CSE544: Assignment 1 Solutions

February 10, 2020

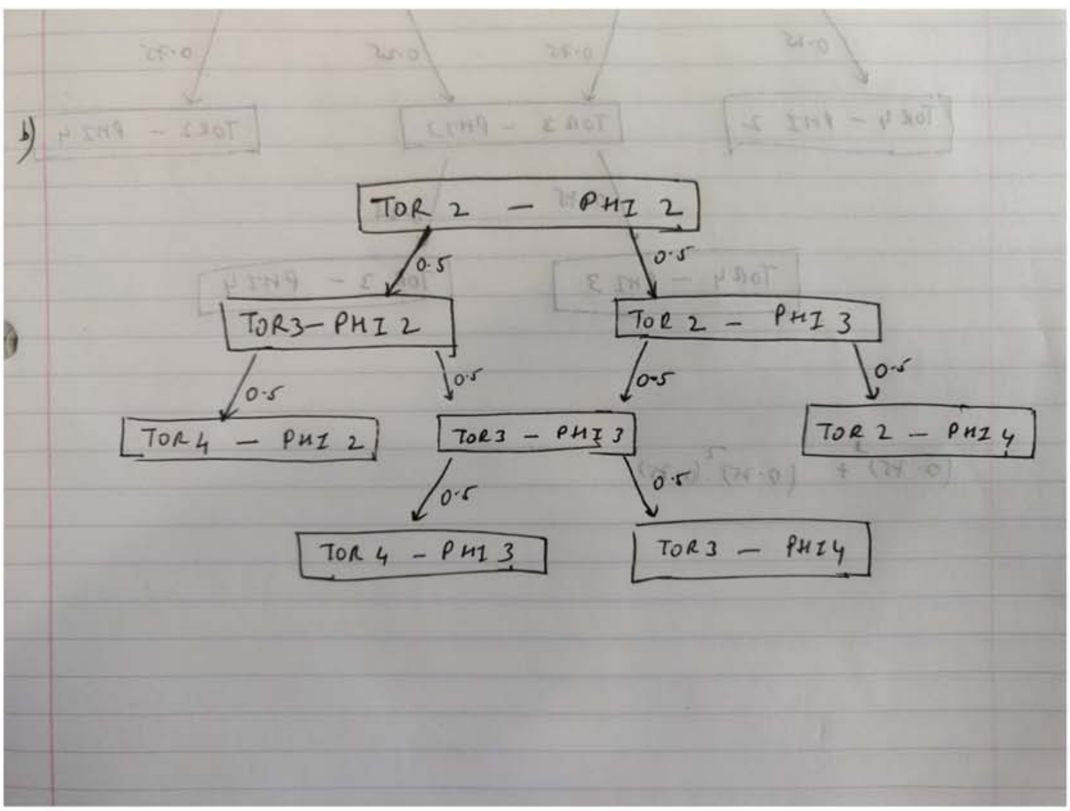
1 Problem 1

(a)

$$P = \binom{4}{2} . (0.5)^2 . (0.5)^2 = 0.375$$

(b)

Illustrated in the plot below:

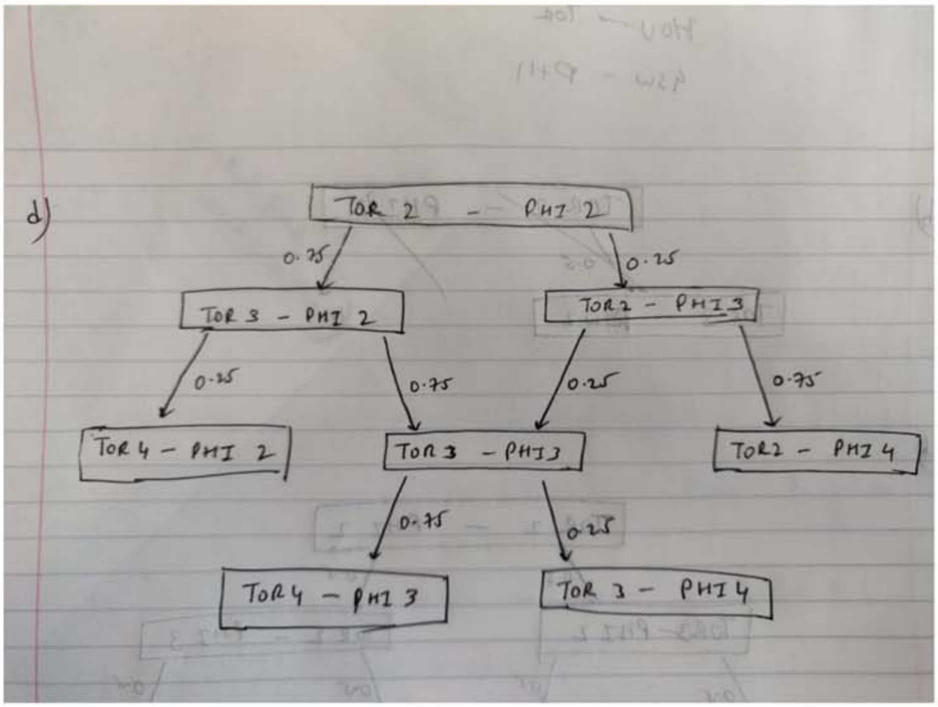


(c)

$$P[TOR(4-3)] = (0.5)^3 + (0.5)^3 = 0.25$$

(d)

