

Let X_1, X_2, \dots, X_n be a random sample from any distribution with mean μ and variance σ^2 .

$$\sigma^2 = \text{Var}[X] := E[(X - \mu)^2]$$

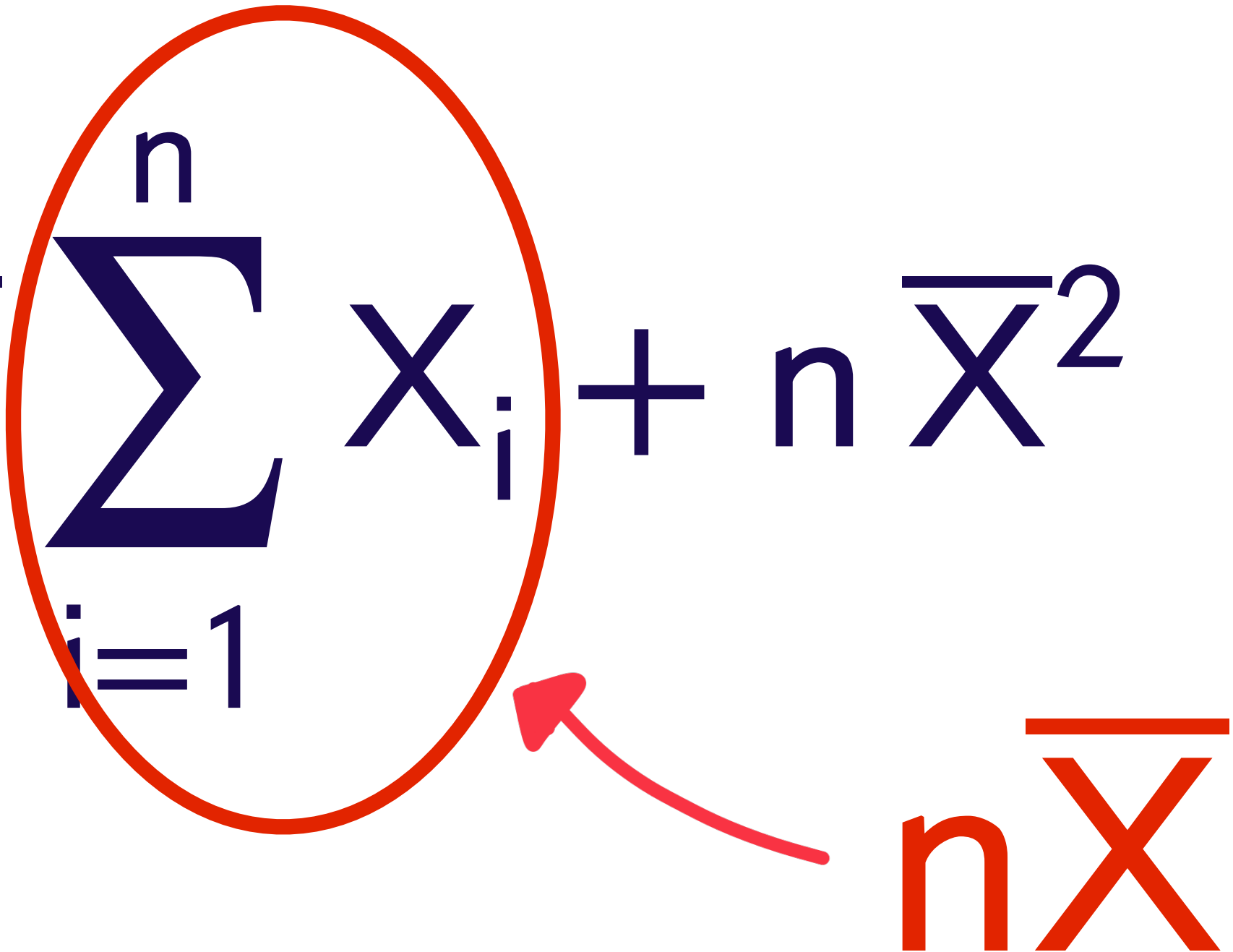
To estimate this from the sample, we could use

$$\tilde{S}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}$$

“sample variance”

