CSE544: Assignment 6 Solutions

April 27, 2020

Problem 1

(a)

Given $X_1, X_2, ..., X_n \sim N(\theta, \sigma^2)$ — σ is known.

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $se^2 = \sigma^2/n$

Prior distribution for $\theta \sim N(a, b^2)$. We have,
f$(\theta) = (2\pi b^2)^{\frac{-1}{2}}.exp(\frac{-(\theta-a)^2}{2b^2})$ $--$ **1**
$f(\mathbf{x}|\theta) = (2\pi\sigma^2)^{\frac{-1}{2}}.\Pi_{i=1}^{n} exp(\frac{-(x_i-\theta)^2}{2\sigma^2})$ $--$ **2**
$f(\theta|\mathbf{x}) \propto f(\mathbf{x}|\theta).f(\theta)$
$Using\ 1\ and\ 2;$
$f(\theta|\mathbf{x}) = (2\pi b^2)^{\frac{-1}{2}}.exp(\frac{-(\theta-a)^2}{2b^2}).(2\pi\sigma^2)^{\frac{-1}{2}}.\Pi_{i=1}^{n} exp(\frac{-(x_i-\theta)^2}{2\sigma^2})$
$= exp(\frac{-1}{2}\{\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{\sigma^2} + \frac{(\theta-a)^2}{b^2}\})$
$= exp(\frac{-1}{2}\{\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i^2 + \theta^2 - 2x_i\theta) + \frac{(\theta-a)^2}{b^2}\})$
$Ignoring\ constants;$
$= exp(-\frac{\theta^2 n}{2\sigma^2} + \frac{2\theta\sum_{i=1}^{n} x_i}{\sigma^2} - \frac{\theta^2}{2b^2} - \frac{a^2}{2b^2} + \frac{\theta a}{b^2})$
$= exp(\theta^2(-\frac{n}{2\sigma^2} - \frac{1}{2b^2}) + \theta(\frac{n\overline{X}}{\sigma^2} + \frac{a}{b^2}) + constant)$ $--$ **3**
$For\ a\ Normal\ distribution\ with\ response\ y\ with\ mean\ x\ and\ variance\ y^2\ we\ have$
$g(r) = (2\pi y^2)^{\frac{-1}{2}} exp\{(r-x)^2/2y^2\}$
$\propto exp\{\frac{-1}{2}r^2 y^{-1} + rx/y + constant\}$ $--$ **4**
$Comparing\ equations$ **3** and **4**
$x = y^2(\frac{a}{b^2} + \frac{n\overline{X}}{\sigma^2})$ $--$ **5**;
$y^2 = (\frac{1}{b^2} + \frac{n}{\sigma^2})^{-1}$ $--$ **6**
$Solving\ for\ y$
$y^2 = (\frac{1}{b^2} + \frac{1}{se})^{-1}$
$y^2 = \frac{b^2.se^2}{b^2+se^2}$ $--$ **7**
$Putting$ **7** in **5**;
$x = \frac{b^2.se^2}{b^2+se^2}.\frac{b^2.\overline{X}+a.se^2}{b^2.se^2}$
$Thus,\ we\ have:\ x = \frac{b^2.\overline{X}+a.se^2}{b^2+se^2};\ y^2 = \frac{b^2.se^2}{b^2+se^2}$
Hence Proved!

(b)

Finding an interval $C = (c, d)$ such that $P(\theta \in C|\mathbf{x}) = (1-\alpha)$.

Choose c and d such that: $P(\theta < c|\mathbf{x}) = 0.025$ and $P(\theta > d|\mathbf{x}) = 0.025$

$$\begin{aligned}
P(d < \theta < c|\mathbf{x}) &= P(\frac{(d-x)}{y} < \frac{(\theta - x)}{y} < \frac{(c-x)}{y}|\mathbf{x}) \\
&= P(\frac{(d-x)}{y} < Z < \frac{(c-x)}{y}) = (1-\alpha) \;\; --\mathbf{I} \\
&From\ definition\ of\ (1-\alpha)\ C.I; \\
&P(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}) = (1-\alpha) \;\; --\mathbf{II} \\
&Comparing \;\; --\mathbf{I}\ and \;\; --\mathbf{II} \\
&c = x + y.z_{\frac{\alpha}{2}}; \qquad d = x - y.z_{\frac{\alpha}{2}} \\
&Posterior\ interval = (x - y.z_{\frac{\alpha}{2}}, x + y.z_{\frac{\alpha}{2}})
\end{aligned}$$

