A6 Animation Written Problem

Problem 1: Interpolating in 2D. On another page are pictures of a square object (with its edges marked so you can tell them apart) under three pairs of linear transformations. For each pair:

1. *Write the 2x2 transformation matrices for each pose; call them M0 and M1.*

Answer:

$$\text{Pair1:}\ M_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M_0 = \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\text{Pair2:}\ M_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M_0 = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

$$\text{Pair3:}\ M_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M_0 = \begin{bmatrix} \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

2. *Write the matrix* $M_{0.5}^{lin}$ *by linearly interpolating element wise halfway between M0 and M1.*

Answer:

$$\text{Pair1:}\ M_{0.5}^{lin} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\text{Pair2:}\ M_{0.5}^{lin} = \begin{bmatrix} \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

$$\text{Pair3:}\ M_{0.5}^{lin} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

3. *Write each of these two poses in the form Mi = RiSi, where Ri is a rotation and Si is a symmetric matrix with positive entries.*

Answer:

$$\text{Pair1:}\ M_0 = R_0 S_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \qquad R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad S_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

$$M_1 = R_1 S_1 = \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad S_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

Pair2: $$M_0 = R_0 S_0 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix} \qquad R_0 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad S_0 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$M_1 = R_1 S_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix} \qquad R_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad S_1 = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Pair3: $$M_0 = R_0 S_0 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix} \qquad R_0 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad S_0 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$M_1 = R_1 S_1 = \begin{bmatrix} \frac{3\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \qquad R_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad S_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

4. Use this decomposition to interpolate a matrix $M_{0.5}^{pol}$ that is halfway between M0 and M1.

Answer:

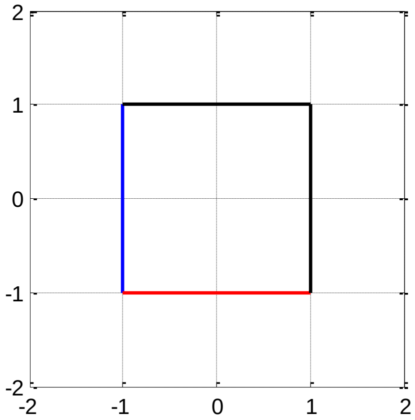
Pair1: $$S_{0.5}^{lin} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \qquad R_{0.5}^{pol} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad M_{0.5}^{pol} = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

Pair2: $$S_{0.5}^{lin} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_{0.5}^{pol} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_{0.5}^{pol} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

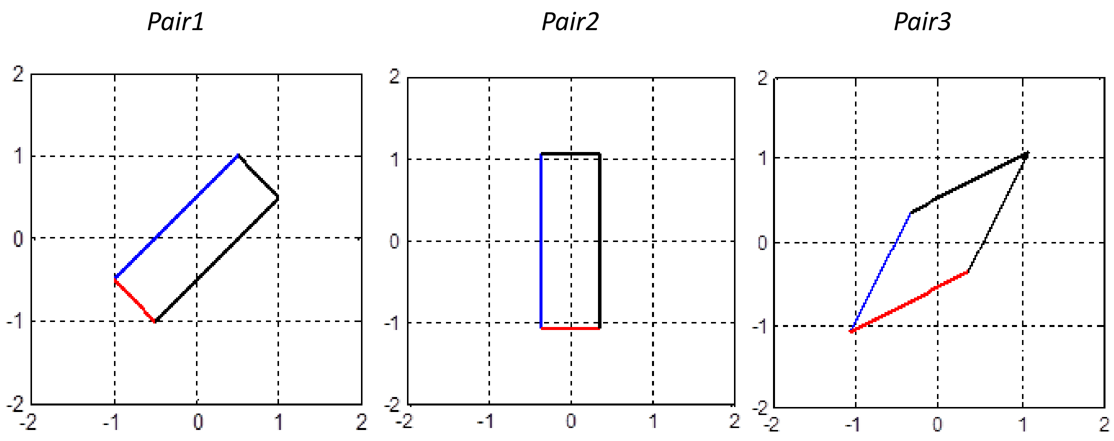
Pair3: $$S_{0.5}^{lin} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \qquad R_{0.5}^{pol} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_{0.5}^{pol} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