- Currently, before numerical observations, this is a random variable.
- It has its own distribution, its own mean, and its own variance.

$$\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i^2 - 2\bar{x}x_i + \bar{x}^2)$$

$$= \sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + n\bar{x}^2$$


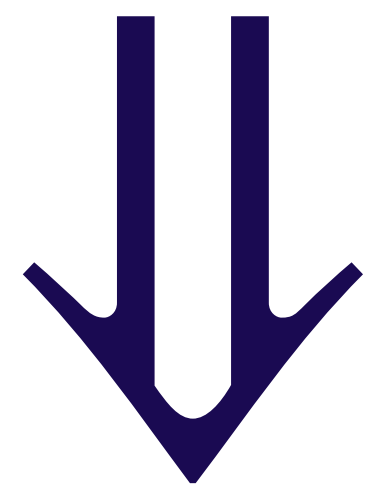
$n\bar{x}$

$$\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2$$

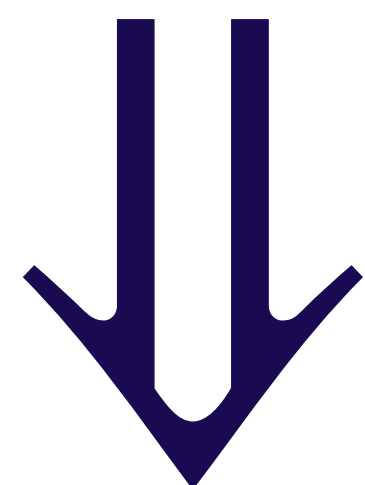
$$= \sum_{i=1}^n x_i^2 - n \left(\sum_{i=1}^n x_i / n \right)^2$$

$$= \sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i \right)^2}{n}$$

$$\text{Var}[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$



$$\begin{aligned} E[X^2] &= \text{Var}[X] + (E[X])^2 \\ &= \sigma^2 + \mu^2 \end{aligned}$$



$$E[\tilde{S}^2] = \dots = \frac{n-1}{n} \sigma^2$$

“sample variance”

Variant of Sample Variance:

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n - 1}$$

$$S^2 = \frac{n}{n - 1} \tilde{S}^2$$

$$\Rightarrow E[S^2] = \frac{n}{n - 1} E[\tilde{S}^2]$$

$$= \frac{n}{n - 1} \frac{n - 1}{n} \sigma^2 = \sigma^2$$

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n - 1}$$

is an **unbiased** estimator of σ^2 .

$$(i.e. E[S^2] = \sigma^2)$$

This is the sample variance that we will use.

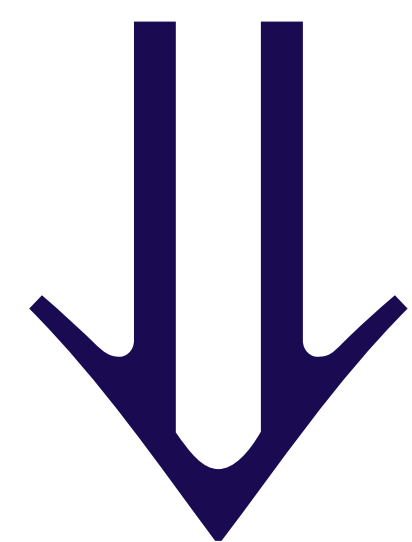
$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

For the normal distribution, these are independent!

