Graphical Inginuety: Spectral Solutions to Combinatorial Conondrums

Garrett J. Kepler

April 22nd, 2024

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Preface

SGT Talk Thursday 4/25

Reimagining Spectral Graph Theory

Dr. Stephen Young

Algorithms, Combinatorics, and Optimization Team Lead
Pacific Northwest National Laboratory

April 25, 2024 at 4:20
Spark 335

Zoom ID: 992 1304 8021 Passcode: 483214

Refreshments at 3:30pm
Neill 216 (Hacker Lounge)

Background

Basics

Definition

The adjacency matrix of a graph G, $A(G) = A = [a_{ij}]$, is the $n \times n$ matrix such that

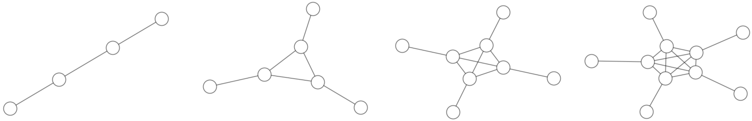
$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ adjacent to } j \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, 2, \cdots, n$

Definition

Throughout, the spectrum of a graph G will refer to the collection of eigenvalues of A, $\sigma(A)$, with the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Examples I



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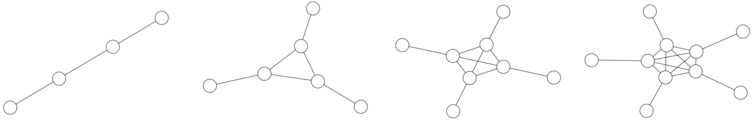

Adjacency spectra starting from the left where $\phi = \frac{1+\sqrt{5}}{2}$:

- $\sigma(A) = \{-\phi, -\frac{1}{\phi}, \frac{1}{\phi}, \phi\}$

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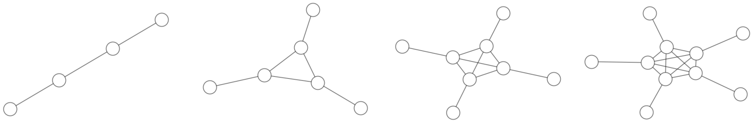

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- $\sigma(A) = \{-\phi, -\phi, 1-\sqrt{2}, \frac{1}{\phi}, \frac{1}{\phi} 1+\sqrt{2}\}$
- $\sigma(A) = \{-\phi, -\phi, -\phi, -.302\cdots, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, 3.302\cdots\}$
- $\sigma(A) = \{-\phi, -\phi, -\phi, -\phi, .236\cdots, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, 4.236\cdots\}$

Examples II

- Let $[n] = \{1, 2, \cdots, n\}$.
- Let nodes of graph G correspond to distinct k-element subsets of $[n]$.
- If $|i \cap j| = 0$ for $i, j \in V(G)$, then let i be adjacent to j i.e. $(i, j) \in E(G)$.

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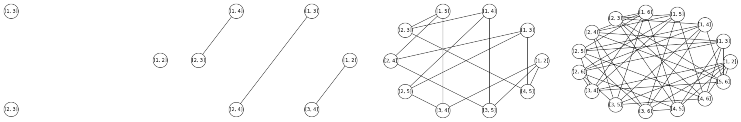

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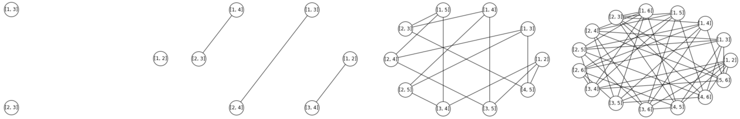

Figure: The Kneser Graphs $K(n, k)$ with $\sigma(A) = \{(-1)^j \binom{n-k-j}{k-j} \mid j = 0, 1, 2, \cdots, k\}$

Structure and Spectra

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- $\sum_{i=1}^{n} \lambda_i = 0$
- $\mathrm{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2 = 2m$ where m is the number of edges in the graph
- $\Delta(G) \geq \lambda_1 \geq \bar{d}(G)$ where $\bar{d}(G)$ is the average degree in G and $\Delta(G)$ is the max. Also, $\lambda_1 = \bar{d}(G)$ for k-regular graphs where $\bar{d} = k$.

Hoffman's Bound

Theorem

Let $G = (V, E)$ be finite, k-regular graph on n nodes with adjacency matrix A. Let $S \subset V(G)$ be an independent set of vertices in G. Then

$$\frac{|S|}{n} \leq \frac{-\lambda_n}{k - \lambda_n}$$

Proof

- Let $\mathbb{R}^n$ be equipped with inner product

$$\langle u, v \rangle = \frac{1}{n} \sum_{x \in V(G)} u(x)v(x)$$

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- Using the orthonormal basis of eigenvectors for A, $\{v_1, v_2, \cdots, v_n\}$, we can rewrite 1_S as: $1_S = \sum_{i=1}^{n} \alpha_i v_i$.

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- Let $\mathbf{j}$ be the all ones vector. Note that $\alpha_1 = \langle 1_S, \mathbf{j} \rangle = \frac{|S|}{n} = \langle 1_S, 1_S \rangle$

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- Since all v_i are normalized, $\langle 1_S, 1_S \rangle = \sum_{i=1}^n \alpha_i^2$

Proof

- Lastly, since $0 = \langle A1_S, 1_S \rangle = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \geq d\alpha_1^2 + \lambda_n \sum_{i=2}^{n} \alpha_i^2 = d\alpha_1^2 + \lambda_n(\alpha_1 - \alpha_1^2)$

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- Thus, $\frac{|S|}{n} = \alpha_1 \leq \frac{-\lambda_n}{d-\lambda_n}$

Problem I

Erdös-Ko-Rado

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Question

For $n, k \in \mathbb{N}$, what is the maximum size of an intersecting family of k-element subsets of an n-element set?

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- Note that the Kneser Graph is $\binom{n-k}{k}$-regular with spectrum $\{(-1)^j\binom{n-k-j}{k-j} \mid j = 0, 1, 2, \cdots, k\}$ making $\lambda_n = -\binom{n-k-1}{k-1}$.

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- So, using Hoffman's Bound,

$$\frac{|S|}{n} \leq \frac{-\lambda_n}{k-\lambda_n} \longrightarrow \frac{\alpha}{\binom{n-k}{k}} \leq \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}}$$

$$\iff \alpha \leq n\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}$$

Problem II

Dimer Problem

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Short answer: take the permanent! ($\text{per}(A) = p^2$)

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- Likewise, $\det(A) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\right) = \sum_{L \subset H} (-1)^{c(L)}$ where $c(L)$ is the number of components in L.

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- A can be factored as $A_m \otimes \mathbb{I}_n + A_n \otimes \mathbb{I}_m$.

Count

- Take $A* = A_m \otimes \mathbb{I}_n + iA_n \otimes \mathbb{I}_m$ (multiplying each horizontal edge by i.
- Then, since $(-1)^{c(L)} = i^{h(L)}$, $\text{per}(A) = \det(A*)$ and so $\det(A*) = p^2$
- Thus, $p^2 = \prod_{i=1}^{n} \lambda_i$ for $\lambda_i \in \sigma(A*)$ i.e.

$$p^2 = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1} l\right)\right)$$

Problem 3: Open

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- Let $k < h < n$ with $k << n$ and consider a partially constructed permutation of the type $\mathbf{R}^{(k)}$:

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The limit

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Question

What about for permutations π such that $j \leq \pi(i) - i \leq k$?