5. Draw the two intermediate states of the object (you can print out the PDF and draw on it if you want).

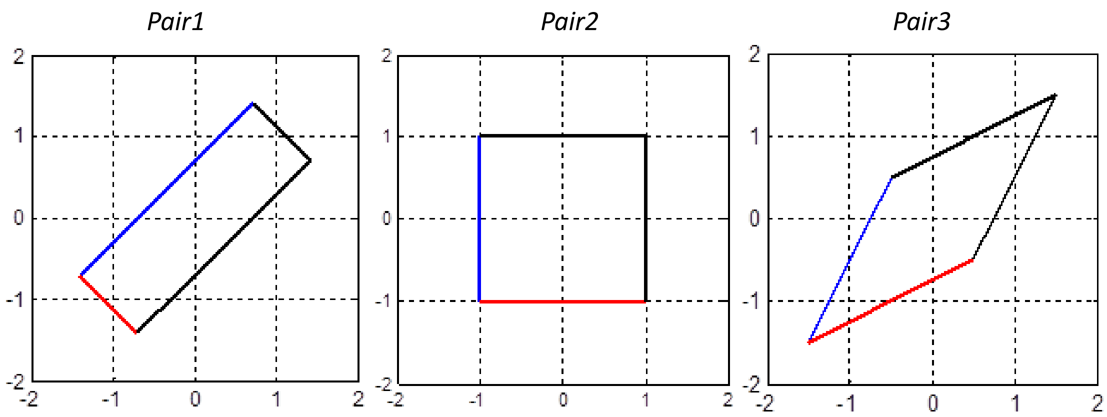
Answer:



Intermediate States of Linear Interpolating (t=0.5)



Intermediate States of Polar Decomposition Interpolating (t=0.5)



Problem 2: Interpolating 3D Rotations

1. Write down the quaternion q1 that corresponds to the identity rotation and the quaternion q2 that corresponds to a rotation of 180 degrees around the x axis.

Answer:

$$M_{q1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad q_1 = [1, \quad 0, \quad 0, \quad 0]$$

$$M_{q2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad q_2 = [0, \quad 1, \quad 0, \quad 0]$$

2. Using the formula for spherical linear interpolation, generate the quaternion q3 that is one fourth of the way from q1 to q2. What are the axis and angle of the rotation q3 represents?

Answer:

$$\varphi = \cos^{-1}(q_1 \cdot q_2)$$

$$q_3 = \frac{q_1\, sin(1-t)\, \varphi + q_2\, sin\, t\, \varphi}{sin\varphi} \quad , t = 0.25$$

$$q_3 = [\cos(22.5°)\, , sin(22.5°), 0,0]$$

The axis and angle represents of *q3* a rotation of 45 degrees around the orientation vector (1,0,0).
$q_3 = [0.924, \quad 0.383, \quad 0, \quad 0]$

3. Convert q3 to a 3x3 rotation matrix M(q3). Verify that starting with the identity and applying this matrix 4 times results in the rotation corresponding to q2.

Answer:

$$M_{q3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix}$$

$$M_{q1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_{q2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$M_{q2} = M_{q3}M_{q3}M_{q3}M_{q3}M_{q1}$$

MATLAB simulation:

```
>> M1

M1 =

     1     0     0
     0     1     0
     0     0     1
```

```
>> M2

M2 =

     1     0     0
     0    -1     0
     0     0    -1
```

```
>> M3

M3 =

    1.0000         0         0
         0    0.7071   -0.7071
         0    0.7071    0.7071
```

```
>> M3*M3*M3*M3*M1

ans =

    1.0000         0         0
         0   -1.0000         0
         0         0   -1.0000
```

4. Write down the matrix corresponding to a rotation of 90 degrees around x followed by a rotation of 90 degrees around y, and convert it to a quaternion q4. Do the same with y and then z, producing q5.

