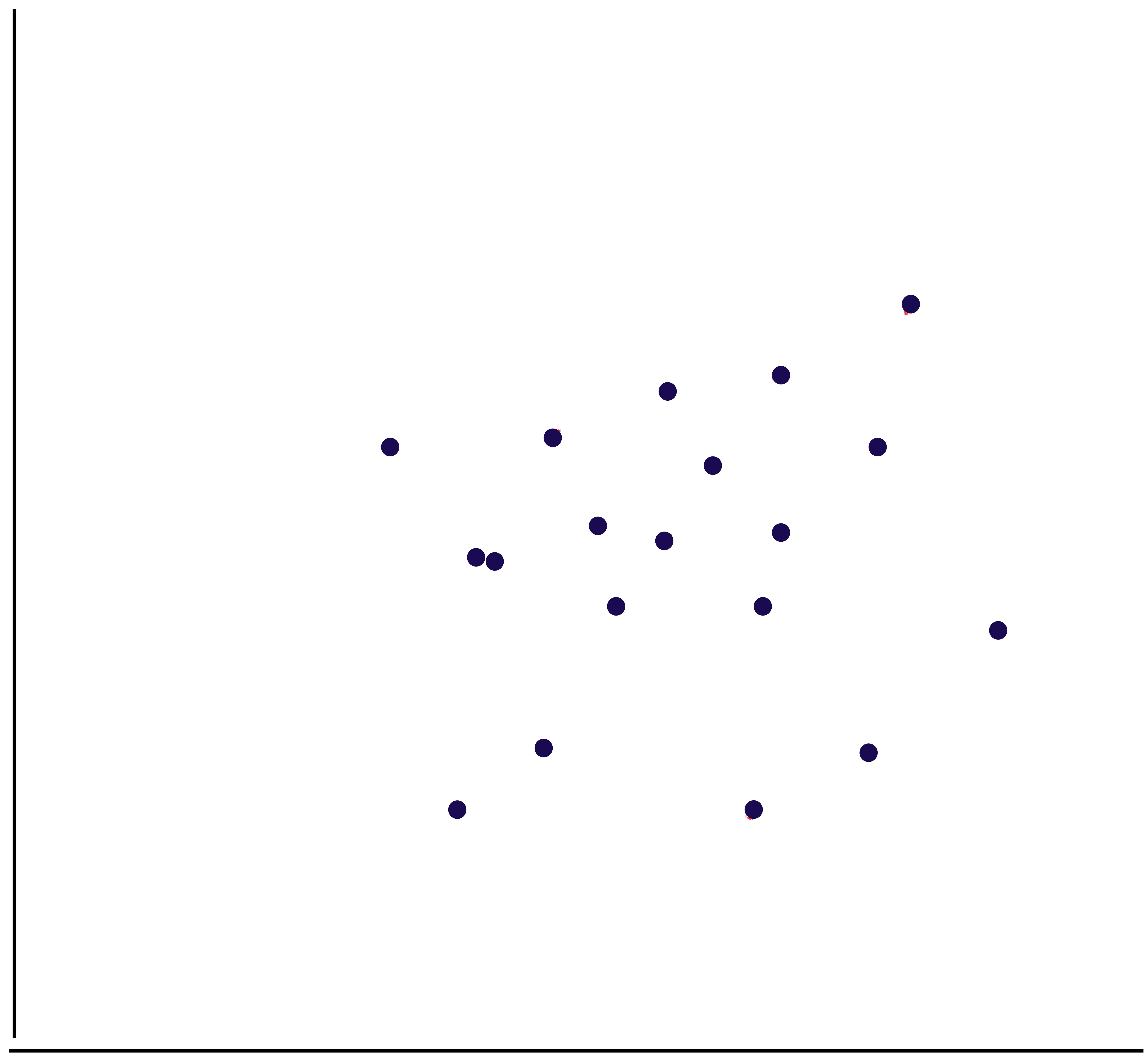
Example: Positive Covariance



Example: Larger Positive Covariance



Example: Covariance Close to Zero



- There is no upper/lower bound on covariance.
- Also, it is in weird units.
- Both problems can be addressed using correlation.

$$\mathrm{Corr}(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(x)\mathrm{Var}(Y)}}$$

- **Notation:** $\rho_{X,Y} = \dfrac{\sigma_{X,Y}}{\sigma_X \sigma_Y}$
- **Property:** $-1 \leq \rho_{X,Y} \leq 1$

Covariance Alternate:

$$\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E[XY] - \mu_Y \underbrace{E[X]}_{\mu_X} - \mu_X \underbrace{E[Y]}_{\mu_Y} + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y$$

$$= E[XY] - E[X]E[Y]$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

- **X and Y independent**

$$\Rightarrow \quad \mathrm{Cov}(X,Y) = 0$$

- $\mathrm{Cov}(X,Y) = 0$ **does not imply X and Y are independent.**

- $\mathrm{Cov}(X,Y) = 0$ **if and only if**

$$\mathrm{Corr}(X,Y) = 0$$

- **If** $\mathrm{Corr}(X,Y) = 0$**, we say that X and Y are uncorrelated.**

independent $\Rightarrow$ uncorrelated

uncorrelated $\not\Rightarrow$ independent

Previous Example:

x	-1	0	1
$P(X = x)$	$1/4$	$1/2$	$1/4$

$$E[X] = 0 \quad \Rightarrow \quad E[X]E[Y] = 0$$

Define $Y = X^2$. Then $E[XY] = 0$

$$\mathrm{Var}[X+Y] = ?$$

Let Z = X+Y.

Note that

$$\mu_Z = E[Z] = E[X] + E[Y]$$
$$= \mu_X + \mu_Y$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[Z] = E[(Z-\mu_Z)^2]$$

$$= E[((X+Y) - (\mu_X + \mu_Y))^2]$$

$$= E[((X-\mu_X) + (Y-\mu_Y))^2]$$

$$\mathrm{Var}[X+Y] = \mathrm{E}[((X-\mu_X)+(Y-\mu_Y))^2]$$

$$= \mathrm{E}[(X-\mu_X)^2] + 2\mathrm{E}[(X-\mu_X)(Y-\mu_Y)] + \mathrm{E}[(Y-\mu_Y)^2]$$

$$= \mathrm{Var}[X] + 2\mathrm{Cov}(X,Y) + \mathrm{Var}[Y]$$

If X and Y are independent

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$