Since $x \to \overline{X}$ $and\ y \to se\ as\ n\ \to \infty$

$Posterior\ interval = (\overline{X} \pm z_{\frac{\alpha}{2}}.se)$

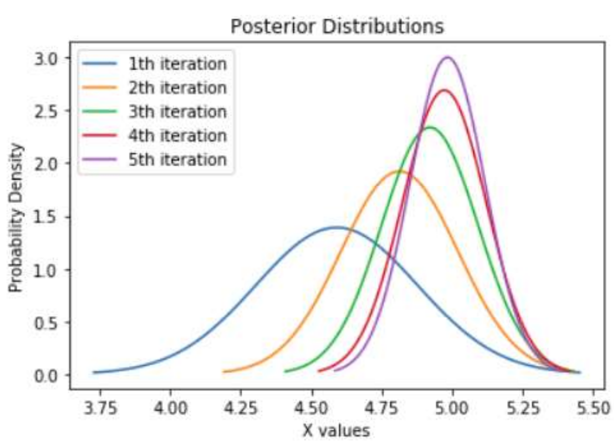
This is the frequentist confidence interval.

Problem 2

(a)

```
a6_q3('q2_sigma3.dat', 9)
```

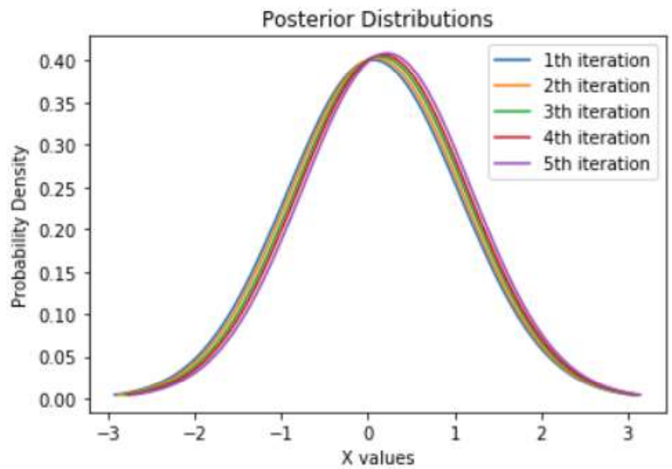
```
1th iteration: Mean = 4.590762414332327 Variance = 0.08256880733944953
2th iteration: Mean = 4.813523613446215 Variance = 0.0430622009569378
3th iteration: Mean = 4.921256878168492 Variance = 0.02912621359223301
4th iteration: Mean = 4.97283741207765 Variance = 0.022004889975550123
5th iteration: Mean = 4.983966097849453 Variance = 0.01768172888015717
```



(b)

```
a6_q3('q2_sigma100.dat', 10000)
```

```
1th iteration: Mean = 0.05871624147982311 Variance = 0.9900990099009901
2th iteration: Mean = 0.09500866961681816 Variance = 0.9803921568627452
3th iteration: Mean = 0.13822626152242073 Variance = 0.970873786407767
4th iteration: Mean = 0.17121883350740297 Variance = 0.9615384615384617
5th iteration: Mean = 0.2189182449674514 Variance = 0.9523809523809524
```



(c)

- When data has low variance, the posterior tends to converge i.e, move away from the prior.
- However, with high variance posterior remains close to the prior.

Problem 3

(a)

First we define the fitted equation to be an equation:

$$\hat{Y} = \beta_0 + \beta_1 X$$

Now, for each observed response Y_i, with a corresponding predictor variable X_i, so we would like to minimize the sum of the squared distances of each observed response to its fitted value.

$$SSE = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)^2$$

Thus, we set the partial derivatives of $SSE(\beta_0, \beta_1)$ with respect β_0 and β_1 equal to zero

$$\frac{dSSE}{d\beta_0} = \sum_{i=1}^{n} 2(-1)(Y_i - \beta_0 - \beta_1 X_i) = 0$$
$$\Rightarrow \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i) = 0$$

$$\frac{dSSE}{d\beta_1} = \sum_{i=1}^{n} 2(-X_i)(Y_i - \beta_0 - \beta_1 X_i) = 0$$
$$\Rightarrow \sum_{i=1}^{n} X_i(Y_i - \beta_0 - \beta_1 X_i) = 0$$

