Digital Communication

Unit No. 1

Random Processes & Noise

Course 2019

SUBJECT : DIGITAL COMMUNICATION (304181)

UNIT I : RANDOM SIGNAL & NOISE

TE (E&TC)

SEMESTER - I AY 2025-2026

LECTURE - I

CO1 : Apply The Knowledge Of Probability And Statistical Calculations On Random Signal Analysis.

Dr. S. M. Jagtap
MVPS'S KBTCOE , Nashik

Objectives

To study Mathematical definition of a random process, Stationary processes, Mean, Correlation & Covariance function

To study Properties of strictly stationary process and Wide Sense stationary Process

Unit 1: Contents

- **Random Processes**:
  - Mathematical definition of a random process.
  - Stationary processes.
  - Mean, Correlation and Covariance Functions
  - Ergodic process
  - Transmission of a random process through a LTI filter
  - Power spectral density (PSD)
- **Mathematical Representation of Noise:**
  - Some Sources of Noise
  - Frequency-domain Representation of Noise
  - Superposition of Noises
  - Linear Filtering of Noise
  - In-phase and Quadrature components of noise
  - Representation of Noise using Orthonormal Coordinates

References

- T1: Taub, Schilling and Saha, “Taub’s Principles of Communication Systems”, 4th Edition, McGraw-Hill.
- T2: Modern Digital & Analog Communication Systems, by B. P. Lathi 4th Ed.
- R5: Simon Haykin, —Digital Communication Systems‖, John Wiley & Sons, Fourth Edition.

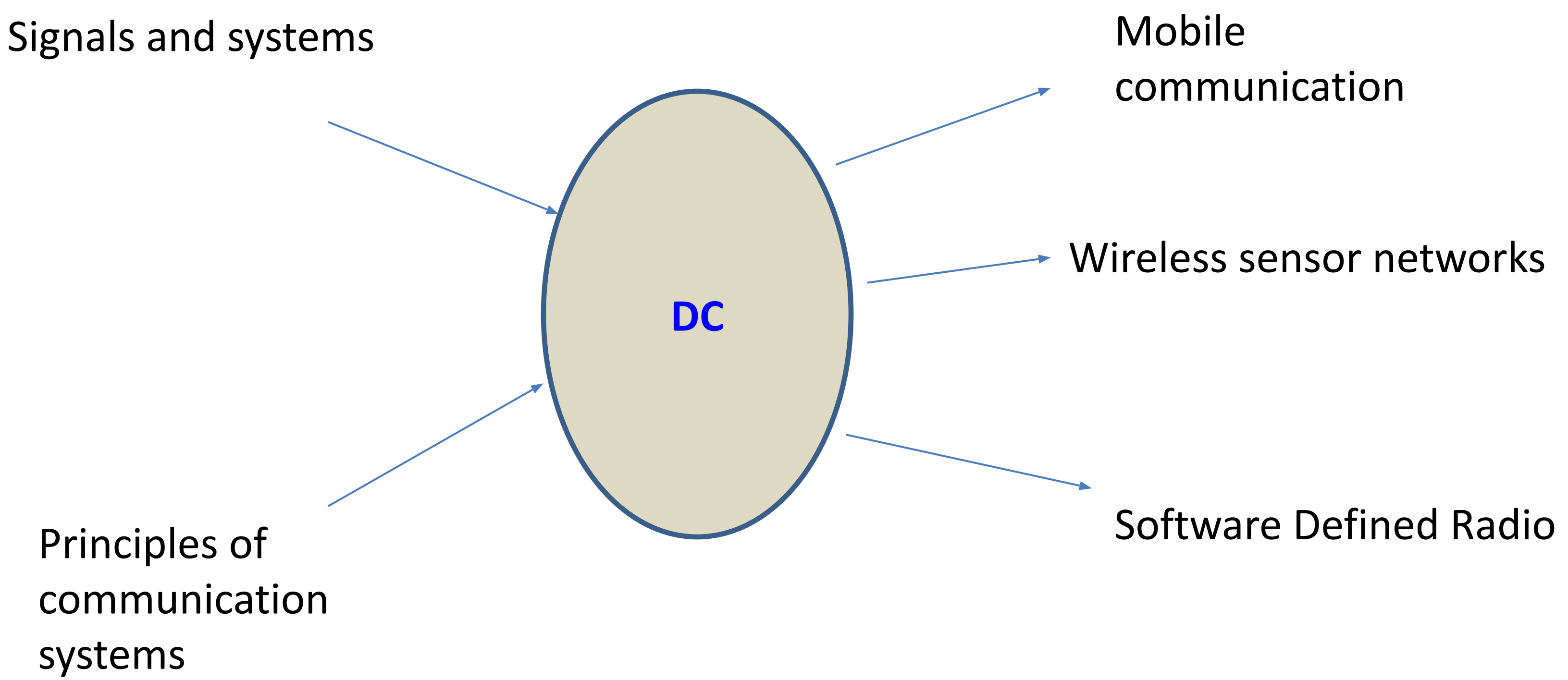
Unit 1: Objectives

- To understand random signals and random process.
- To define and calculate mean, correlation and covariance functions of random process.
- To describe and define various types of random process e.g. Ergodic and Gaussian process etc.
- To understand transmission of a random process through a linear filter.
- To understand mathematical representation of noise and study its in-phase and quadrature components.

Pre-requisites

- Signals and their classification.
- Probability theory, Random variables, PDF and CDF
- LTI system
- Noise and its types.
- Fourier Transform and Fourier series representation of signals

Mapping of Digital communication



Importance of Random Processes

- Random variables and processes talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random variable
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability

Some live examples of RP

- Performance of any player in a cricket match.
- Number of Covid-19 patients per day.

Contents

Unit III : Random Signal & Noise

- Introduction, Mathematical definition of a random process, Stationary processes, Mean, Correlation & Covariance function
- Wide Sense stationary Process
- Ergodic processes
- Transmission of a random process through a LTI filter, Power spectral density, Gaussian process
- Noise, Narrow band noise, Representation of narrowband noise in terms of in phase & quadrature components.

Deterministic and random processes :

- both continuous functions of time (usually)

- Deterministic processes :

  physical process is represented by explicit mathematical relation

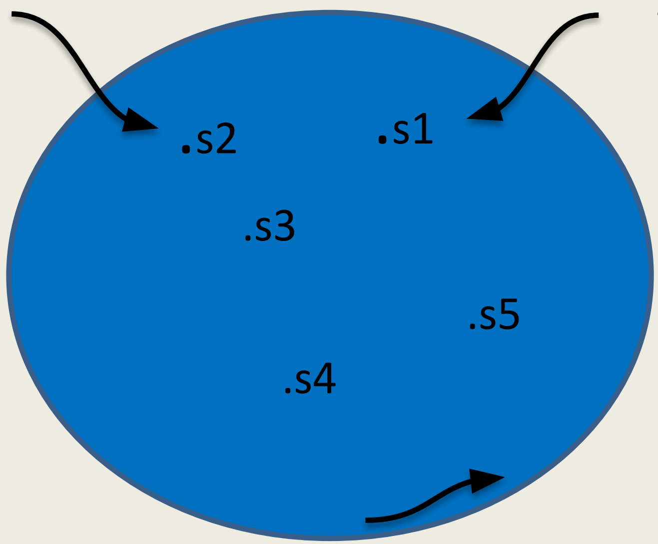
- Random processes :

  result of a large number of separate causes. Described in probabilistic terms and by properties which are averages

Random Process Or Stochastic Process

- **Sample Space is collection of all possible sample points**
- **Sample Space of a random variable X representing temperature at 12 O'clock**

