

Lemma 0.27: Composition of Bijections is a Bijection

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Lemma 0.27: Let A , B , and C be sets and suppose that there are bijective correspondences between A and B , and between B and C . Then there is a bijective correspondence between A and C .

Proof: Suppose there are bijections $f : A \rightarrow B$ and $g : B \rightarrow C$, and define $h = (g \circ f) : A \rightarrow C$. We will show that h is a bijection.¹

We first show that h is surjective, that is that h is onto. Recall that since f and g are both bijections (and hence surjections), we have that $f(A) = B$ and $g(B) = C$. Therefore we also have that

$$\begin{aligned} h(A) &= (g \circ f)(A) \\ &\stackrel{\text{def}}{=} \{c \in C \mid (g \circ f)(a) = c, \text{ for some } a \in A\} \\ &= \{c \in C \mid g(f(a)) = c, \text{ for some } a \in A\} \\ &= \{c \in C \mid g(b) = c, \text{ for some } b \in f(A)\} \\ &\stackrel{\text{def}}{=} g(f(A)) \\ &= g(B) \\ &= C \end{aligned}$$

and hence h is also surjective.² ✓

Next we will show that h is injective. That is, we will show that if $h(a) = h(a')$, then we must have that $a = a'$. Suppose that $h(a) = h(a')$. By our definition of h this means that $g(f(a)) = g(f(a'))$. However, both f and g are injective (since they are bijections) and so

$$\begin{aligned} g(f(a)) = g(f(a')) &\implies f(a) = f(a') \\ &\implies a = a', \end{aligned}$$

and hence h is injective. ✓

Since h is both surjective (onto) and injective (1-to-1), then h is a bijection, and the sets A and C are in bijective correspondence. □

¹Note that we have never explicitly shown that the composition of two functions is again a function. To save on time and ink, we are leaving that proof to be independently verified by the reader.

²In this argument, I claimed that the sets $\{c \in C \mid g(f(a)) = c, \text{ for some } a \in A\}$ and $\{c \in C \mid g(b) = c, \text{ for some } b \in f(A)\}$ are equal. In general, the proper way to show that two sets, X and Y , are equal is to show that both $X \subset Y$ and $Y \subset X$. Try showing this double inclusion explicitly for the sets mentioned above. One way should be immediately clear, and the other requires that f is onto.

What makes this a “good” proof?

Constructing a well-written proof is one of the more difficult aspects of a being a successful math student. Much like other types of writing, proof writing is somewhat of an art, and it takes a lot of reworking to get it perfect. The hallmark of a good proof is the perfect blend of clarity and elegance. It is easy to find many extreme examples in either direction; either with writing that is very terse and hard to follow, or that is so wordy and clunky that it is easy to get lost.

It can be hard to pinpoint what exactly was done right in any particular proof, and there is certainly no “formula” that can guarantee you a well-written argument. To that end I will merely point out some of the aspects of my proof that I feel are applicable in a broader sense, and that I feel are good goals to have when writing most proofs.

- The proof begins with a restatement of the initial hypotheses. Restating the theorem word-for-word is not always necessary, but you should always provide the reader with the proper set-up for the theorem. Listing all your hypotheses and assumptions/suppositions is a good way to begin any proof.
- I gave names to all of the important objects (e.g. points, functions, sets, etc.) that I will be referencing later in the proof. This lets us quickly refer to the objects by name, rather than *describing* them each time, or having to refer to them in some other nebulous way.
- Before I get into the heart of the proof, I briefly mention what it is that we’re trying to prove. If you just dive right in and don’t tell the reader where you intend on going, the reader is much more likely to get lost in the details, never having seen the intended goal.
- I “recall” the important aspects of the functions f and g when they are needed (for example, that being a bijection also includes being a surjection.) This is more of an issue of style, however; I prefer to remind my readers of important facts while they go, rather than just hoping they remember them on their own. That being said, there is a fine balance between telling the reader too much, and not telling them enough. A reader might feel like you are talking down to them if you constantly remind them of even the most simple facts; but leaving the reader entirely to follow along on their own can leave them feeling lost. You will have to find your own balance as you hone your proof-writing skills throughout this course.
- It is important to use definitions and notation well. There will be *a lot* of symbology and terminology in this class, all of which is there for a particular reason. Every definition or symbol we see is there to help us prove or understand something, and most of them are incredibly specific. Unlike an English essay, where you can use many words to elaborate on a topic, or to say the same thing in many different ways, proof writing needs to be razor-sharp. And this is really only possible with precise definitions and an agreed upon set of notation. **Set notation is particularly important.** Please make sure you are as comfortable writing with it as you are with writing English sentences.
- When I have shown a particularly important part of the argument, I emphasized it with a checkmark. This is my style, and certainly not expected of you. Try developing your own style while writing.
- After everything has been shown, and we have reached the end of the argument, I summarize the results and recap with what I was trying to prove from the beginning. The symbol \square or \blacksquare is commonly used at the end of a proof to indicate to the reader that a proof has been finished. This breaks up the proof visually, especially when it is inside a textbook, among other writing and expositions. The letters *Q.E.D.* are also frequently used, which stand for the Latin phrase, “quod erat demonstrandum” (“that which was to be demonstrated”).