The we could get 2 normal equations:

$$\beta_0 n + \beta_1 \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$

$$\beta_0 \sum_{i=1}^{n} X_i + \beta_1 \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i Y_i$$

For the first normal equation, we could get

$$\beta_0 = \frac{\sum_{i=1}^{n} Y_i - \beta_1 \sum_{i=1}^{n} X_i}{n}$$

Substitute into the second normal equation, yields,

$$
\begin{aligned}
\frac{\sum_{i=1}^n Y_i - \beta_1 \sum_{i=1}^n X_i}{n} \sum_{i=1}^n X_i + \beta_1 \sum_{i=1}^n X_i^2 &= \sum_{i=1}^n X_i Y_i \\
\beta_1 (\sum_{i=1}^n X_i^2 - \frac{(\sum_{i=1}^n X_i)^2}{n}) &= \sum_{i=1}^n X_i Y_i - \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n} \\
\beta_1 (\sum_{i=1}^n X_i^2 - 2\frac{(\sum_{i=1}^n X_i)^2}{n} + \frac{(\sum_{i=1}^n X_i)^2}{n}) &= \sum_{i=1}^n X_i Y_i - \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n} - \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n} + \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n} \\
\beta_1 (\sum_{i=1}^n X_i^2 - 2\sum_{i=1}^n X_i \frac{\sum_{i=1}^n X_i}{n} + \sum_{i=1}^n (\frac{\sum_{i=1}^n X_i}{n})^2) &= \sum_{i=1}^n X_i Y_i - \sum_{i=1}^n X_i \bar{Y} - \sum_{i=1}^n Y_i \bar{X} + \sum_{i=1}^n \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n^2} \\
\beta_1 \sum_{i=1}^n (X_i^2 - 2X_i \frac{\sum_{i=1}^n X_i}{n} + (\frac{\sum_{i=1}^n X_i}{n})^2) &= \sum_{i=1}^n X_i Y_i - \sum_{i=1}^n X_i \bar{Y} - \sum_{i=1}^n Y_i \bar{X} + \sum_{i=1}^n \bar{X}\bar{Y} \\
\beta_1 \sum_{i=1}^n (X_i^2 - \bar{X})^2 &= \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y}) \\
\Rightarrow \hat{\beta}_1 &= \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2}
\end{aligned}
$$

Thus we could have

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

(b)

First, we rewrite $\hat{\beta}_1$ as

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})Y_i}{S_{xx}} = \sum_{i=1}^n \frac{X_i - \bar{X})Y_i}{S_{xx}} = \sum_{i=1}^n c_i Y_i$$

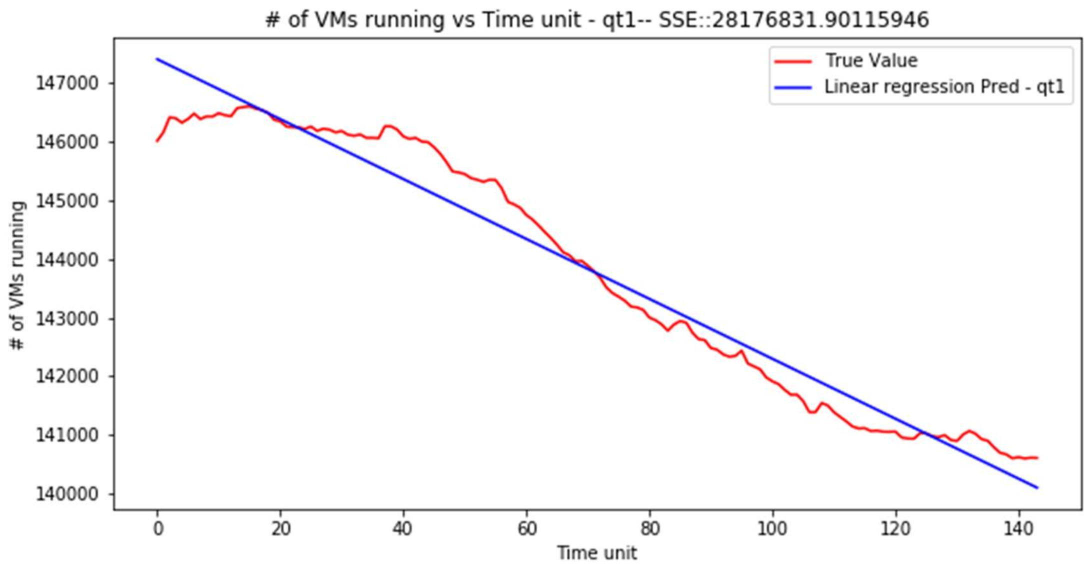
and we could have $\sum_{i=1}^n c_i = \sum_i \frac{X_i - \bar{X}}{S_{xx}} = \frac{n\bar{X} - n\bar{X}}{S_{xx}} = 0$. Also, $E[\epsilon_i] = 0$. Then, we have