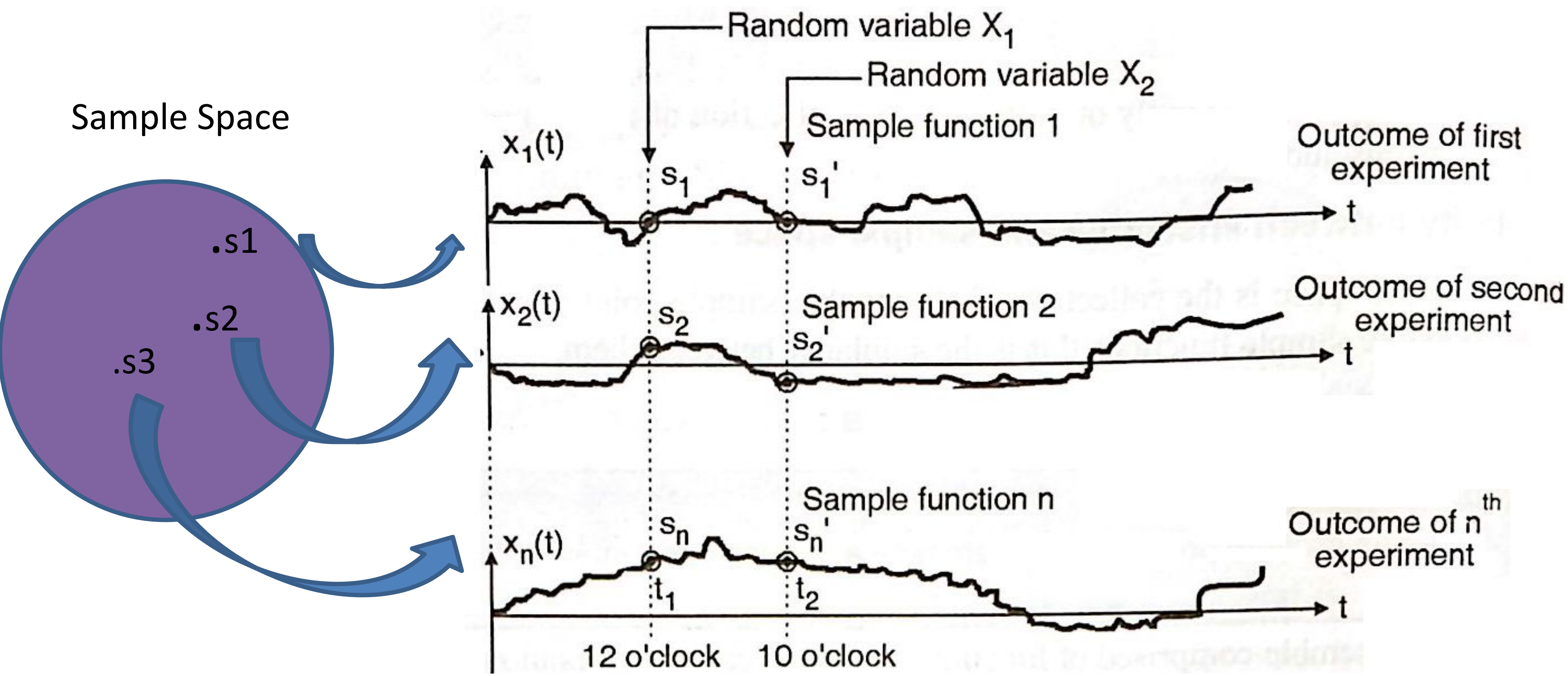
Temperature at 12 on day 2



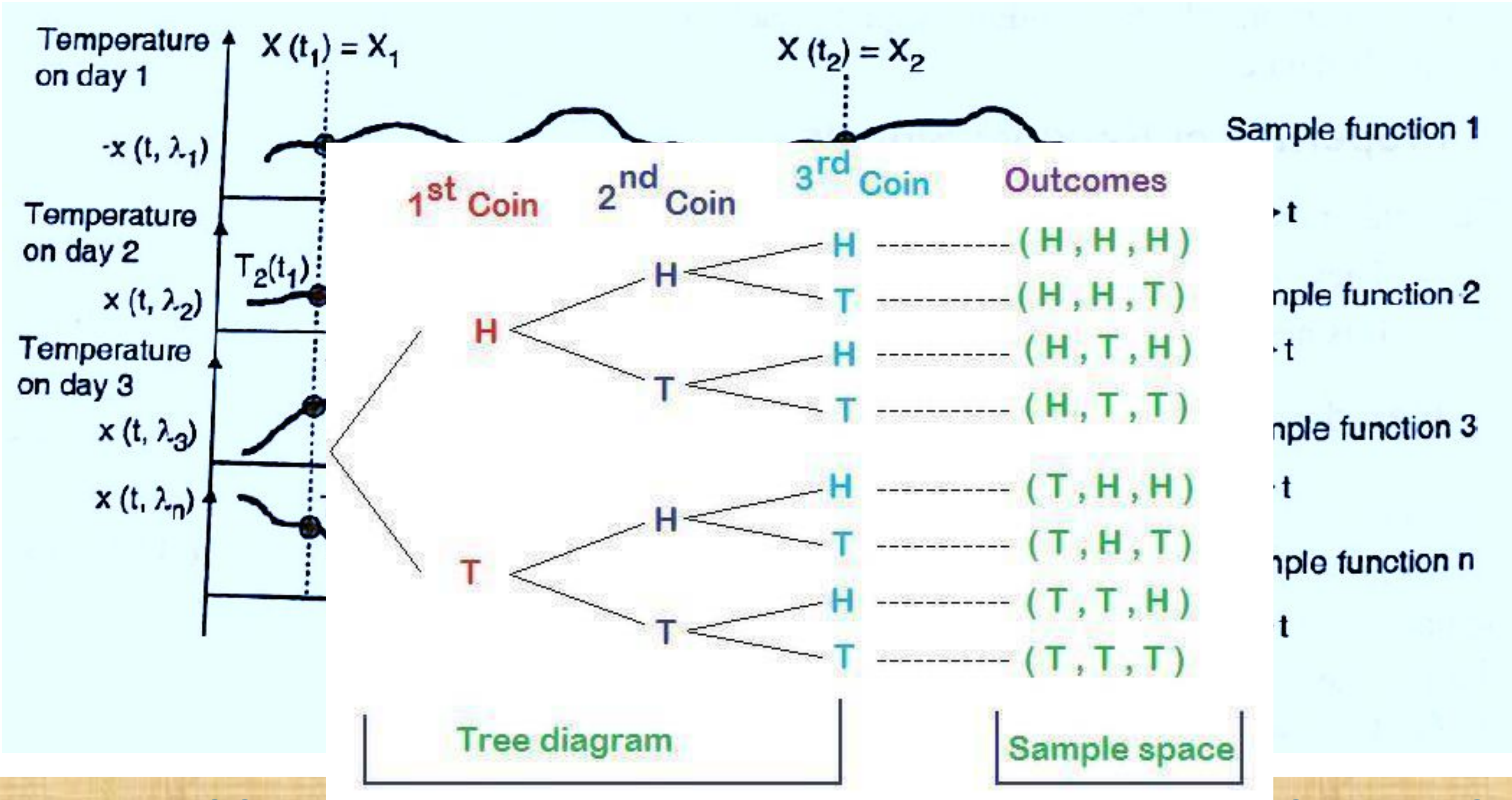
Temperature at 12 on day 1

Sample Space

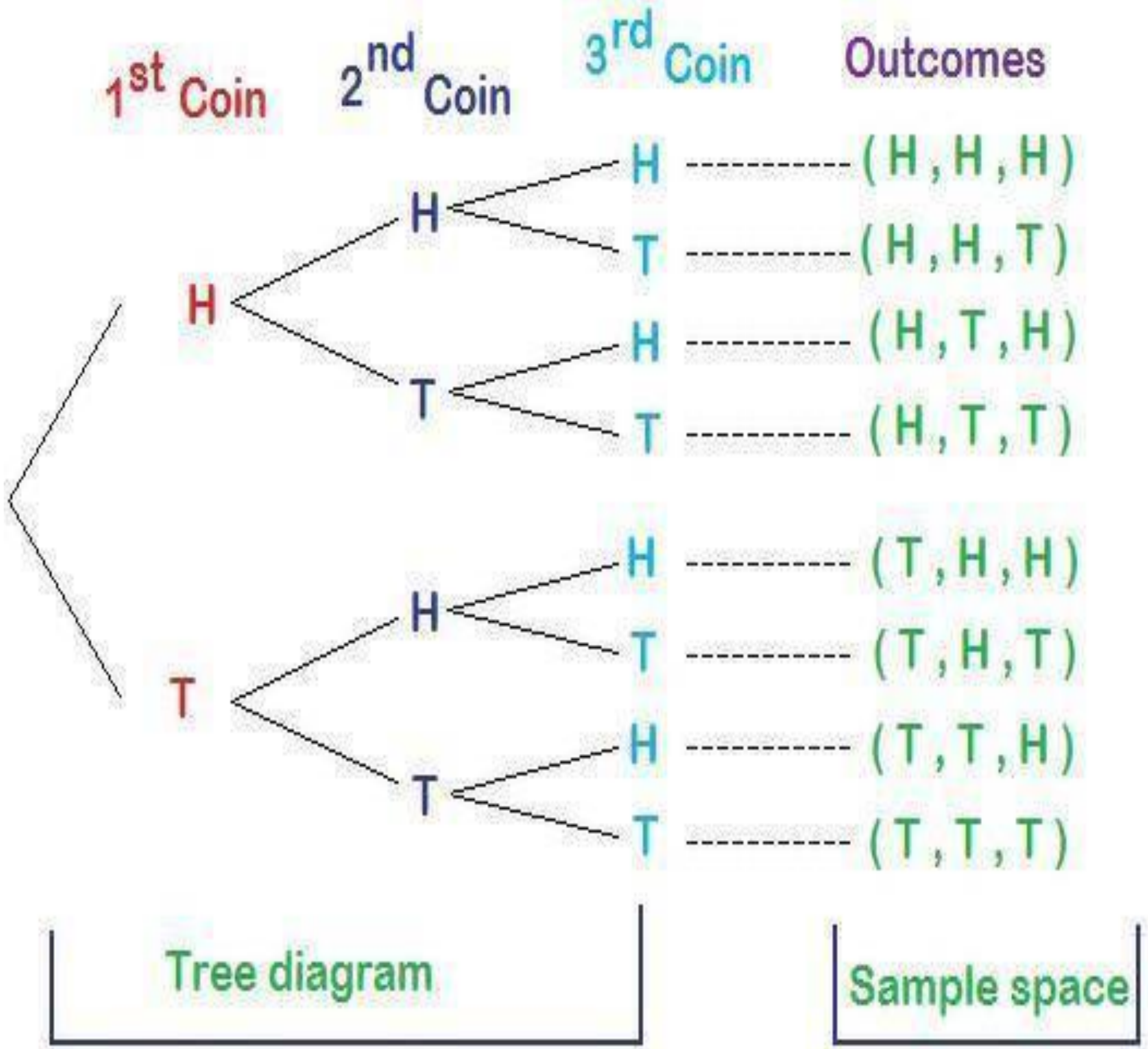
Ensemble of A Sample Function

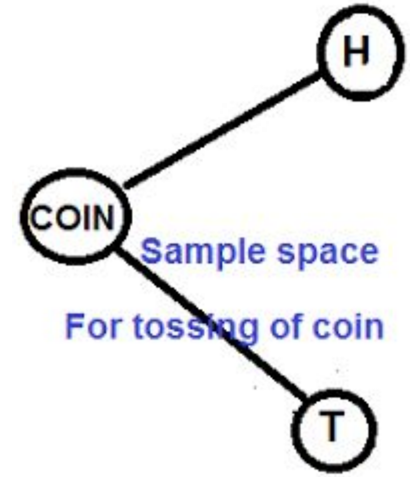


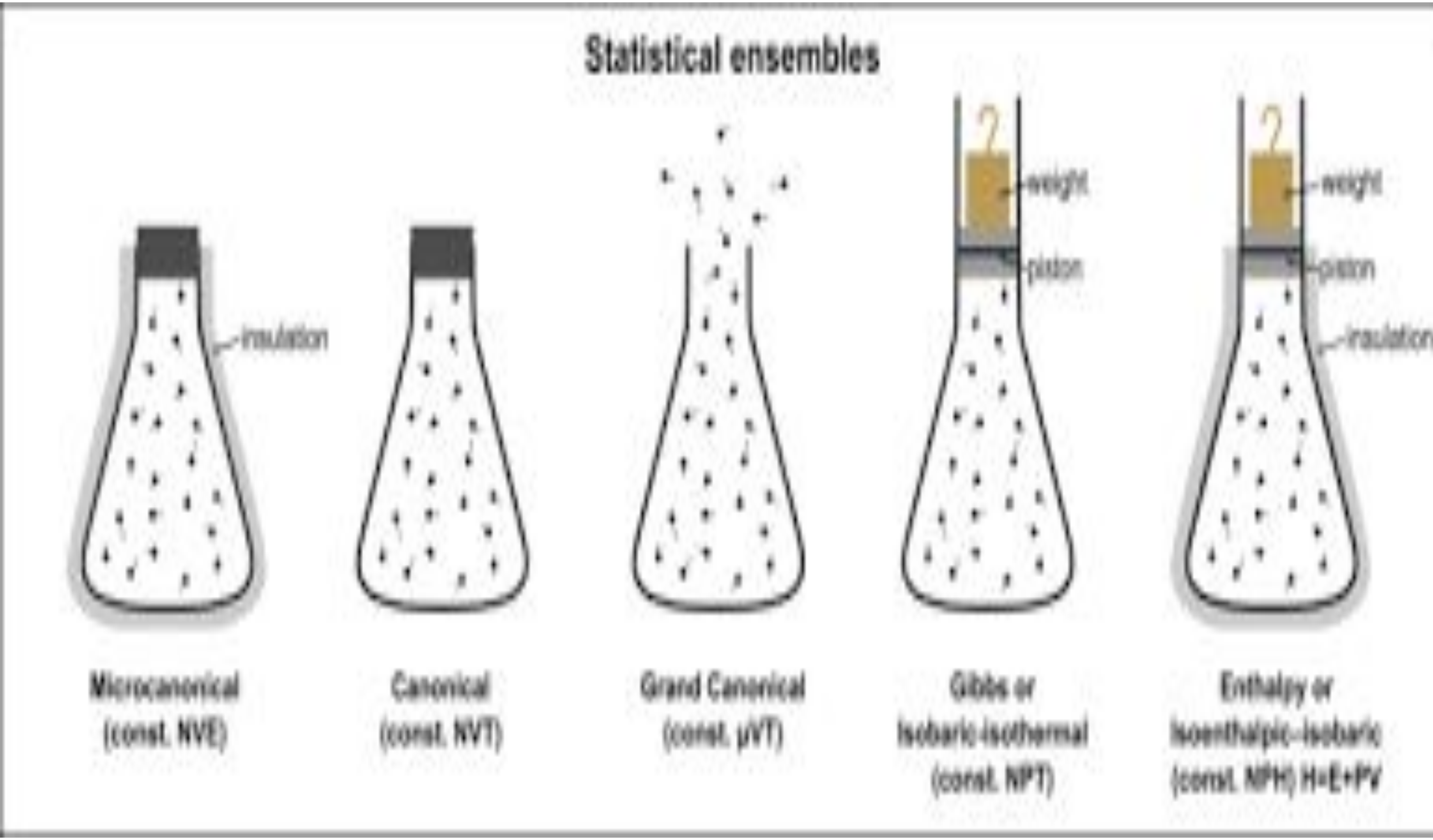
Sample Function



- **Ensemble means family or collection of all the possible sample function.**


1st Coin
2nd Coin
3rd Coin
Outcomes
H
T
H
T
H
T
H
T
H
T
H
T
H
T
(H,H,H)
(H,H,T)
(H,T,H)
(H,T,T)
(T,H,H)
(T,H,T)
(T,T,H)
(T,T,T)
Tree diagram
Sample space


H
COIN
Sample space
For tossing of coin
T


Statistical ensembles
insulation
weight
piston
weight
piston
insulation
Microcanonical
(const. NVE)
Canonical
(const. NVT)
Grand Canonical
(const. μVT)
Gibbs or
Isobaric-isothermal
(const. NPT)
Enthalpy or
Isoenthalpic-isobaric
(const. NPH) H=E+PV

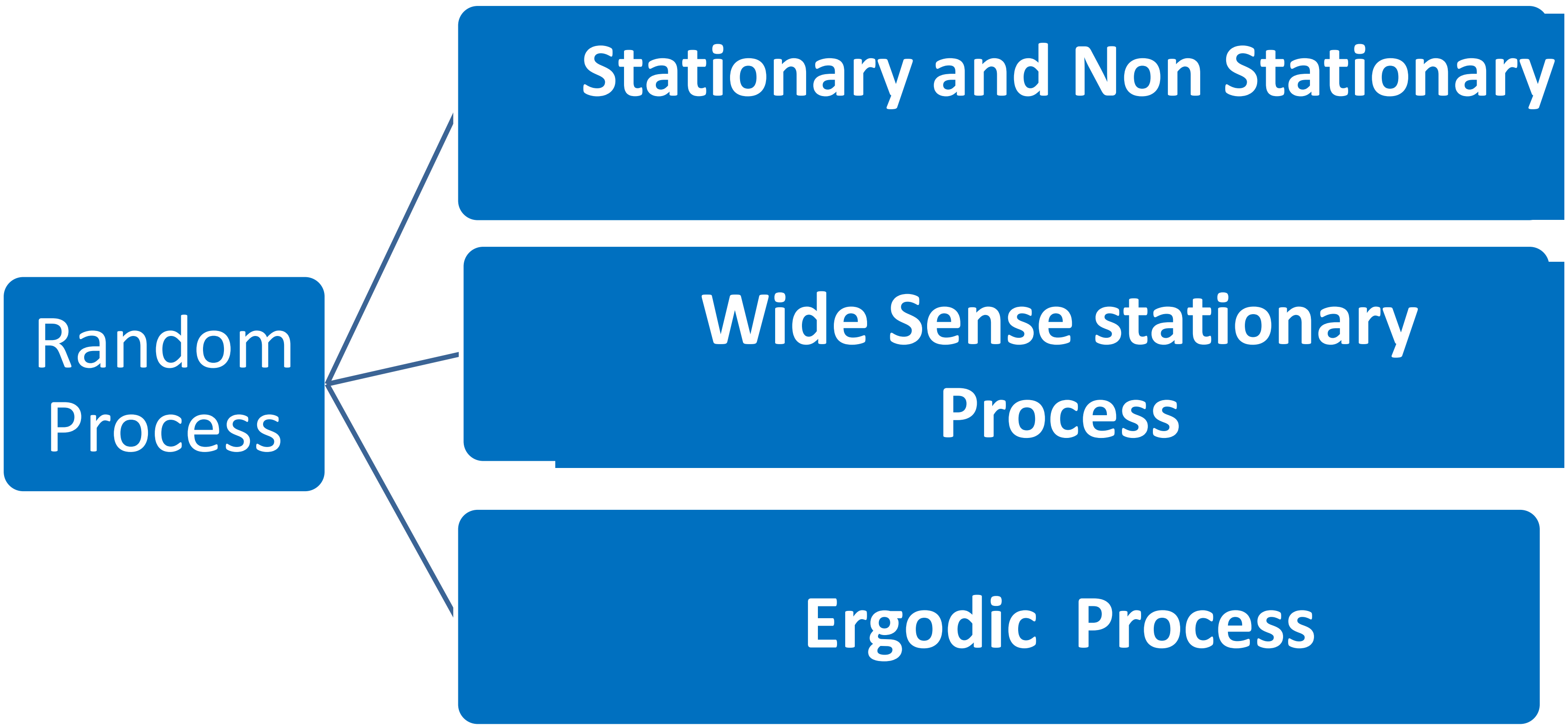
Time Average

- **Time average of a random process is defined as the statistical average obtained by considering time t as a variable.**