Aside:

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

independent

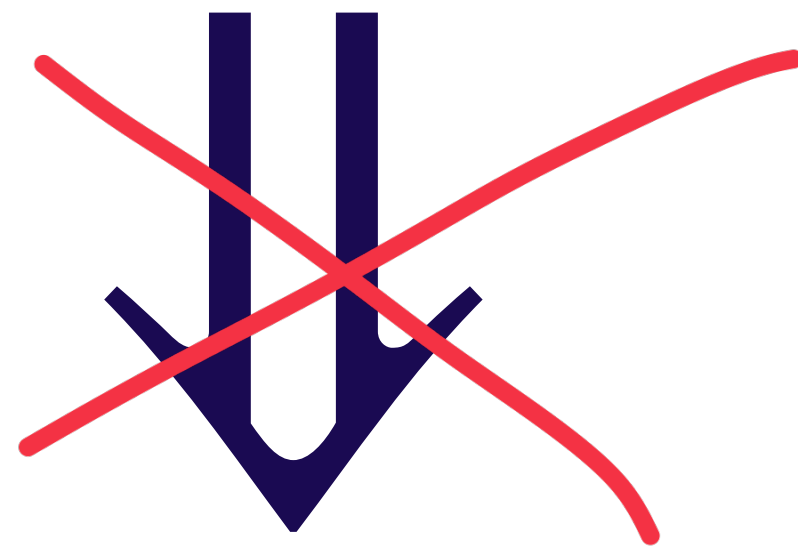


$$X_1 + X_2 \sim \chi^2(n_1 + n_2)$$

Aside:

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

independent



$$X_1 - X_2 \sim \chi^2(n_1 - n_2)$$

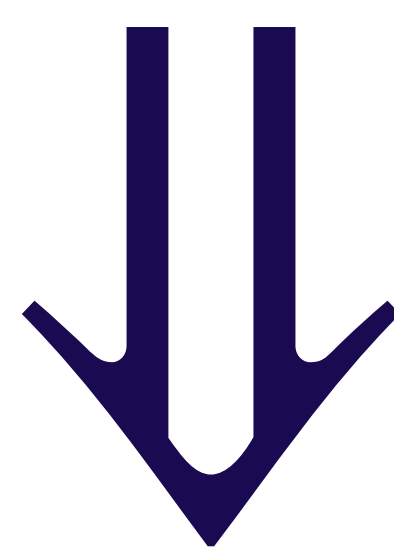
However,

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

$$X_3 \sim ?$$

$$X_1 = X_2 + X_3$$

and X_2 and X_3 independent



$$X_3 = X_1 - X_2 \sim \chi^2(n_1 - n_2)$$

$$\sum_{i=1}^n (x_i - \mu)^2$$

$$= \sum_{i=1}^n (x_i - \bar{X} + \bar{X} - \mu)^2$$

$$= \sum_{i=1}^n (x_i - \bar{X})^2$$

$$+ 2(\bar{X} - \mu) \sum_{i=1}^n (x_i - \bar{X})$$

$$+ n(\bar{X} - \mu)^2$$

$$= 0$$

$$\begin{aligned} & \sum x_i - n\bar{X} \\ & \sum x_i - \sum x_i \end{aligned}$$

$$\sum_{i=1}^n (x_i - \mu)^2$$

$$= \sum_{i=1}^n (x_i - \bar{X} + \bar{X} - \mu)^2$$

$$= \sum_{i=1}^n (x_i - \bar{X})^2 + n (\bar{X} - \mu)^2$$

Let X_1, X_2, \dots, X_n be a random sample from the **normal** distribution with mean μ and variance σ^2 .

$$\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2}$$


$$= \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} =$$


$$\sum_{i=1}^n$$

$$\left(\underbrace{\frac{X_i - \mu}{\sigma}}_{N(0, 1)} \right)^2$$

$\chi^2(n)$




$\chi^2(1)$



$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2}$$

$$= \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} = \left(\frac{\bar{X} - \mu}{\underbrace{\sigma/\sqrt{n}}_{N(0,1)}} \right)^2$$