$$
\begin{aligned}
E[\hat{\beta}_1] &= \sum_{i=1}^n c_i E[Y_i] \\
&= \sum_{i=1}^n c_i E[\beta_0 + \beta_1 X_i + \epsilon_i] \\
&= \beta_0 \sum_{i=1}^n c_i + \beta_1 \sum_{i=1}^n c_i X_i + \sum_{i=1}^n c_i E[\epsilon_i] \\
&= \beta_1 \sum_{i=1}^n \frac{(X_i - \bar{X})X_i}{\sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \beta_1 \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \beta_1
\end{aligned}
$$

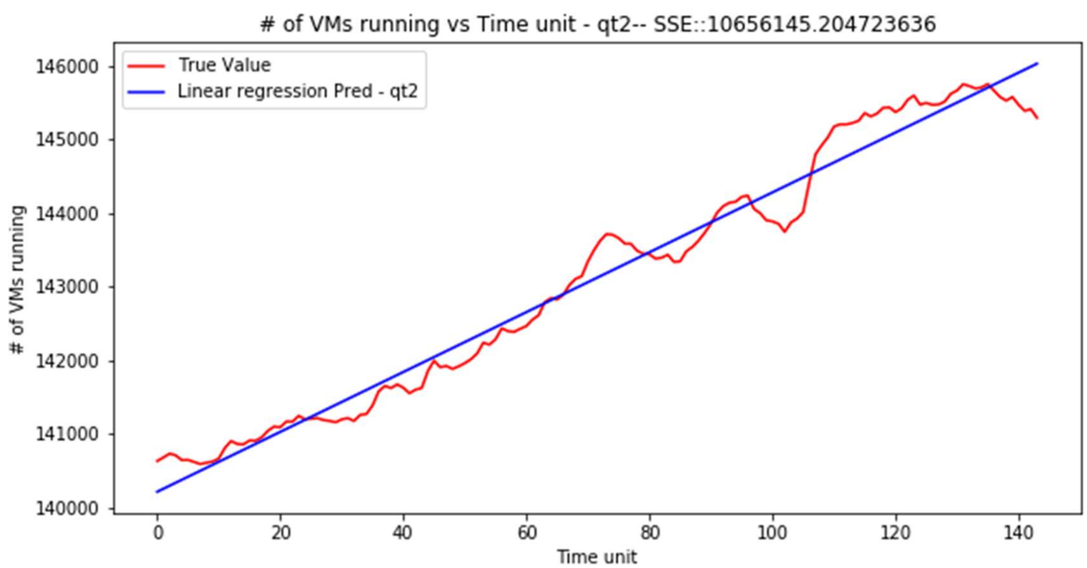
$$
\begin{aligned}
E[\hat{\beta}_0] &= E[\bar{Y} - \hat{\beta}_1 \bar{X}] \\
&= E[\frac{\sum_{i=1}^{n} Y_i}{n} - \frac{\sum_{i=1}^{n} \hat{\beta}_1 X_i}{n}] \\
&= \frac{\sum_{i=1}^{n} E[\beta_0 + \beta_1 X_i - \hat{\beta}_1 X_i]}{n} \\
&= \frac{\sum_{i=1}^{n} (\beta_0 + \beta_1 X_i - \beta_1 X_i)}{n} \\
&= \beta_0
\end{aligned}
$$