- **The Time average of any sample function x(t) can be defined as,**

$$m_x(T) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\, dt$$

- **In the time average we have time 't' treated as a variable. This the difference between time average and ensemble average.**


Random Process
Stationary and Non Stationary
Wide Sense stationary Process
Ergodic Process

Random Processes: significance

Randomness or unpredictability is a fundamental property of information.

For example,

- The speech waveform recorded by a microphone,
- The signal received by communication receiver
- The temperature of a certain city at noon or
- The daily record of stock-market data represents random variables that change with time.

- ***How do we characterize such data?*** Such data are characterized as ***random*** or ***stochastic processes***.

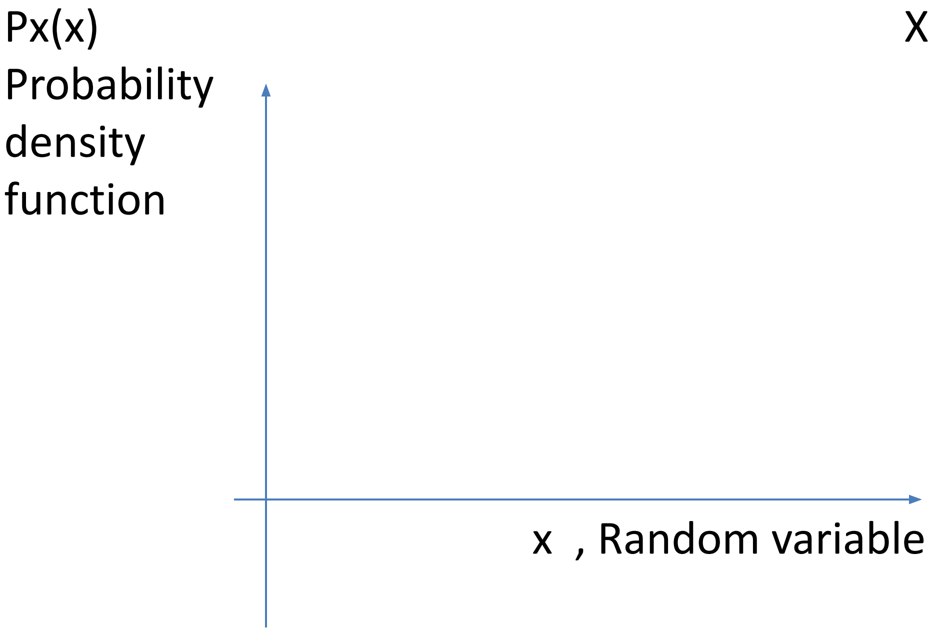
Random Processes: significance contd.