Illustrated in the plot below:



(e)

$$P[TOR(4-3)] = (0.75)^3 + (0.25)^2.(0.75) = 0.46875$$

(f)

Part a: $N = 10^3$: 0.391

$N = 10^4$:0.3811

$N = 10^5$:0.37473

$N = 10^6$:0.375474

$N = 10^7$: 0.3749454

Part c $N = 10^3$: 0.26598

$N = 10^4$:0.24166

$N = 10^5$:0.24857

$N = 10^6$:0.25069

$N = 10^7$: 0.249722

Part e

$N = 10^3$: 0.44161

$N = 10^4$: 0.47182

$N = 10^5$:0.47079

$N = 10^6$:0.46909

$N = 10^7$: 0.46874

2 Problem 2

(a)

Let E_i be the event that you pick iPhone i at the ith step. We need to find $P(\cup_{i=1}^n E_i)$. By the principle of inclusion-exclusion we have

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} P(E_i \cap E_j \cdots \cap E_n)$$

Note that $P(E_i) = 1/n$ for all i. One way to see this is by using the full sample space: there are $n!$ possible orderings of the iPhones, all equally likely, and $(n-1)!$ of these are favorable to E_i.

Similarly

$$P(E_i \cap E_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

,

$$P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

and so on.

In the inclusion-exclusion formula, there are n terms involving one event, $\binom{n}{2}$ terms involving two events, $\binom{n}{3}$ terms involving three events, and so forth. By the symmetry of the problem, all n terms of the form $P(E_i)$ are equal, all $\binom{n}{2}$ terms of the form $P(E_i \cap E_j$ are equal. Therefore one has

$$\begin{aligned} P(\cup_{i=1}^n E_i) &= \frac{n}{n} - \frac{\binom{n}{2}}{n(n-1)} + \frac{\binom{n}{3}}{n(n-1)(n-2)} - \cdots + (-1)^{n+1}\frac{1}{n!} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!} \end{aligned}$$

(b)

(n, N)

$(5, 10^2)$: 0.67

$(5, 10^3)$: 0.643

$(5, 10^4)$: 0.631

$(5, 10^5)$: 0.63315

$(20, 10^2)$: 0.58

$(20, 10^3)$: 0.612

$(20, 10^4)$: 0.6224

$(20, 10^5)$: 0.63184

3 Problem 3

Let us introduce some notation:

$M_R \rightarrow Machine\ Red\ is\ good$
$M_G \rightarrow Machine\ Green\ is\ good$
$W_R \rightarrow Win\ on\ Machine\ Red$
$W_G \rightarrow Win\ on\ Machine\ Green$

Translating the above notation into probabilities specified in the question:

$$
\begin{aligned}
P(M_R) &= 0.5 & P(M_G) &= 0.5 \\
P(W_R|M_R) &= 0.5 & P(W_G|M_G) &= 0.5 \\
P(\neg W_R|M_R) &= 0.5 & P(\neg W_G|M_G) &= 0.5 \\
P(W_R|M_R) &= 0.1 & P(W_G|\neg M_G) &= 0.1 \\
P(\neg W_R|\neg M_R) &= 0.9 & P(\neg W_G|\neg M_G) &= 0.9
\end{aligned}
$$

(a)

We need to compute: $P(\neg M_R|\neg W_R)$

Applying Bayes' theorem;

$$
\begin{aligned}
P(\neg M_R|\neg W_R) &= \frac{P(\neg W_R|\neg M_R).P(\neg M_R)}{P(\neg W_R)} \\
&= \frac{0.9 * 0.5}{P(\neg W_R)} \\
P(\neg W_R) &= P(\neg W_R|\neg M_R).P(\neg M_R) + P(\neg W_R|M_R).P(M_R) - -Total\ probability\ theorem \\
P(\neg W_R) &= (0.9 + 0.5) * (0.5) \\
&= 0.7
\end{aligned}
$$

(b)

From the question, we know that each play of the machines is independent and also conditionally independent of the results of prior plays.

Probability to compute: $P(M_G|(W_R, W_R))$

Here, (W_R, W_R) denotes result of two prior outcomes.

