Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and known variance σ^2.

Consider testing the simple versus simple hypotheses

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu = \mu_1$$

where μ_0 and μ_1 are fixed and known.

Suppose that $\mu_0 < \mu_1$.

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

The Test:

Reject H_0**, in favor of** H_1 **if**

$$\overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

came from the probability of making a Type I error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

The Test:

Reject H_0, in favor of H_1 if

$$\overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Question:

What about the Type II error?

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

	Your Decision: fail to reject H_0	Your Decision: reject H_0
H_0 true		Type I error
H_0 false	Type II error	

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = \mathrm{P}(\text{Type II Error})$$

$$= \mathrm{P}(\text{Fail to Reject } H_0 \text{ when false})$$

$$= \mathrm{P}\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \text{ when } \mu = \mu_1 \right)$$

$$= \mathrm{P}\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}; \mu_1 \right)$$

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- **It is locked in!**

$$\beta = P\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}; \mu_1 \right)$$

$$\overline{X} \sim N(\mu_1, \sigma^2/n)$$

$$= P\left(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}}; \mu_1 \right)$$

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = P\left(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} ; \mu_1 \right)$$

$$= P\left(Z \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = P\left(Z \leq \underbrace{\frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}}}_{} \right)$$

This is a fixed number, so compute the probability and that's your β!

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = P\left(Z \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

We could create the entire test starting from the "β point of view" and then α would be locked in.

Type II Error

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu = \mu_1$$
$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = P\left(Z \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

If we want to set both α and β we would have to free up the sample size as another unknown. (c and n)

Type II Error

	Your Decision: fail to reject H_0	Your Decision: reject H_0
H_0 true	✓	Type I error
H_0 false	Type II error	✓

Note: $\beta \neq 1 - \alpha$

Composite vs Composite

$$X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)\,, \quad \sigma^2 \textbf{ known}$$

$$H_0 : \mu \leq \mu_0 \quad \textbf{vs} \quad H_1 : \mu > \mu_0$$

$$H_0 : \mu \leq \mu_0$$
$$H_1 : \mu > \mu_0$$

Step One:

Choose an estimator for μ.

$$\hat{\mu} = \overline{X}$$

Step Two:

Give the "form" of the test.

Reject H_0, in favor of H_1 if $\overline{X} > c$, where c is to be determined.

$$H_0 : \mu \leq \mu_0$$
$$H_1 : \mu > \mu_0$$

Step Three:

Find c.

$$\alpha = \mathrm{P}(\text{Type I Error})$$
$$= \mathrm{P}(\text{Reject } H_0 \text{ when true})$$
$$= \mathrm{P}(\overline{X} > c \text{ when } \mu \leq \mu_0)$$
$$=$$

The definitions we have used for α and β are for simple hypotheses only.

Definitions:

- The level of significance or "size" of a test is denoted by α and is defined by

$$\alpha = \max_{\mu \in \mathrm{H}_0} \mathrm{P}(\text{Type I Error}) = \max_{\mu \in \mathrm{H}_0} \mathrm{P}(\text{Reject } \mathrm{H}_0;\, \mu)$$

$$\beta = \max_{\mu \in \mathrm{H}_1} \mathrm{P}(\text{Type II Error}) = \max_{\mu \in \mathrm{H}_1} \mathrm{P}(\text{Fail to Reject } \mathrm{H}_0;\, \mu)$$

Definitions:

- $1 - \beta$ is known as the **power of the test**

$$1 - \beta = 1 - \max_{\mu \in H_1} P(\text{Fail to Reject } H_0;\, \mu)$$

$$= \min_{\mu \in H_1} \left(1 - P(\text{Fail to Reject } H_0;\, \mu)\right)$$

$$= \min_{\mu \in H_1} P(\text{Reject } H_0;\, \mu)$$

High power is good!