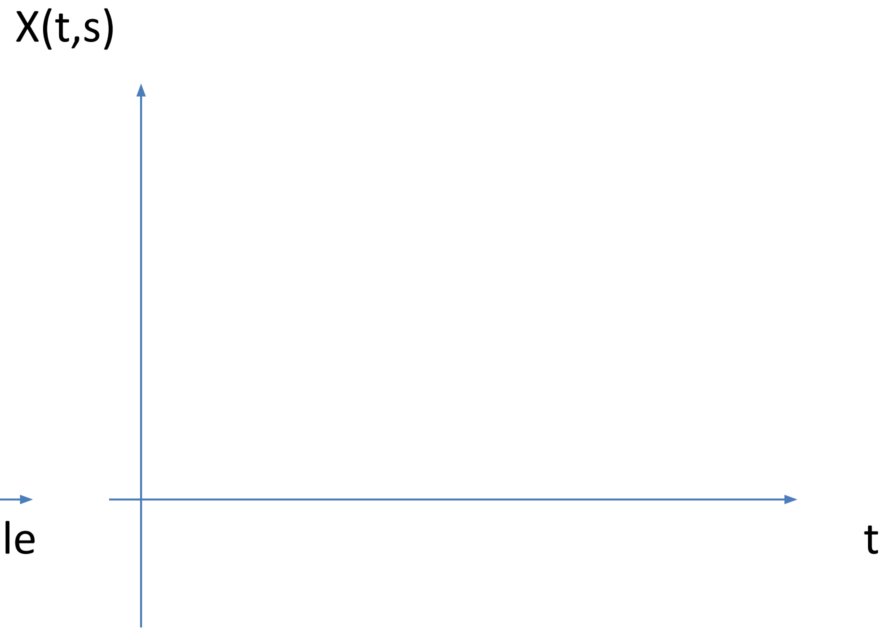
- A random variable maps each sample point in the sample space to a point in the real line.

- A random process maps each sample point to a waveform.

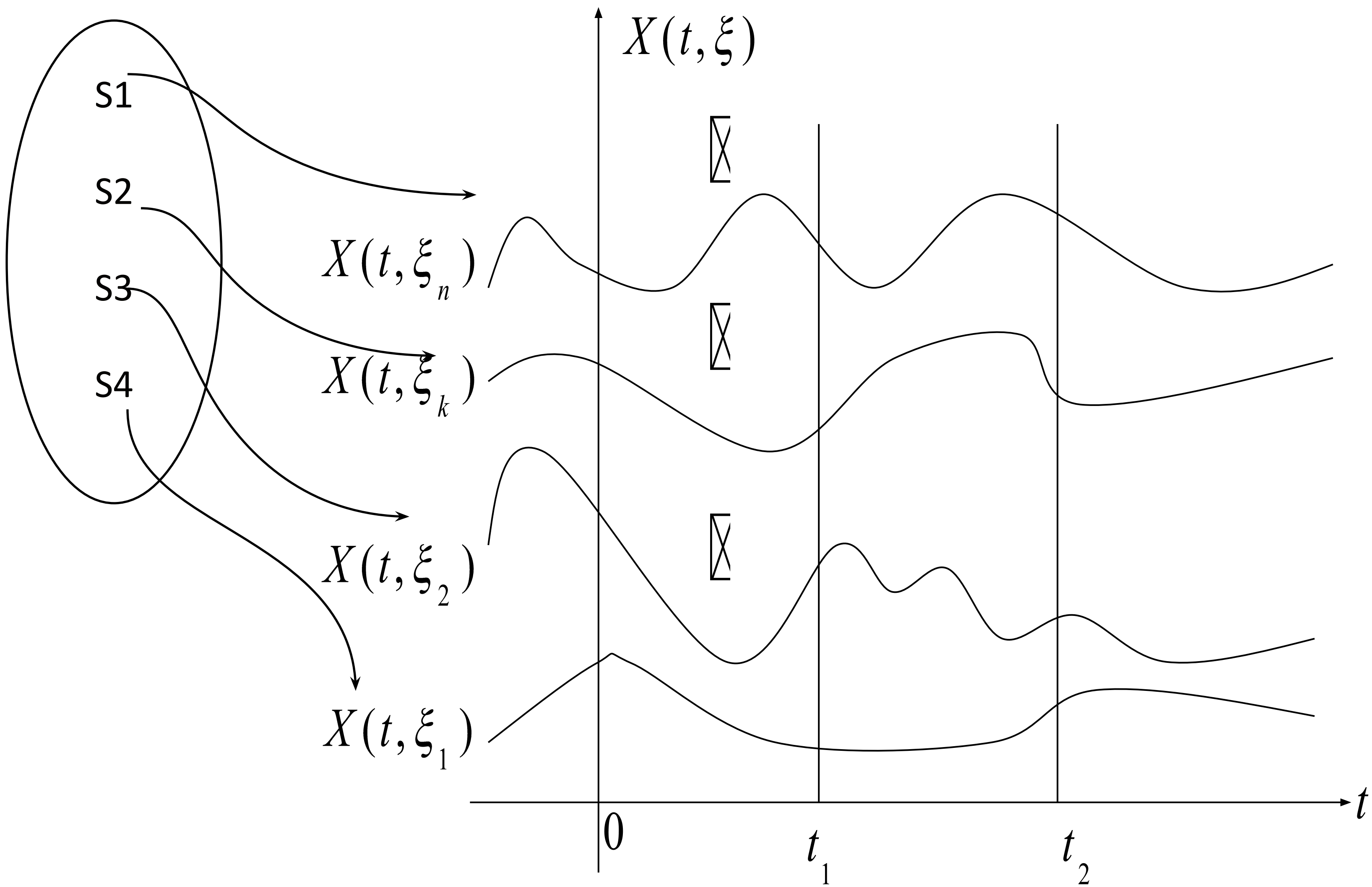
- Thus a random process is a function of the sample point 's' and index variable 't' and may be written as $X(t,s)$ for fixed $t=t_0$ $X(t_0, s)$ is a random variable.

Representing RP





e.g. Temperature records for the day



Properties of Random Signals

- The random signals mentioned earlier have two properties

1. They are functions of time and defined over some time interval
2. It is not possible to describe exactly the waveform of this signal with respect to time

Ensemble and sample function

- The collection of all possible waveform is known as Ensemble. (sample space in random variable)

- Each waveform in the collection is a sample function (sample point)

- Amplitudes of all the sample functions at $t=t_0$ is ensemble statistics.

Introduction

- The statistical analysis is essential because, the signals used in communication systems are random in nature.
- The knowledge of random variables , mean , variance, standard deviation and random process is essential in order to perform the statistical analysis of a communication system.
- The most important issue in the statistical analysis of a communication system is the characterization of the random signal such as voice signal, TV signals, electrical noise, computer data etc.

Classification of random processes

- Stationary and Non-stationary
- Wide-Sense or Weakly Stationary
- Ergodic

Strictly Stationary random process :

- Ensemble averages do not vary with time
- The statistical characterization of the process is time invariant.
- The PDFs obtained at any instants must be identical.
- The Autocorrelation function must be

$$R_x(t_1,t_2) = R_x(t_2 - t_1)$$

Wide sense Stationary random process :

- Mean value is constant
- Autocorrelation function is independent of the shifts

$$R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(\tau)$$

- All stationary processes are wide-sense stationary but converse is not true.

Ergodic random process :

- Ensemble averages are equal to time averages of any sample function.

$$E\left[\mu_X(T)\right] = \overline{x(t)} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \mu_X\,dt = \mu_X$$

- stationary process in which averages from a single record are the same as those obtained from averaging over the ensemble
- Most stationary random processes can be treated as Ergodic

Terminology Describing Random Processes

- A *stationary* random process has statistical properties which do not change at all time
- A *wide sense stationary* (WSS) process has a mean and autocorrelation function which do not change with time
- A random process is *Ergodic* if the time average always converges to the statistical average
- Unless specified, we will assume that all random processes are WSS and Ergodic

Ergodic

Example: $X(t) = A\cos(\omega t + \phi)$ where ϕ is uniform $\left[-\pi, \pi\right]$

Ensemble average: $m_X(t) = E\left[X(t)\right] = AE\left[\cos(\omega t + \phi)\right] = 0$

Time average:

$$\left\langle X(t)\right\rangle_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \frac{A}{2T}\int_{-T}^{T}\cos(\omega t + \phi)\,dt = \frac{A}{2T}\frac{\sin(\omega t + \phi)}{\omega}\Bigg|_{-T}^{T}$$

