Let  $X_1, X_2, ..., X_n$  be a random sample from the normal distribution with mean  $\mu$  and known variance  $\sigma^2$ .

Consider testing the simple versus simple hypotheses

$$H_0: \mu = \mu_0$$
  $H_1: \mu = \mu_1$ 

where  $\mu_0$  and  $\mu_1$  are fixed and known.

Suppose that  $\mu_0 < \mu_1$ .

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

#### The Test:

Reject H<sub>0</sub>, in favor of H<sub>1</sub> if

$$\overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

came from the probability of making a Type I error

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 

 $\mu_0 < \mu_1$ 

#### The Test:

Reject H<sub>0</sub>, in favor of H<sub>1</sub> if

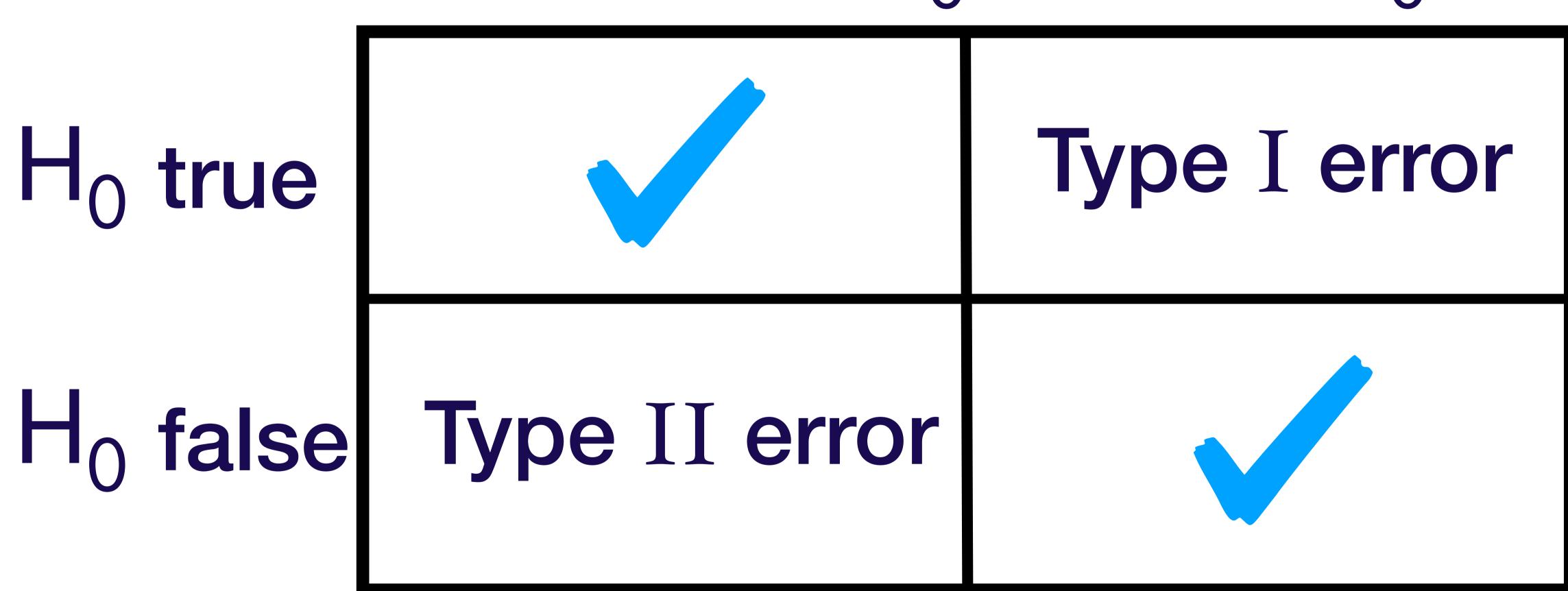
$$\overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

### Question:

What about the Type II error?

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

# Your Decision fail to reject $H_0$ reject $H_0$



It is locked in!

$$\beta = P(Type II Error)$$

= P(Fail to Reject H<sub>0</sub> when false)

$$= P\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \text{ when } \mu = \mu_1\right)$$

$$= P\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}; \mu_1\right)$$

 $H_0: \mu = \mu_0$ 

$$H_1: \mu = \mu_1$$

$$\mu_0 < \mu_1$$

It is locked in!

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

$$\beta = P\left(\overline{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}; \mu_1\right)$$

$$\overline{X} \sim N(\mu_1, \sigma^2/n)$$

$$= P\left(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}}; \mu_1\right)$$

It is locked in!

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

$$= P \left( \frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}} \right) = \frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}}$$

$$= P \left( Z \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

It is locked in!

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

$$\beta = P \left( Z \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma / \sqrt{n}} \right)$$

This is a fixed number, so compute the probability and that's your  $\beta$ !

It is locked in!

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

$$\beta = P \left( Z \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma / \sqrt{n}} \right)$$

We could create the entire test starting from the " $\beta$  point of view" and then  $\alpha$  would be locked in.