Problem 4

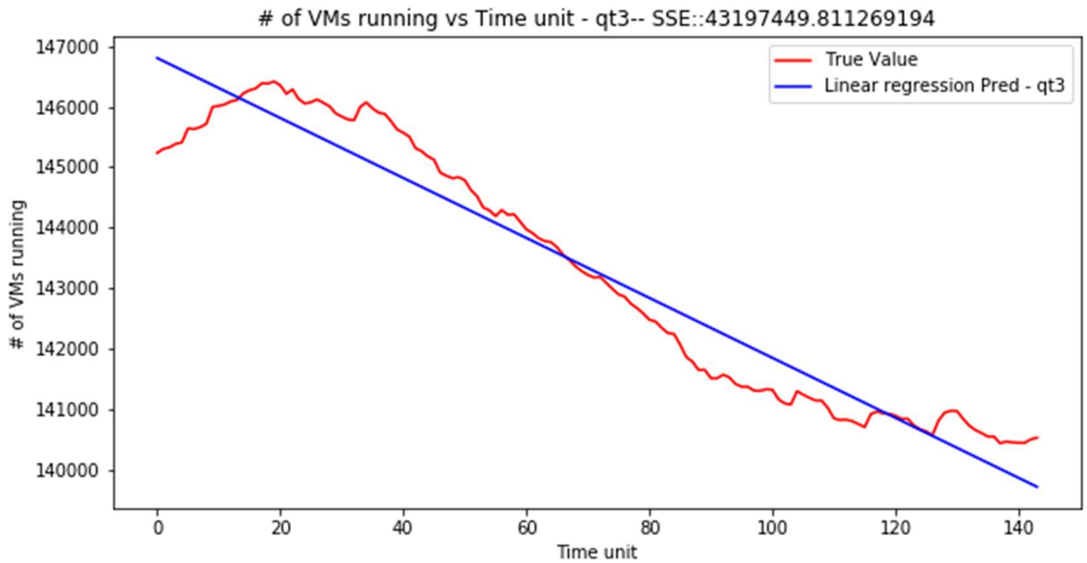
(a)



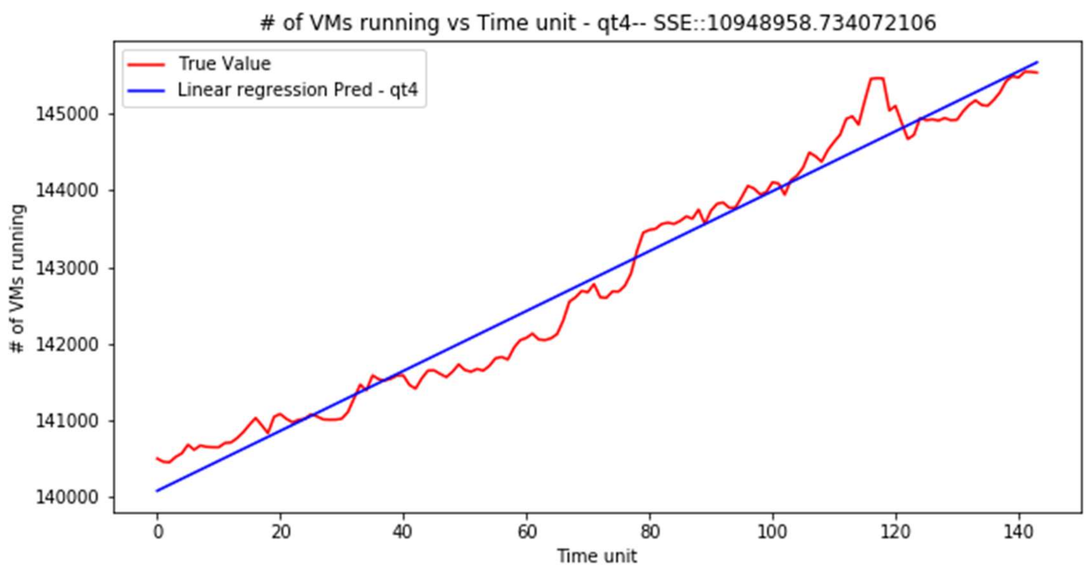
# of VMs running vs Time unit - qt1-- SSE::28176831.90115946
True Value
Linear regression Pred - qt1
# of VMs running
Time unit



# of VMs running vs Time unit - qt2-- SSE::10656145.204723636
True Value
Linear regression Pred - qt2
# of VMs running
Time unit



# of VMs running vs Time unit - qt3-- SSE::43197449.811269194
True Value
Linear regression Pred - qt3
# of VMs running
Time unit
147000
146000
145000
144000
143000
142000
141000
140000
0
20
40
60
80
100
120
140



# of VMs running vs Time unit - qt4-- SSE::10948958.734072106
True Value
Linear regression Pred - qt4
# of VMs running
Time unit
145000
144000
143000
142000
141000
140000
0
20
40
60
80
100
120
140

Problem 5

a)

$H \equiv RV$ for the soil type.