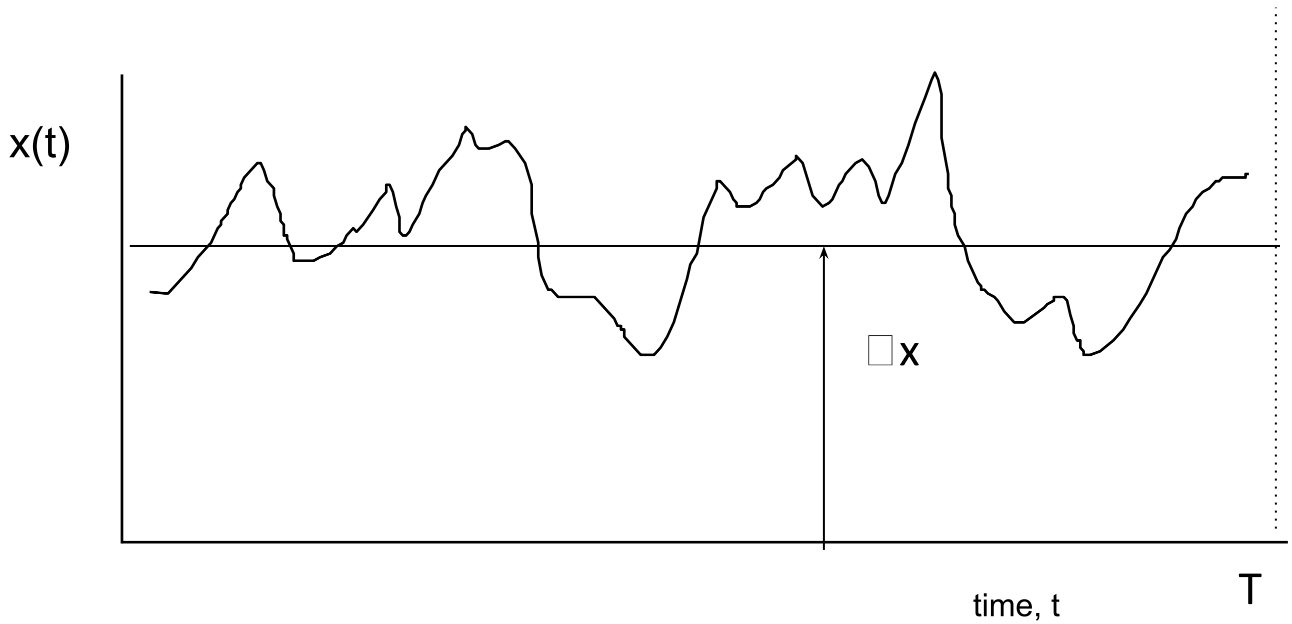
$$= \frac{A}{\omega 2T}\left[\sin(\omega T + \phi) - \sin(-\omega T + \phi)\right]$$

$$= \frac{2A}{\omega 2T}\sin\omega T\cos\phi; \qquad \lim_{T\to\infty}\frac{A}{\omega T}\sin\omega T\cos\phi = 0$$

Time average = Ensemble average
therefore mean ergodic

Mean, Correlation and covariance function

- Mean value :



$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

Mean, Correlation and covariance function

- Mean value of a stationary random process is a constant.

$$\mu_X(t) = \mu_X$$

- Autocorrelation function of a random process X(t)is given as

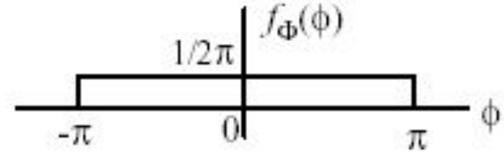
$$R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

Example:

$X = A\cos(\omega t_0 + \phi)$ ϕ is a uniform random variable: $[-\pi, \pi]$

$X = g(\phi)$

$E[X] = E_\phi[g(\phi)] = \int g(\phi) f_\Phi(\phi)\, d\phi$



(a) Mean

$$
\begin{aligned}
m_X &= E\left[A\cos\left(\omega t_0 + \phi\right)\right] \\
&= A\int_{-\pi}^{\pi} \cos\left(\omega t_0 + \phi\right)\frac{1}{2\pi} d\phi \\
&= \frac{A}{2\pi}\cdot \sin\left(\omega t_0 + \phi\right)\Big|_{-\pi}^{\pi} \\
&= \frac{A}{2\pi}\left[\sin\left(\omega t_0 + \pi\right) - \sin\left(\omega t_0 - \pi\right)\right] \\
&= \frac{A}{2\pi}\left[\sin\omega t_0 \cos\pi + \cos\omega t_0 \overset{0}{\cancel{\sin\pi}} - \sin\omega t_0 \cos\pi + \cos\omega t_0 \overset{0}{\cancel{\sin\pi}}\right] = 0
\end{aligned}
$$

(b) Variance

$$\sigma_X^2 = E\left[A^2\cos^2\left(\omega t_0+\phi\right)\right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2\left(\omega t_0+\phi\right)d\phi = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1+\cos\left(2\omega t_0+2\phi\right)}{2}d\phi$$

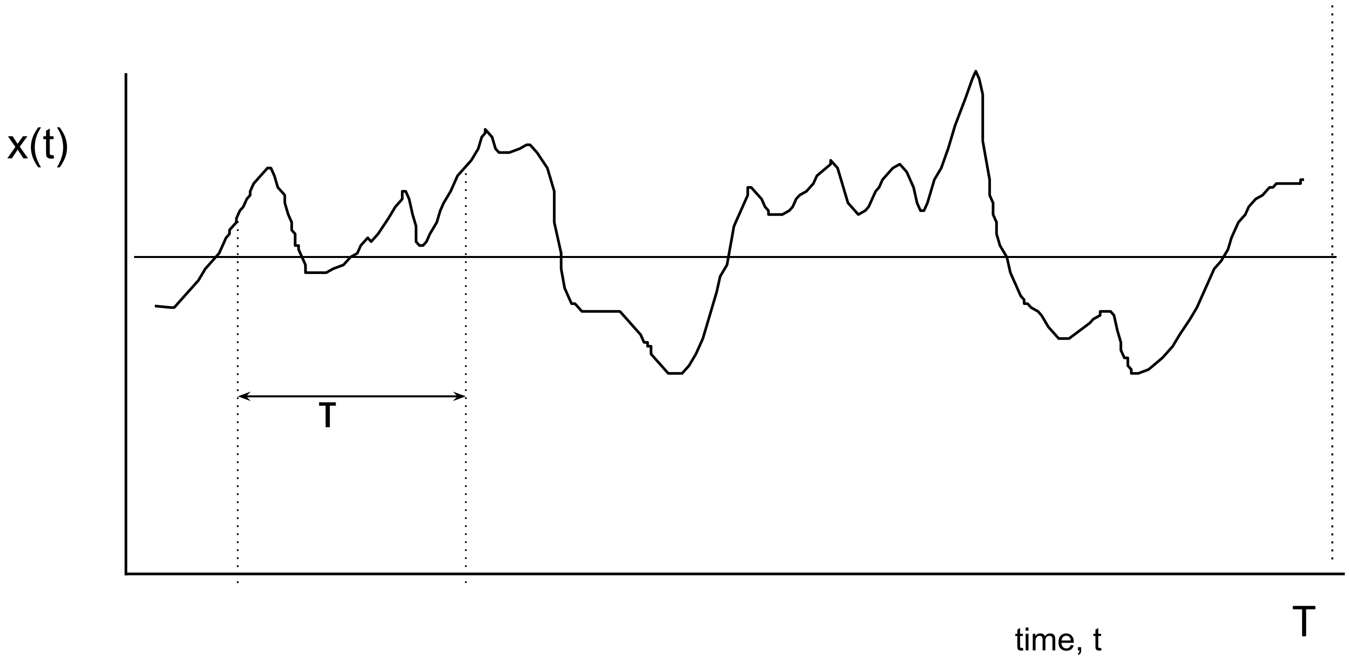
$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}d\phi + \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\cos\left(2\omega t_0+2\phi\right)d\phi$$

$$= \frac{A^2}{4\pi}\cdot\phi\Bigg|_{-\pi}^{\pi} + \frac{A^2}{4\pi}\frac{\sin\left(2\omega t_0+2\phi\right)}{2}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin\left(2\omega t_0+2\pi\right)-\sin\left(2\omega t_0-2\pi\right)\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin 2\omega t_0 - \sin 2\omega t_0\right] = \frac{A^2}{2}$$

• Autocorrelation :



- The autocorrelation, or autocovariance, describes the general dependency of x(t) with its value at a short time later, x(t+τ)

Autocorrelation properties

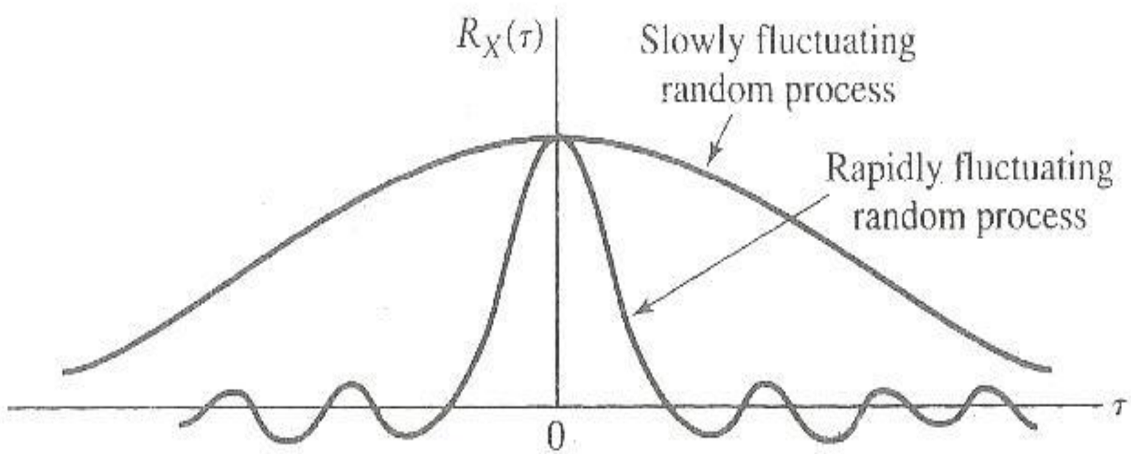
Properties of the Autocorrelation Function†

1. $R_X(\tau) = R_X(-\tau)$ Symmetry

2. $R_X(0) = E\left[X^2(t)\right] \geq 0$ Power of W.S.S. process

3. $\left|R_X(\tau)\right| \leq R_X(0)$ Maximum value

Autocorrelation function of slowly and rapidly fluctuating random processes



Mean, Correlation and covariance function

- Auto-covariance function of a stationary random process X(t)is given as

$$C_X(t_1, t_2) = E\left[(X(t_1) - \mu_X)(X(t_2) - \mu_X)\right]$$
$$= R_X(t_2 - t_1) - \mu_X^2$$