Applying Bayes' theorem;

$$
\begin{aligned}
P(M_G|(W_R, W_R)) &= \frac{P((W_R, W_R)|M_G).P(M_G)}{P((W_R, W_R))} \\
P(M_G|(W_R, W_R)) &= \frac{P((W_R|M_G).P(W_R|M_G).P(M_G)}{P((W_R, W_R))} --From\ the\ definition\ of\ conditional\ independence. \\
P(M_G|(W_R, W_R)) &= \frac{P((W_R|M_G).P(W_R|M_G).P(M_G)}{P((W_R).P(W_R))} --As\ each\ play\ of\ machines\ is\ independent. \\
P(M_G|(W_R, W_R)) &= \frac{P((W_R|\neg M_R).P(W_R|\neg M_R).P(M_G)}{P(W_R).P(W_R)} \\
P(M_G|(W_R, W_R)) &= \frac{(0.1).(0.1).(0.5)}{P(W_R).P(W_R)}; \qquad P(W_R) = (0.5+0.1)*(0.5) = 0.3 --- Total\ probability\ theorem. \\
P(M_G|(W_R, W_R)) &= \frac{(0.1).(0.1).(0.5)}{(0.3).(0.3)} \\
P(M_G|(W_R, W_R)) &= 0.055555556
\end{aligned}
$$

Alternative solution

Here, we do not consider the independence assumption and compute $P((W_R, W_R))$ only based on conditional independence assumption among different plays.

$$
\begin{aligned}
P((W_R, W_R)) &= P((W_R, W_R)|M_R).P(M_R) + P((W_R, W_R)|\neg M_R).P(\neg M_R) \\
P((W_R, W_R)) &= P(W_R|M_R).P(W_R|M_R).P(M_R) + P(W_R|\neg M_R).P(W_R|\neg M_R).P(M_R) - via\ conditional\ independence \\
P((W_R, W_R)) &= (0.5).(0.5).(0.5) + (0.1).(0.1).(0.5) = 0.18 + 0.02 = 0.2
\end{aligned}
$$

$$
\begin{aligned}
P(M_G|(W_R, W_R)) &= \frac{(0.2).(0.2).(0.5)}{(0.2)} \\
&= 0.13
\end{aligned}
$$

(c)

Similar to **(b)**,

We need to compute; $P(\neg M_R|(\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R))$ Applying Bayes' theorem;

$$
\begin{aligned}
P(\neg M_R|(\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R)) &= \frac{P((\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R)|\neg M_R).P(\neg M_R)}{P(\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R)} \\
&= \frac{(0.9^n).(0.5)}{P(\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R)} --- From\ conditional\ independence \\
P(\neg W_R) &= (0.5+0.9)*(0.5) = 0.7 --- Total\ probability\ theorem. \\
&= \frac{(0.9^n).(0.5)}{(P(\neg W_R)^n} --- As\ each\ play\ of\ machines\ is\ independent. \\
&= \frac{(0.9^n).(0.5)}{(0.7)^n}
\end{aligned}
$$

For at least 95% confidence that Red machine is bad given jackpot failed for n times;

$$\begin{aligned}
P(\neg M_R|(\neg W_R, \neg W_R, \xrightarrow{n\ times} \neg W_R)) &\geq \frac{95}{100} \\
(\frac{9}{7})^n.(0.5) &\geq \frac{95}{100} \\
(\frac{9}{7})^n &\geq \frac{95}{50} \\
Applying\ logarithmonbothsides; n(\log 9 - \log 7) &\geq (\log 95 - \log 50) \\
n(0.2513144) &\geq (0.64185) \\
n &\geq \frac{0.64185}{0.2513144} \\
n &\geq 2.553
\end{aligned}$$

Therefore, after playing the Red machine **3 times** and failing, one can be at least 95% confident that Red machine is bad.

4 Problem 4

(a)

Let A and R be the variables for "the owner have an above-average lifespan" and "the ring is the One Ring", respectively. Then,

$P(A|R) = 0.95$, $P(\bar{A}|R) = 0.05$, $P(\bar{A}|\bar{R}) = 0.75$ and $P(A|\bar{R}) = 0.25$. Then,

$$\begin{aligned}
P(R|A) &= \frac{P(A|R)P(R)}{P(A)} \\
&= \frac{P(A|R)P(R)}{P(A|R)P(R) + P(A|\bar{R})P(\bar{R})} \\
&= \frac{0.95 \cdot 10^{-4}}{0.95 \cdot 10^{-4} + 0.25 \cdot (1 - 10^{-4})} \\
&= \frac{0.95}{0.95 + 0.25 * 9999} \\
&= 0.00038
\end{aligned}$$

(b)

