Axisymmetric flow patterns in compressible ideal fluids

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(Dated: June 12, 2021)

Abstract

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I. INTRODUCTION

II. ANALYTICAL FORMULATION

The conservations of mass, and momentum, for a fluid with density ρ and flow $\vec{v}$, may be expressed through

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}), \qquad \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = -\frac{\nabla p}{\rho}, \tag{1}$$

respectively, where the pressure p and density ρ may be related via the sound speed

$$v_{\mathrm{s}} = \sqrt{\frac{dp}{d\rho}}, \tag{2}$$

with the total derivative taken in the adiabatic regime. When the flow intensity is neglibible with respect to the sound speed, Eqs. (1) and Eq. (2) are automatically satisfied by the constant and uniform density ρ_0 and pressure p_0. Consider the linear perturbations ρ_1, p_1, and $\vec{v}_1$ of density, pressure, and flow, respectively, about such an equilibrium. In that case, the first and second of Eqs. (1) furnish

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \vec{v}_1, \qquad \frac{\partial \vec{v}_1}{\partial t} = -\frac{\nabla p_1}{\rho_0}, \tag{3}$$

respectively, and Eq. (2) gives

$$\nabla p_1 = v_{\mathrm{s}}^2 \nabla \rho_1. \tag{4}$$

Combining Eqs. (3) with Eq. (4), we deduce

$$\frac{\partial^2 \vec{v}_1}{\partial t^2} = v_{\mathrm{s}}^2 \nabla \left(\nabla \cdot \vec{v}_1\right). \tag{5}$$

We now express the perturbation flow in terms of a scalar potential.

Let us take the curl of Eq. (5). Since the curl of the gradient vanishes, we derive

$$\frac{\partial^2}{\partial t^2} \nabla \times \vec{v}_1 = 0. \tag{6}$$

Suppose that the first-order time-derivative of the perturbation vorticity $\nabla \times \vec{v}_1$ also vanishes. As a consequence, ignoring all integration constants, Eq. (6) implies

$$\nabla \times \vec{v}_1 = 0. \tag{7}$$

Now, once the perturbation flow is conservative, Eq. (7) enables us to express $\vec{v}_1$ in terms of a scalar potential, say Φ,

$$\vec{v}_1 = \nabla\Phi. \tag{8}$$

Plugging Eq. (8) back in Eq. (5), we find

$$\nabla\frac{\partial^2\Phi}{\partial t^2} = \nabla\left(v_{\rm s}^2\nabla^2\Phi\right). \tag{9}$$

One more time, ignoring any integration constant, Eq. (9) leads to the widely-recognized wave equation

$$\frac{\partial^2\Phi}{\partial t^2} = v_{\rm s}^2\nabla^2\Phi \tag{10}$$

with no source nor sink term.

In this work, we are interested in describing axisymmetric flow patterns in compressible ideal fluids. Adopting cylindrical coordinates (r, θ, z), the property of axisymmetry means that the azimuthal angle θ-derivatives of scalar functions vanish. Let us express the scalar potential Φ in the axisymmetric form

$$\Phi\left(r, z - v_{\rm ph}t\right) = \phi\left(r\right)\exp\left[\imath k\left(z - v_{\rm ph}t\right)\right], \tag{11}$$

where $v_{\rm ph}$ is the phase speed of fluid oscillations with wavenumber k along the axial z-direction. Substituting Eq. (11) in Eq. (8), we get the noncircular flow

$$\vec{v}_1 = \hat{r}v_r + \hat{z}v_z. \tag{12}$$

It follows from Eq. (12) that the so-called streamline equation may be expressed through

$$\frac{dr}{v_r} = \frac{dz}{v_z} \tag{13}$$

at a fixed instant. We have three situations.

III. SUBSONIC MODES

The condition $v_{\rm ph} < v_{\rm s}$ determines the propagation of subsonic modes in the fluid. In that situation, substituting Eq. (11) in Eq. (10), we deduce the well-known zero-order modified Bessel equation,

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \phi = 0 \tag{14}$$

for the potential amplitude $\phi(x)$, where $\gamma x = kr$, with

$$\gamma = \frac{1}{\sqrt{1 - v_{\rm ph}^2/v_{\rm s}^2}} > 1. \tag{15}$$

The general solution of Eq. (14) may be expressed in terms of the zero-order first I_0 and second K_0 kind modified Bessel functions,

$$\phi(x) = \phi_0 I_0(x) + \varphi_0 K_0(x), \tag{16}$$

where ϕ_0 and φ_0 are constants.

The solution (16) is not finite along the cylindrical z-axis because in the limit $x \to 0$, the function $K_0(x)$ has a logarithmic singularity. To overcome this difficulty, we choose $\varphi_0 = 0$, in such a way that Eq. (16) simplifies to

$$\phi(x) = \phi_0 I_0(x). \tag{17}$$

Substituting Eq. (17) in Eq. (11), we derive the full expression

$$\Phi(x, z - v_{\rm ph}t) = \phi_0 I_0(x) \exp\left[\imath k (z - v_{\rm ph}t)\right], \tag{18}$$

for the scalar potential. Therefore, the perturbation flow follows from substituting Eq. (18) in Eq. (8),

$$\vec{v}_1 = k\phi_0 \left[\frac{\hat{r}}{\gamma} I_1(x) + \imath \hat{z} I_0(x)\right] \exp\left[\imath k (z - v_{\rm ph}t)\right], \tag{19}$$

where $I_1(x) = I_0'(x)$ is the first-order first-kind modified Bessel function, with the prime denoting a first-order x-derivative. Let us consider the perturbation streamlines in the fluid. Substituting the real part of Eq. (19) in Eq. (13), we find

$$\frac{I_0(x)}{I_1(x)} dx + \frac{\cot(kz)}{\gamma^2} d(kz) = 0 \tag{20}$$

at the initial instant $t = 0$. Integrating Eq. (20), we get the curve family

$$\ln\left|\frac{kr}{\gamma} I_1\left(\frac{kr}{\gamma}\right)\right| + \frac{\ln|\sin(kz)|}{\gamma^2} = C, \tag{21}$$

where a given value of the constant C defines a particular streamline, with $kr > 0$, and $kz \neq n\pi$ for n integer.