The two hypotheses are: $\boldsymbol{H_0}: H = 0$ and $\boldsymbol{H_1}: H = 1$ with $P(H = 0) = p$ and $P(H = 1) = (1 - p)$

Observations of water concentration metric $\boldsymbol{w} = \{w_1, \dots w_n\}$

$f_W(w|H = 0) = N(w; -\mu, \sigma^2)$ and $f_W(w|H = 1) = N(w; \mu, \sigma^2)$

Also $w_i s$ are conditionally independent of each other given the hypothesis/soil type.

$$P(H = 0|\boldsymbol{w}) = \frac{P(\boldsymbol{w}|H = 0)P(H = 0)}{P(\boldsymbol{w})} \qquad By\ Bayes\ theorem$$

$$\Rightarrow \quad P(H = 0|\boldsymbol{w}) = \frac{P(H=0)}{P(\boldsymbol{w})} \prod_{i=1}^{n} f_W(w_i|H = 0) \qquad \because (w_i|H = h) \perp (w_i|H = h)$$

$$\Rightarrow \quad P(H = 0|\boldsymbol{w}) = c.p.\exp\left(-\frac{\sum_i (w_i + \mu)^2}{2\sigma^2}\right)$$

We choose $H_0 (C = 0)$ if $P(H = 0|\boldsymbol{w}) \geq P(H = 1|\boldsymbol{w})$, i.e.

$$c.p.\exp\left(-\frac{\sum_i (w_i + \mu)^2}{2\sigma^2}\right) \geq c.(1 - p).\exp\left(-\frac{\sum_i (w_i - \mu)^2}{2\sigma^2}\right)$$

$$\Rightarrow \exp\left(-\frac{\sum_i (w_i + \mu)^2 - \sum_i (w_i - \mu)^2}{2\sigma^2}\right) \geq \frac{(1 - p)}{p}$$

$$\Rightarrow \exp\left(-\frac{2\mu \sum_i w_i}{\sigma^2}\right) \geq \frac{(1 - p)}{p}$$

$$\left(\sum_i w_i\right) \leq \frac{\sigma^2}{2\mu} \ln\left(\frac{p}{1 - p}\right)$$

(b)

For $P(H_0) = 0.1$, the hypothesis selected are: 0 1 0 0 1 0 1 1 0 1

For $P(H_0) = 0.3$, the hypothesis selected are: 0 1 0 0 1 0 1 1 0 1

For $P(H_0) = 0.5$, the hypothesis selected are: 0 1 0 0 1 0 1 1 0 1

For $P(H_0) = 0.8$, the hypothesis selected are: 0 1 0 0 1 0 1 1 0 1

(c) We choose H_0 i.e. $C = 0$ iff

$$\left(\sum_i w_i\right) \leq \frac{\sigma^2}{2\mu} \ln\left(\frac{p}{1 - p}\right)$$

$\Rightarrow$ We choose $H_1\ iff$

$$\left(\sum_i w_i\right) > \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right)$$

$$P(C=0|H=1) = P\left(\left(\sum_i w_i\right) \leq \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) |\ (H=1)\right)$$

$\because\ w_i|(H=0) \sim N(-\mu, \sigma^2)$

$$\Rightarrow \left(\sum_i w_i\right)|(H=0) \sim N(-n\mu, n\sigma^2)$$

$$\Rightarrow \left(\sum_i w_i\right)|(H=1) \sim N(n\mu, n\sigma^2)$$

$$\Rightarrow P(C=0|H=1) = \Phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) - n\mu}{\sqrt{n\sigma^2}}\right) \ \because if\ X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$$

Similarly,

$$P(C=1|H=0) = P\left(\left(\sum_i w_i\right) > \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) |\ (H=0)\right)$$

$$\Rightarrow P(C=1|H=0) = 1 - \Phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) + n\mu}{\sqrt{n\sigma^2}}\right)$$

$$\therefore AEP = (1-p).\Phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) - n\mu}{\sqrt{n\sigma^2}}\right) + p.\left(1 - \Phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) + n\mu}{\sqrt{n\sigma^2}}\right)\right)$$