<u>Example</u>: $X(t) = A\cos(\omega t + \phi)$, ϕ is uniform $[-\pi, \pi]$

Find $m_X(t)$ and $R_X(t_1,t_0)$.

$$m_X(t) = E\left[X(t)\right] = A\int_{-\pi}^{\pi}\cos(\omega t+\phi)\frac{d\phi}{2\pi} = 0, \text{ independent of time}$$

$$R_X(t_1,t_0) = E\left[X(t_1)X(t_0)\right] = \int_{-\pi}^{\pi} A\cos(\omega t_1+\phi)A\cos(\omega t_0+\phi)\frac{d\phi}{2\pi}$$

$$= \frac{A^2}{2\pi}\cdot\frac{1}{2}\int_{-\pi}^{\pi}\cos(\omega(t_1-t_0))\,d\phi + \frac{A^2}{2\pi}\cdot\frac{1}{2}\int_{-\pi}^{\pi}\cos(\omega(t_1+t_0)+2\phi)\,d\phi$$

$$= \frac{A^2}{2}\cos(\omega(t_1-t_0)), \text{ function of } t_1-t_0$$

For $t_1 - t_0 = \tau$,

$$R_X(t_1,t_0) = \frac{A^2}{2}\cos(\omega\tau) = R_X(\tau), \quad \tau \text{ is called the lag}$$

Jointly Stationary Properties

- Properties

$$R_{xy}(\tau) = R_{xy}(-\tau)$$

$$\left|R_{xy}(\tau)\right| \le \sqrt{R_x(0)\,R_y(0)}$$

$$\left|R_{xy}(\tau)\right| \le \tfrac{1}{2}\left[R_x(0) + R_y(0)\right]$$

- Uncorrelated:

$$R_{\mathrm{xy}}(\tau) = \overline{x(t)\,y(t+\tau)} = \overline{x}\;\overline{y}$$

- Orthogonal:

$$R_{\mathrm{xy}}(\tau) = 0$$

Power Spectral Density (PSD) Function

- Fourier transform of the autocorrelation function

$$S_X(f) = \mathcal{F}\left[R_X(\tau)\right] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df$$

- Properties:
  - $R_X(0) = E\left[X^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\, df$
  - $S_X(f)$ is real and even symmetric: $S_X(f) = S_X(-f)$
  - $S_X(f) \geq 0$

Spectral density

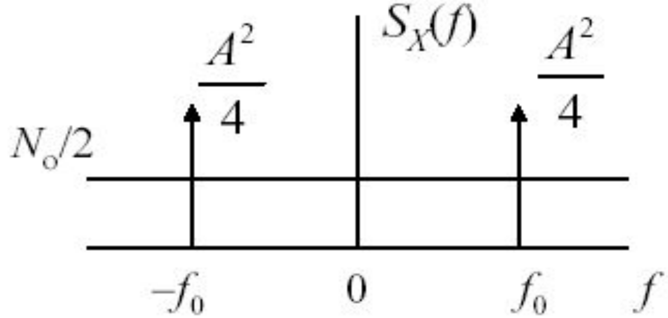
Example: $X(t) = A\cos(2\pi f_0 t + \phi) + W(t)$

$W(t)$ is white, $N_0/2$; ϕ is uniform $[-\pi, \pi]$; W, ϕ independent

$$R_X(\tau) = \frac{A^2}{2}\cos 2\pi f_0 \tau + \frac{N_0}{2}\delta(\tau)$$

$$S_X(f) = \mathrm{F}\left[R_X(\tau)\right] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \frac{A^2}{2}\mathrm{F}\left[\cos 2\pi f_0 \tau\right] + \frac{N_0}{2}\mathrm{F}\left[\delta(\tau)\right] = \frac{A^2}{4}\delta(f - f_0) + \frac{A^2}{4}\delta(f + f_0) + \frac{N_0}{2}$$



Transmission of a random process through a Linear filter

$$\mu_Y(t) = E\left[Y(t)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau) X(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)]\, d\tau$$

The mean of the output random process Y(t) is given as

$$= \mu_X \int_{-\infty}^{\infty} h(\tau)\, d\tau = \mu_X H(0)$$

Points to remember

- Mean of random process Y(t) produced at the output of a LTI system in response to input random process X(t) equals to the mean of X(t) multiplied by the dc response of the system.
- Autocorrelation function of the random process Y(t) is a constant.

Filtering of random signals

Frequency Domain Analysis

$$R_X(\tau) \Leftrightarrow S_X(f) \qquad R_Y(\tau) \Leftrightarrow S_Y(f)$$

$$R_{XY}(\tau) \Leftrightarrow S_{XY}(f) \qquad R_{YX}(\tau) \Leftrightarrow S_{YX}(f)$$

$$h(\tau) \Leftrightarrow H(f) \qquad h(-\tau) \Leftrightarrow H^*(f)$$

Cross-spectral density function:

$$R_{YX}(\tau) = h(\tau) * R_X(\tau) \quad \Leftrightarrow \quad S_{YX}(f) = H(f) S_X(f)$$

Output power spectral density function

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau) \quad \Leftrightarrow \quad \boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

Mathematical Representation of Noise

Some sources of Noise

- Unwanted waves that tend to disturb the transmission and processing of signals in communication systems.

- Thermal resistor noise: randomness in voltage that appears across the resistor terminals
- Shot noise: randomness of emission of electrons or current pulse generated at any time instant.

- Additive noise: noise that added to the signal.
- Fading: noise that multiply the signal.

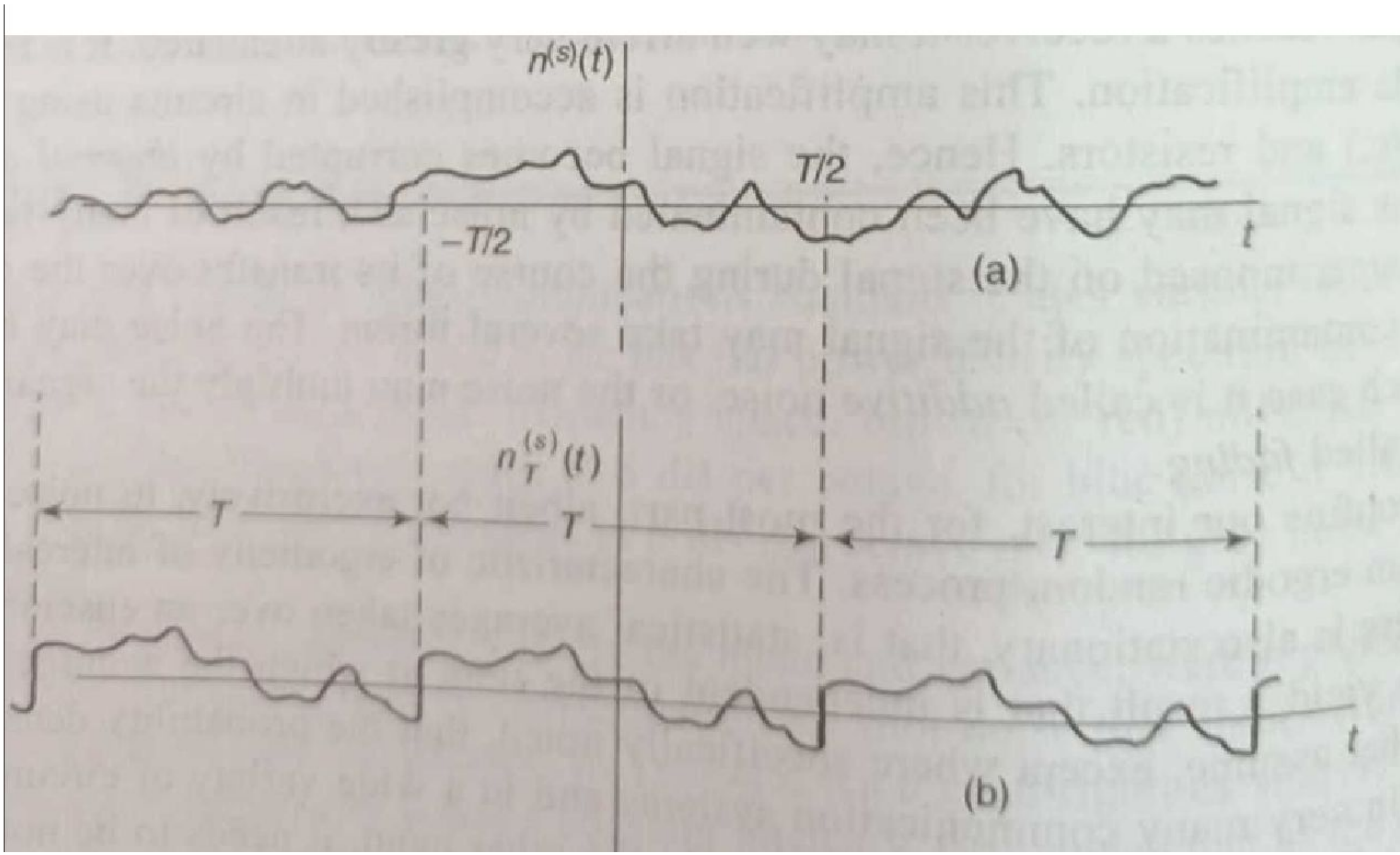
Frequency –Domain Representation of Noise

- Noise is passed through Filters and takes a form of random noise process.
- a noise sample function ***$n_s(t)$*** of such a process appears somewhat similar to a sine wave of frequency 'f' having random amplitude and phase