IV. SONIC MODES

The condition $v_{\mathrm{ph}} = v_{\mathrm{s}}$ determines the propagation of sonic modes in the fluid. In that situation, substituting Eq. (11) in Eq. (10), we deduce the differential equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0, \tag{22}$$

for the potential amplitude $\phi\left(r\right)$. The general solution of Eq. (22) is given by

$$\phi\left(r\right) = \phi_0 \ln\left(\frac{r}{r_0}\right), \tag{23}$$

where ϕ_0 and $r_0 > 0$ are constants.

The solution (23) is not finite along the cylindrical z-axis because in the limit $r \to 0$, the function $\phi\left(r\right)$ has a singularity. Substituting Eq. (23) in Eq. (11), we derive the full expression for the scalar potential,

$$\Phi\left(r, z - v_{\mathrm{s}}t\right) = \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left[\imath k\left(z - v_{\mathrm{s}}t\right)\right]. \tag{24}$$

Therefore, the perturbation flow follows from substituting Eq. (18) in Eq. (8),

$$\vec{v}_1 = k\phi_0 \left[\frac{\hat{r}}{kr} + \imath\hat{z}\ln\left(\frac{r}{r_0}\right)\right] \exp\left[\imath k\left(z - v_{\mathrm{s}}t\right)\right]. \tag{25}$$

Let us consider the perturbation streamlines in the fluid. Substituting the real part of Eq. (25) in Eq. (13), we find

$$\frac{r}{r_0}\ln\left(\frac{r}{r_0}\right) d\left(\frac{r}{r_0}\right) + \frac{\cot\left(kz\right)}{\gamma^2} d\left(kz\right) = 0 \tag{26}$$

at the initial instant $t = 0$, where

$$\gamma = kr_0 > 0. \tag{27}$$

Integrating Eq. (27), we get the curve family

$$\frac{k^2r^2}{2\gamma^2}\ln\left(\frac{kr}{\gamma}\right) - \frac{k^2r^2}{4\gamma^2} + \frac{\ln\left|\sin\left(kz\right)\right|}{\gamma^2} = C, \tag{28}$$

where a given value of the constant C defines a particular streamline, with $kr > 0$, and $kz \neq n\pi$ for n integer.

V. SUPERSONIC MODES

The condition $v_{\rm ph} > v_{\rm s}$ determines the propagation of supersonic modes in the fluid. In that situation, substituting Eq. (11) in Eq. (10), we deduce the well-known zero-order Bessel equation,

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} + \phi = 0 \tag{29}$$

for the potential amplitude $\phi(x)$, where $\gamma x = kr$, with

$$\gamma = \frac{1}{\sqrt{v_{\rm ph}^2/v_{\rm s}^2 - 1}} > 0. \tag{30}$$

The general solution of Eq. (29) may be expressed in terms of the zero-order first J_0 and second Y_0 kind Bessel functions,

$$\phi(x) = \phi_0 J_0(x) + \varphi_0 Y_0(x), \tag{31}$$

where ϕ_0 and φ_0 are constants.

The solution (31) is not finite along the cylindrical z-axis because in the limit $x \to 0$, the function $Y_0(x)$ has a logarithmic singularity. To overcome this difficulty, we choose $\varphi_0 = 0$, in such a way that Eq. (31) simplifies to

$$\phi(x) = \phi_0 J_0(x). \tag{32}$$

Substituting Eq. (32) in Eq. (11), we derive the full expression

$$\Phi(x, z - v_{\rm ph}t) = \phi_0 J_0(x) \exp[\imath k(z - v_{\rm ph}t)], \tag{33}$$

for the scalar potential. Therefore, the perturbation flow follows from substituting Eq. (33) in Eq. (8),

$$\vec{v}_1 = k\phi_0 \left[-\frac{\hat{r}}{\gamma} J_1(x) + \imath\hat{z} J_0(x)\right] \exp[\imath k(z - v_{\rm ph}t)], \tag{34}$$

where $J_1(x) = -J_0'(x)$ is the first-order first-kind Bessel function, with the prime denoting a first-order x-derivative. Let us consider the perturbation streamlines in the fluid. Substituting the real part of Eq. (34) in Eq. (13), we find

$$-\frac{J_0(x)}{J_1(x)}dx + \frac{\cot(kz)}{\gamma^2} d(kz) = 0 \tag{35}$$

at the initial instant $t = 0$. Integrating Eq. (35), we get the curve family

$$-\ln\left|\frac{kr}{\gamma}J_1\left(\frac{kr}{\gamma}\right)\right| + \frac{\ln|\sin(kz)|}{\gamma^2} = C, \tag{36}$$

where a given value of the constant C defines a particular streamline, with $kr > 0$, and $kz \neq n\pi$ for n integer.

VI. CONCLUSION

ACKNOWLEDGMENTS

FEMS is partially supported by São Paulo Research Foundation (FAPESP) under grant number 17/20192-2. GRAS is supported by Coordination of Superior Level Staff Improvement (CAPES) under a MSc scholarship.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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