Now Let W be the variable for "writing will apear on it". Thus $P(W|R) = 0.9$, $P(\bar{W}|R) = 0.1$, $P(\bar{W}|\bar{R}) = 0.95$ and $P(W|\bar{R}) = 0.05$. Since A and W are conditionally independent tests, we have

$$P(R|WA) = \frac{P(WA|R)P(R)}{P(WA)} = \frac{P(WA|R)P(R)}{P(WA|R)P(R) + P(WA|\bar{R})P(\bar{R})}$$

Since A and W are conditionally independent, therefore,

$$P(R|WA) = \frac{P(W|R)P(A|R)P(R)}{P(WA|R)P(R) + P(WA|\bar{R})P(\bar{R})} = \frac{P(W|R)P(A|R)P(R)}{P(W|R)P(A|R)P(R) + P(W|\bar{R})P(A|\bar{R})P(\bar{R})}$$

Thus,

$$P(R|WA) = \frac{0.9 \cdot 0.95 \cdot 10^{-4}}{0.9 \cdot 0.95 \cdot 10^{-4} + 0.05 \cdot 0.25 \cdot (1 - 10^{-4})} \approx 0.006794$$

5 Problem 5

Way 1

First introduce a random variable I_x, where $I_x = \begin{cases} 1, & if\ X > x \\ 0, & otherwise \end{cases}$.

Now assume $X = \sum\limits_{x=0}^{\infty} I_x$. Then $E[X] = E[\sum\limits_{x=0}^{\infty} I_x] = \sum\limits_{x=0}^{\infty} E[I_x] = \sum\limits_{x=0}^{\infty} P(X > x)$

QED.

Way 2

$$\begin{aligned}
\sum_{x=0}^{\infty} Pr[X > x] &= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} Pr[X = y] \\
&= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} P_x(y) \\
&= \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} Pr[X = y] \\
&= \sum_{y=1}^{\infty} P_x(y) \sum_{x=0}^{y-1} 1 \\
&= \sum_{y=1}^{\infty} P_x(y) y \\
&= E[X]
\end{aligned}$$

6 Problem 6

$$P_X(i) = \frac{e^{-\lambda} \lambda^i}{i!}$$

(a)

$$\begin{aligned}
\sum_{i=0}^{\infty} P_X(i) &= \sum_{i=0}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \\
&= e^{-\lambda}(1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + ...) \\
&= e^{-\lambda} e^{\lambda} \text{ (From Taylor series expansion of } e^{\lambda}) \\
&= 1
\end{aligned}$$

(b)

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{\infty} iP_X(i) = \sum_{i=0}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} \\
&= \sum_{i=1}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=1}^{\infty} i\frac{\lambda^i}{i!} \\
&= \lambda e^{-\lambda}\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda}\sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!} \\
&= \lambda e^{-\lambda}e^{\lambda} = \lambda
\end{aligned}
$$

(c)

$$
\begin{aligned}
E[X^2] &= \sum_{i=0}^{\infty} i^2P_X(i) = \sum_{i=0}^{\infty} i^2\frac{e^{-\lambda}\lambda^i}{i!} \\
&= \sum_{i=1}^{\infty} i^2\frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=1}^{\infty} i^2\frac{\lambda^i}{i!} \\
&= \lambda e^{-\lambda}\sum_{i=1}^{\infty} i\frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda}\sum_{i=0}^{\infty} (i+1)\frac{\lambda^{i}}{i!} \\
&= \lambda e^{-\lambda}[\sum_{i=1}^{\infty} i\frac{\lambda^{i}}{i!} + \sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!}] \\
&= \lambda e^{-\lambda}[\lambda\sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!} + e^{\lambda}] \\
&= \lambda e^{-\lambda}[\lambda e^{\lambda} + e^{\lambda}] = \lambda^2 + \lambda
\end{aligned}
$$

$$
Var(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
$$

7 Problem 7

(a)

$$
\int_1^{+\infty} f_X(x) = \int_1^{+\infty} \alpha x^{-\alpha-1}dx = -x^{-\alpha}|_1^{+\infty} = 0 + 1 = 1
$$

(b)

$$
E[x] = \int_1^{+\infty} x\cdot\alpha\cdot x^{-\alpha-1}dx = \int_1^{+\infty} \alpha x^{-\alpha}dx = \frac{\alpha}{-\alpha+1}x^{-\alpha+1}|_1^{+\infty} = \frac{\alpha}{\alpha-1}
$$

(c)

Similarly, we have

$$E[x^2] = \int_1^{+\infty} \alpha x^{-\alpha+1} dx = \frac{\alpha}{-\alpha+2} x^{-\alpha+2}|_1^{+\infty}$$

Since $1 < \alpha < 2$, then $0 < -\alpha + 2 < 1$. Thus $x^{-\alpha+2} \to +\infty$, so $E[x^2] = +\infty$ and $Var[x] = +\infty$