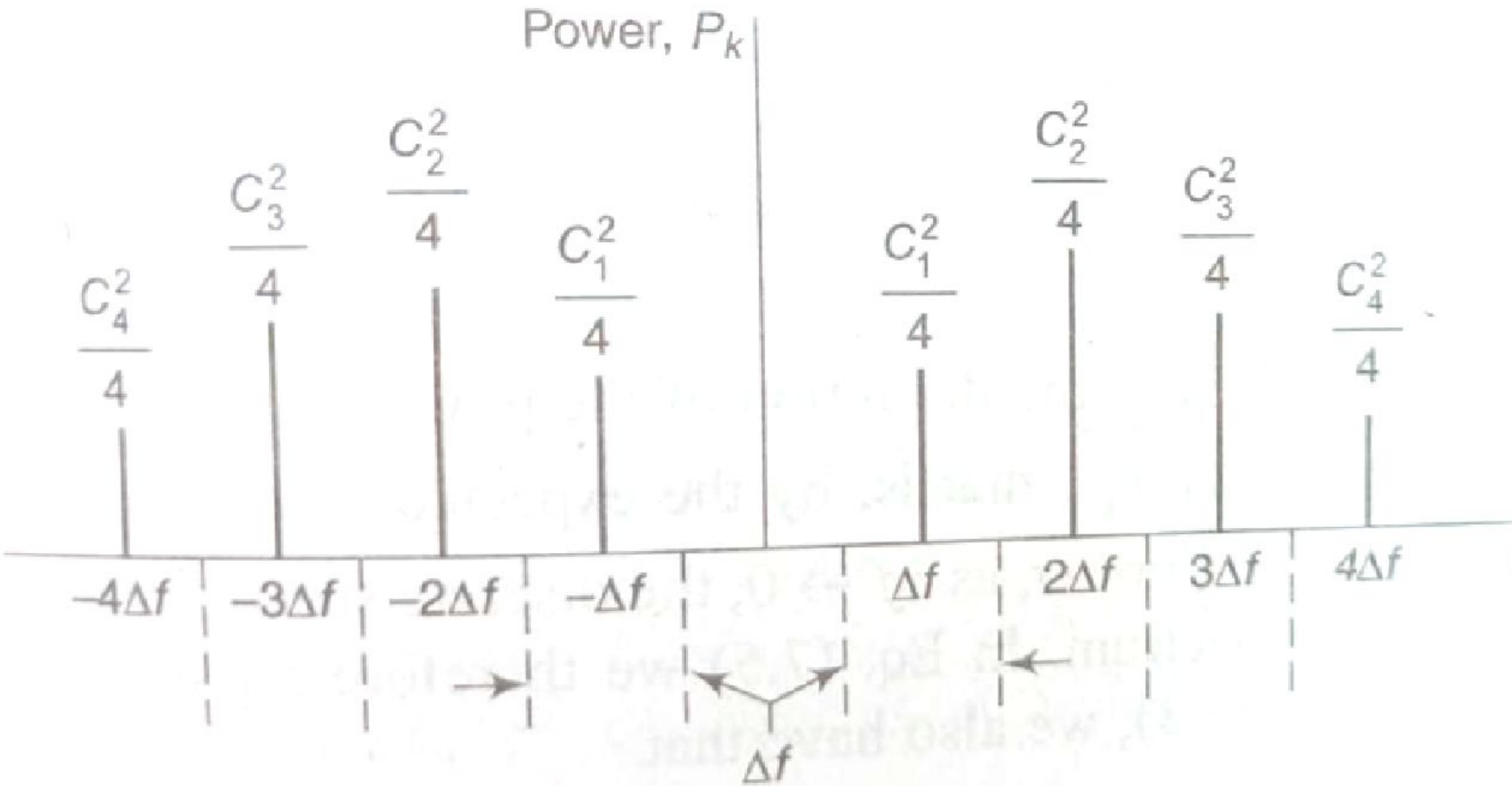
- Periodic wave is expanded using Fourier series:

- $n_s(t) = \sum_{k=1}^{\infty}(a_k \cos 2\pi k\Delta f t + b_k \sin 2\pi k\Delta f t)$ or
- $n_s(t) = \sum_{k=1}^{\infty} c_k \cos(2\pi k\Delta f t + \theta_k)$

A sample noise waveform and periodic noise waveform –T/2 to T/2 $(\Delta f = 1/T)$



Power spectrum of noise waveform



- Considering $T \to \infty \; and \; \Delta f \to 0$ the periodic sample functions of noise revert to the actual noise sample functions.

- Noise spectral density is
    - $G_n(f) = \lim_{\Delta f \to 0} \frac{\overline{c_k^2}}{4\Delta f}$

- *The total noise power is*
    - $P_T = \int_{-\infty}^{\infty} G_n(f) df = 2 \int_0^{\infty} G_n(f) df$

Superposition of Noises

- **superposition of power of two noise processes** n1(t) and n2(t) whose spectral ranges overlap partially or entirely is P12

$$P_{12} = E\{[n_1(t) + n_2(t)]^2\}$$

$$P_{12} = E[n_1^2(t)] + E[n_2^2(t)] + 2E[n_1(t)n_2(t)$$

$$P_{12} = P1 + P2 + 2E[n_1(t)n_2(t)]$$

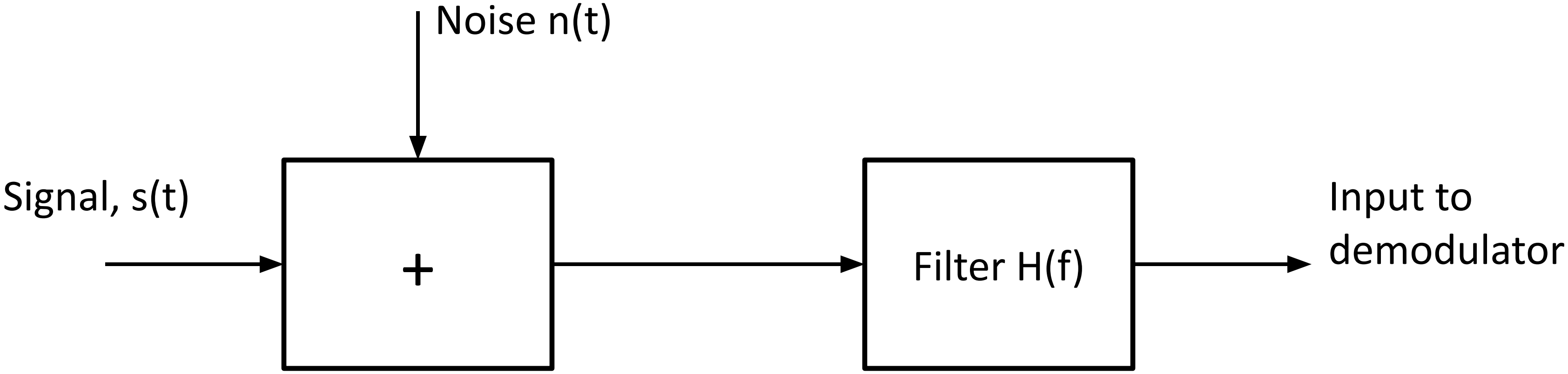
$$P_{12} = P1 + P2 \text{ if noise processes are uncorrelated}$$

- **Mixing noise with sinusoid** (*n(t) cos 2πfot) :* it gives rise to two noise spectral components
- One at sum frequency and one at the difference frequency {(fo+ Δf) and (fo - Δf) resp.}
- Amplitude gets reduced by a factor of 2 w.r.t. the original noise spectral component

Superposition of Noises and

- **Mixing noise with noise** *($n_k(t)n_l(t)$)* : it gives rise to two noise spectral components
- One at sum frequency and one at the difference frequency {(k+l) Δf and (k-l) Δf resp.}
- Spectral components power is equal to ½*P_kP_l

Linear Filtering of Noise



A filter is placed before a demodulator to limit the noise power input to the demodulator

WHITE NOISE

- The primary spectral characteristic of thermal noise is that its power spectral density is *the same* for all frequencies of interest in most communication systems
- A thermal noise source emanates an equal amount of noise power per unit bandwidth at all frequencies—from dc to about 10^{12} Hz.
- Power spectral density G(f) $G_n(f) = \frac{N_0}{2}$ watts/hertz
- Autocorrelation function of white noise is