Answer:

$$M_{q4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad q_4 = [0.5, \quad 0.5, \quad 0.5, \quad -0.5]$$

$$M_{q5} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad q_5 = [0.5, \quad -0.5, \quad 0.5, \quad 0.5]$$

5. Using the formula for spherical linear interpolation, generate the quaternion q6 that is one fourth of the way from q4 to q5. What are the axis and angle of the rotation q6 represents?

Answer:

$\varphi = \cos^{-1}(q_4 \cdot q_5)$

$q_6 = \dfrac{q_4\, sin(1-t)\,\varphi + q_5 \,sin\, t\,\varphi}{sin\varphi} \quad, t = 0.25$

$q_6 = [0.6533, \quad 0.2706, \quad 0.6533, \quad -0.2706]$

The axis and angle of *q6* represents a rotation of **98.42** degrees around the orientation vector **(0.3574, 0.8629, -0.3574)**.

6. *Compute the quaternion q7 for the rotation that rotates from q4 to q6—that is, M(q6) = M(q7)M(q4). (It might be easiest to do this by converting everything to matrices, determining the rotation, and then converting back.) Verify that starting with M(q4) and applying M(q7) four times, you end up at M(q5).*

Answer:

$$M_{q7} = M_{q6}\ M_{q4}^{-1} = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$q_7 = [0.924, \quad 0, \quad 0, \quad 0.383]$

$$M_{q4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, M_{q5} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$M_{q5} = M_{q7} M_{q7} M_{q7} M_{q7} M_{q4}$

MATLAB simulation:

```
>> M4

M4 =

     0     1     0
     0     0    -1
    -1     0     0
```

```
>> M5

M5 =

     0    -1     0
     0     0     1
    -1     0     0
```

```
>> M7

M7 =

    0.7071   -0.7071         0
    0.7071    0.7071         0
         0         0    1.0000
```

```
>> M7*M7*M7*M7*M4

ans =

         0   -1.0000   -0.0001
         0   -0.0001    1.0000
   -1.0000         0         0
```

7. *What are the axis and angle of the rotation q7 represents? Spend a few minutes holding a book and rotating it around its various axes, until it seems clear that this is the right answer.*

Answer:

The axis and angle of *q7* represents a rotation of **45** degrees around the orientation vector **(0, 0, 1)**.

Extra Credit:

Answer:

For frame f_0, f_1, f_2, f_3,we get M_0, M_1, M_2, M_3. The time *t* for the interpolation is determined by how far along frame f lies between $f_1$and f_2.

$$M = \begin{bmatrix} RS & T \\ 0 & 1 \end{bmatrix}$$

Scale:

$$S_t = \left(-\frac{1}{2}t^3 + t^2 - \frac{1}{2}t\right)S_0 + \left(\frac{3}{2}t^3 - \frac{5}{2}t^2 + 1\right)S_1 + \left(-\frac{3}{2}t^3 + 2t^2 + \frac{1}{2}t\right)S_2 + \left(\frac{1}{2}t^3 - \frac{1}{2}t^2\right)S_3$$

Translation:

$$T_t = \left(-\frac{1}{2}t^3 + t^2 - \frac{1}{2}t\right)T_0 + \left(\frac{3}{2}t^3 - \frac{5}{2}t^2 + 1\right)T_1 + \left(-\frac{3}{2}t^3 + 2t^2 + \frac{1}{2}t\right)T_2 + \left(\frac{1}{2}t^3 - \frac{1}{2}t^2\right)T_3$$

Rotation:

For rotation matrixes R_0, R_1, R_2, R_3,we get quaternions q_0, q_1, q_2, q_3. The Catmull-Rom interpolation is a follows:

$\varphi = cos^{-1}(q_1 \cdot q_2)$

$$q_t = \frac{q_0\, sin\left[\left(-\frac{1}{2}t^3 + t^2 - \frac{1}{2}t\right)\varphi\right] + q_1\, sin\left[\left(\frac{3}{2}t^3 - \frac{5}{2}t^2 + 1\right)\varphi\right] + q_2\, sin\left[\left(-\frac{3}{2}t^3 + 2t^2 + \frac{1}{2}t\right)\varphi\right] + q_3\, sin\left[\left(\frac{1}{2}t^3 - \frac{1}{2}t^2\right)\varphi\right]}{sin\varphi}$$