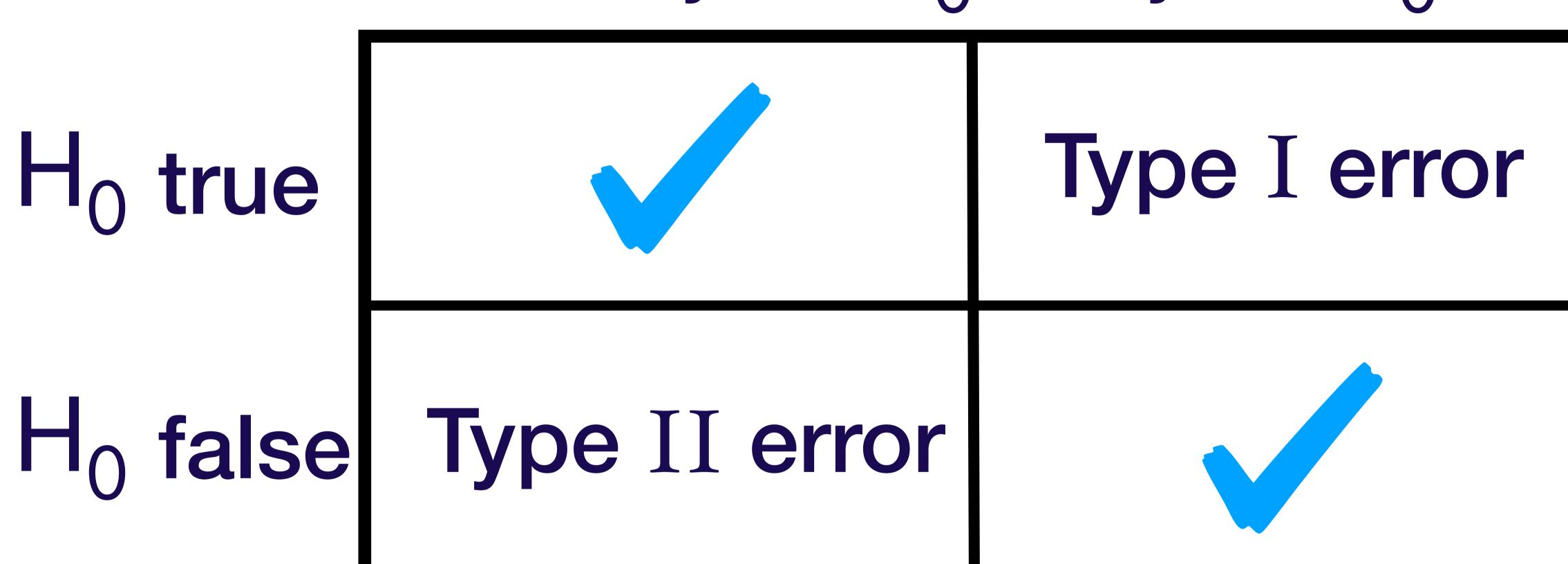
It is locked in!

$$H_0: \mu = \mu_0$$
 $H_1: \mu = \mu_1$ 
 $\mu_0 < \mu_1$ 

$$\beta = P \left( Z \le \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma / \sqrt{n}} \right)$$

If we want to set both  $\alpha$  and  $\beta$  we would have to free up the sample size as another unknown. (c and n)





Note:  $\beta \neq 1 - \alpha$ 

#### Composite vs Composite

$$X_1, X_2, ..., X_n \sim N(\mu, \sigma^2), \sigma^2 \text{ known}$$

$$H_0: \mu \leq \mu_0$$
 vs  $H_1: \mu > \mu_0$ 

 $H_0: \mu \leq \mu_0$ 

$$H_1$$

 $H_1: \mu > \mu_0$ 

#### Step One:

Choose an estimator for  $\mu$ .

#### Step Two:

Give the "form" of the test.

Reject  $H_0$ , in favor of  $H_1$  if X > c, where c is to be determined.

$$H_0: \mu \leq \mu_0$$

$$H_1: \mu > \mu_0$$

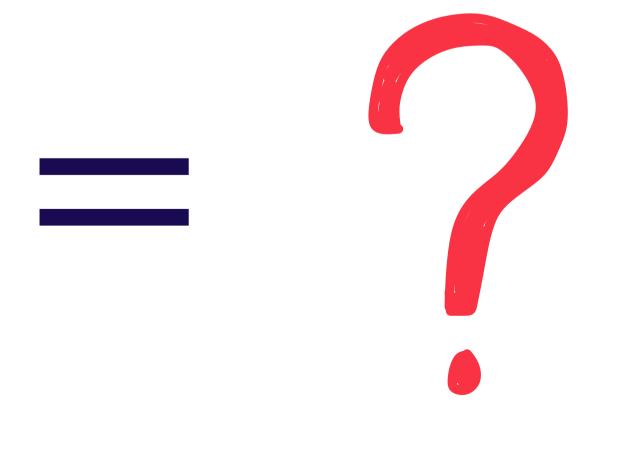
## Step Three:

Find c.

$$\alpha = P(Type I Error)$$

 $= P(Reject H_0 when true)$ 

$$= P(\overline{X} > c \text{ when } \mu \leq \mu_0)$$



The definitions we have used for  $\alpha$  and  $\beta$  are for simple hypotheses only.

#### Definitions:

• The level of significance or "size" of a test is denoted by  $\alpha$  and is defined by

$$\alpha = \max P(Type I Error)$$

= 
$$\max_{\mu \in H_0} P(\text{Reject } H_0; \mu)$$

$$\beta = \max P(Type II Error)$$

=  $\max_{\mu \in H_1} P(Fail to Reject H_0; \mu)$ 

#### Definitions:

 $-1 - \beta$  is known as the power of the test

$$1 - \beta = 1 - \max_{\mu \in H_1} P(Fail to Reject H_0; \mu)$$

$$= \min_{\mu \in H_1} \left( 1 - P(\text{Fail to Reject } H_0; \mu) \right)$$

= 
$$\min_{\mu \in H_1} P(\text{Reject } H_0; \mu)$$
 High power is good!