$$R_n(\tau) = \mathcal{F}^{-1}\{G_n(f)\} = \frac{N_0}{2}\delta(\tau)$$

- The average power P of white noise if infinite

$$P_n = \int_{-\infty}^{\infty} \frac{N_0}{2}\, df = \infty$$

WHITE NOISE

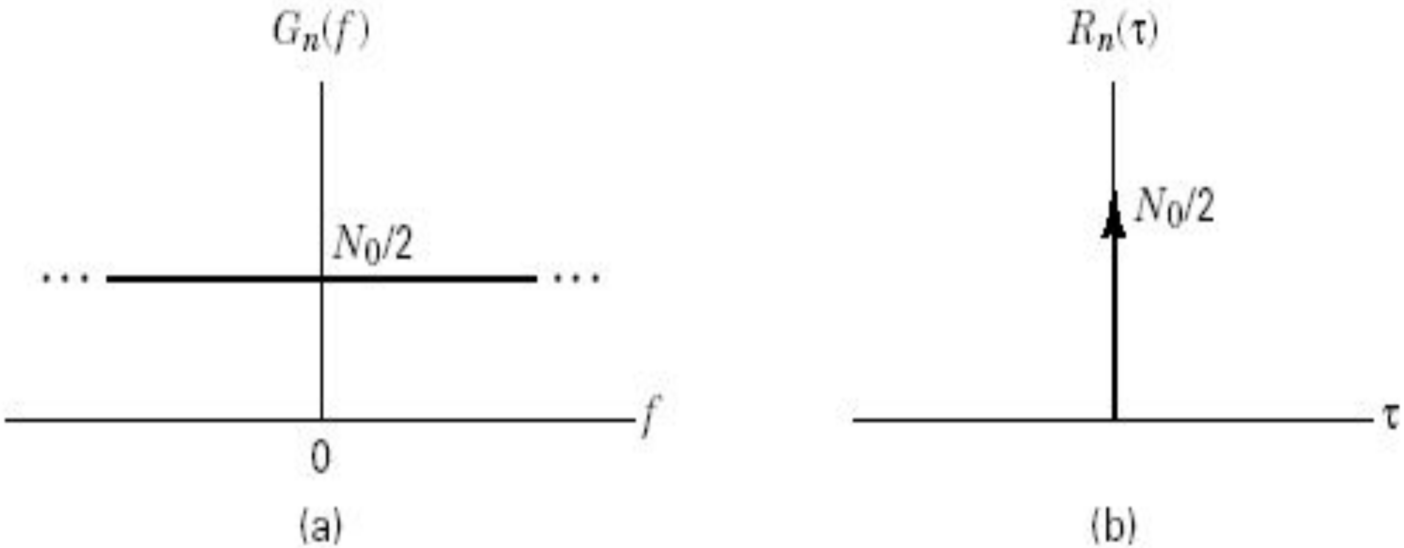
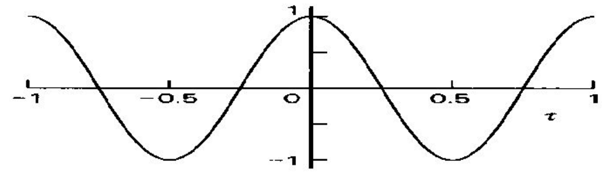
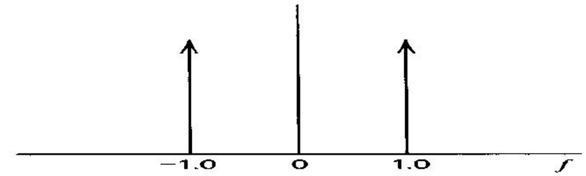
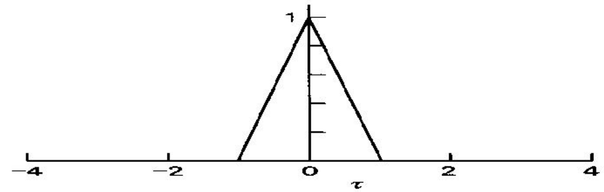
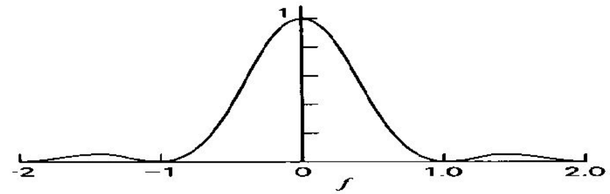
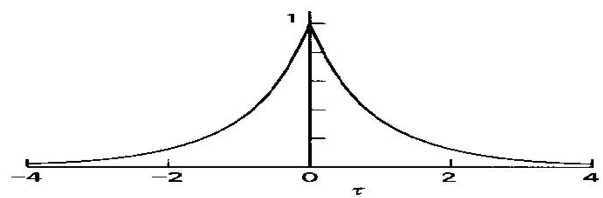
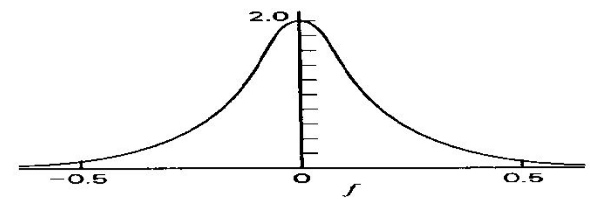
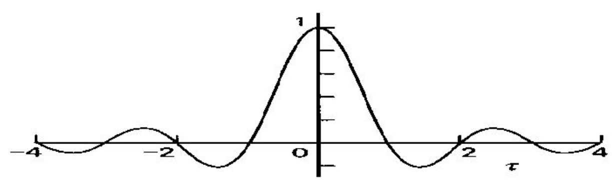
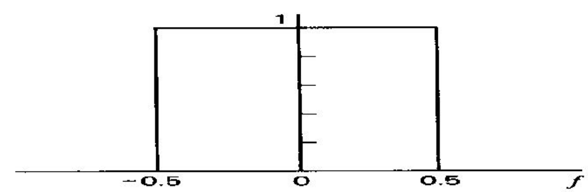
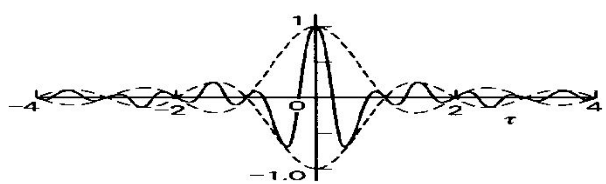
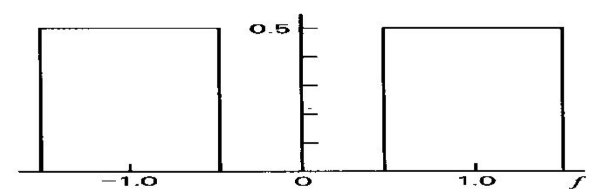
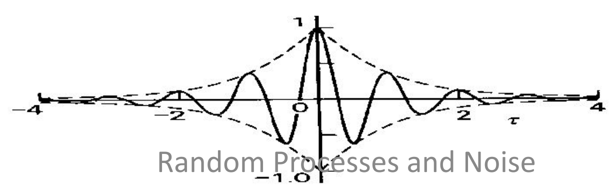
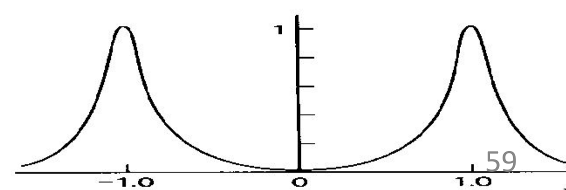


Figure 1.8 (a) Power spectral density of white noise. (b) Autocorrelation function of white noise.

Table 4.1 Graphical Summary of Autocorrelation Functions and Power Spectral Densities of Random Processes of Zero Mean and Unit Variance

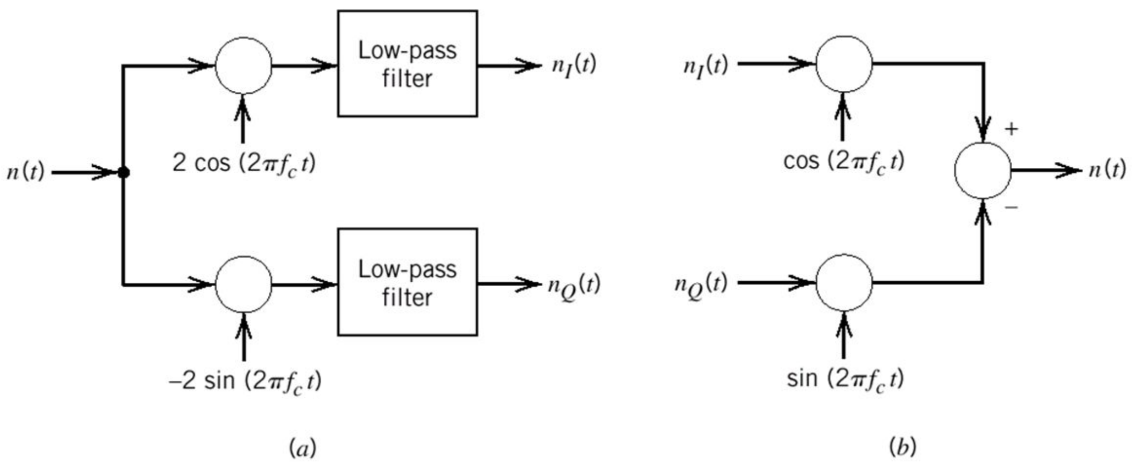
Type of Process, $X(t)$	Autocorrelation Function, $R_X(\tau)$	Power Spectral Density, $S_X(f)$
Sinusoidal process of unit frequency and random phase		
Random binary wave of unit symbol-duration		
RC low-pass filtered white noise		
Ideal low-pass filtered white noise		
Ideal band-pass filtered white noise		
RLC-filtered white noise		

Representation of Narrowband Noise in Terms of In-Phase and Quadrature Components

- Consider a narrowband noise n(t) of bandwidth $2B$ centered on frequency fc, as illustrated in Figure
- We may represent n(t) in the canonical (standard) form:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

where, $n_I(t)$ is ***in-phase*** component of $n(t)$ and $n_Q(t)$ is ***quadrature*** component of .



Quadrature components of Noise

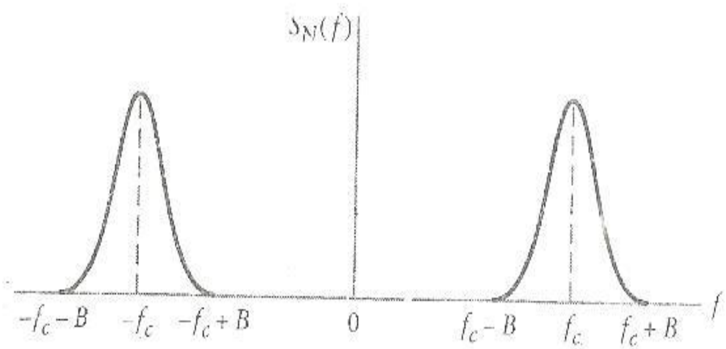
- Here we represent $n(t)$ in terms of envelope and phase components:

$$n(t) = r(t)\cos[2\pi f_c t + \psi(t)]$$

where, $r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$ and $\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$

- $r(t)$ is called the ***envelope*** of $n(t)$, and the $\psi(t)$ is called the ***phase*** of $n(t)$.

PSD of narrow band noise



Representation of noise using orthonormal coordinates/functions

- $n(t) = \sum_{i=0}^{\infty} n_i u_i(t)$
- $n_i(t) = \int_0^T n(t) u_i(t) dt$
- $n_i n_j = \int_0^T n(t) u_i(t) dt \int_0^T n(\lambda) u_j(\lambda) d\lambda$
- $E\left(n_i n_j\right) = \frac{\eta}{2} \int_0^T u_i(t) u_j(t) dt = \frac{\eta}{2}$ if i=j

  = 0 if i≠j

Laboratory experiment

C-4	Simulation study of random processes. Find various statistical parameters of the random process.	1

Matlab simulation:Random_Process_Matlab Code.docx

References

- NPTEL online course on Analog Communication: Random Processes by Prof. Goutam Das
- Hsu, Schaum's outlines, "Analog and Digital communications" second Ed.

Thank You