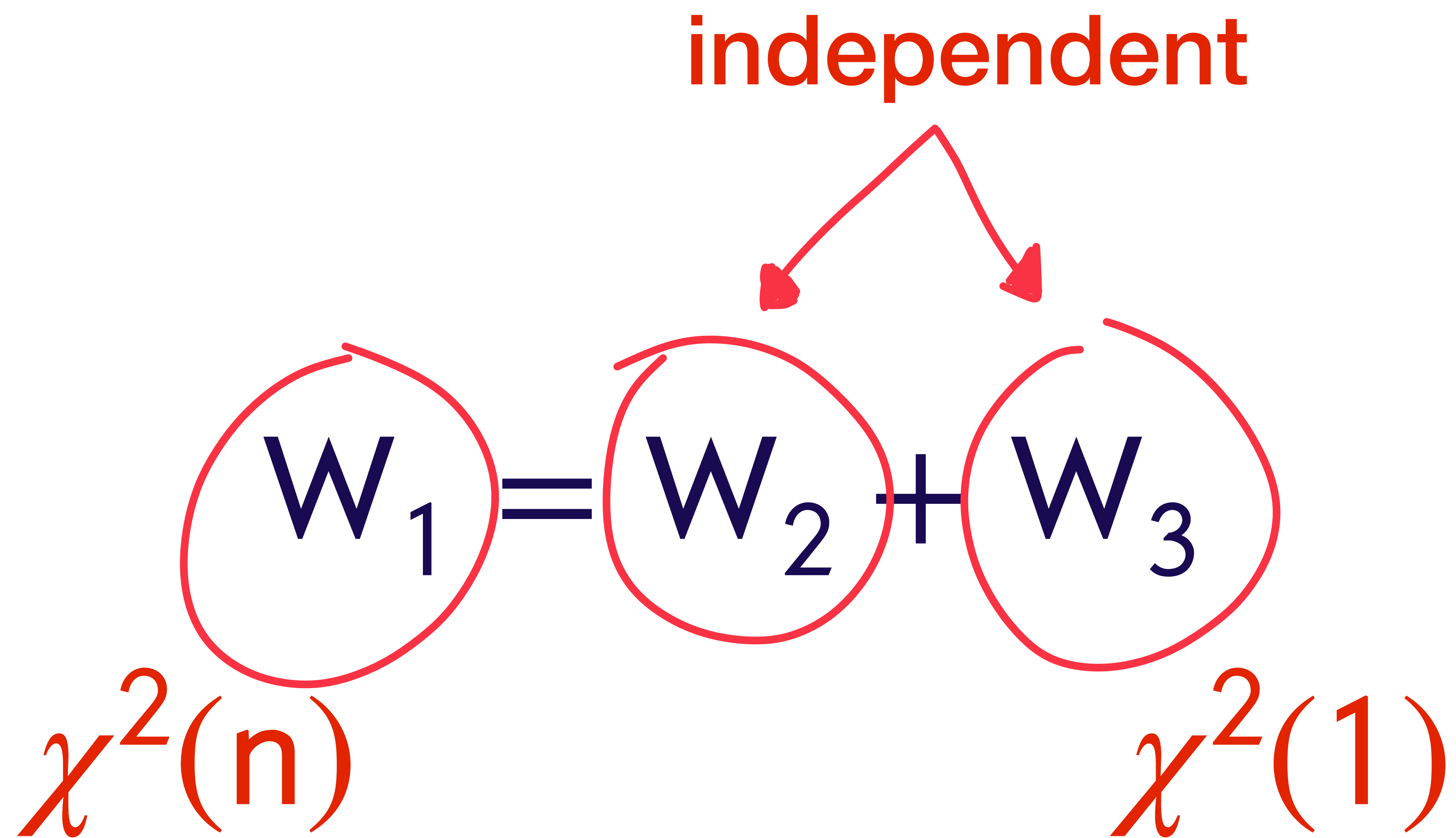
 $\chi^2(1)$

Let X_1, X_2, \dots, X_n be a random sample from the **normal** distribution with mean μ and variance σ^2 .

$$= \frac{(n-1)S^2}{\sigma^2}$$

$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} = \boxed{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2}} + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

Let X_1, X_2, \dots, X_n be a random sample from the **normal** distribution with mean μ and variance σ^2 .



Let X_1, X_2, \dots, X_n be a random sample from the **normal** distribution with mean μ and variance σ^2 .

$$\Rightarrow W_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$