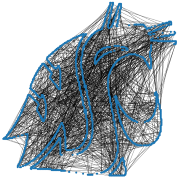

Applications in and Methods of Spectral Graph Theory

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Math 453
November 21st, 2025

Seminars

The Mathematics and Statistics Department at WSU host a number of seminars. The Math 593 Seminar (The Theory and Applications of Discrete Math, Linear Algebra, and Number Theory Seminar) occurs every Monday during the semester at 4:10-5PM in Webster 11. Stop by!

Cat 1

Cat 1

Figure 1: Beetle: Goodest Boy

Eigenvectors of Matrices

What is an eigenvector? The matrix perspective informs us that an eigenvector is a vector whose relationship to a matrix can be described with a scalar. That is,

Definition

If $Ax = \lambda x$ for scalar λ, vector x and matrix A, then x is an *eigenvector* of A corresponding to *eigenvalue* λ.

Eigenvectors of Graphs

Definition

If $Ax = \lambda x$ for scalar λ, vector x and matrix A, then x is an *eigenvector* of A corresponding to *eigenvalue* λ.

The definition of an eigenvector of a graph is natural then:

Definition

If A is a matrix associated to a graph G such that $Ax = \lambda x$, then x is an *eigenvector* of the graph G corresponding to λ.

<u>Note</u>: We did not specify what this matrix A is. This could be the adjacency matrix or the Laplacian or whatever we like!

Laplacian

For this talk, we will be discussing the Laplacian:

Definition

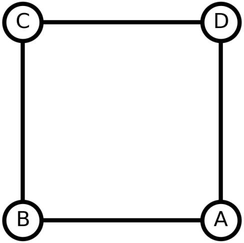
Let G be a (simple, undirected) graph on n nodes. Then, the *Laplacian* matrix of G is the $n \times n$ matrix L with entries

$$L_{ij} = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

where $d(i)$ denotes the degree of the node i and $i \sim j$ means i is adjacent to j.

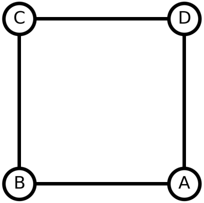
Laplacian Example

As an example, consider the following graph:



Note that each degree is 2. Maybe we label the rows and columns lexicographically. Then, our Laplacian is ...

Laplacian Example



$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Fiedler Vector

One fantastic result from spectral graph theory is that the Laplacian's *second eigenvector* for a connected graph G splits our graph into two *connected* components:

Theorem

Let G be a connected graph with Laplacian L. If L has eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$. Then, the eigenvector x_2 corresponding to λ_2 has the following properties:

- *The induced subgraph $G_- = \{i \in G | x_i < 0\}$ is connected.*
- *Likewise, the induced subgraph $G_{\geq 0} = \{i \in G | x_i \geq 0\}$ is connected.*

Not only that, the Fiedler vector is a good estimator of bottlenecks in graphs!

Example

One fun fact about the Fiedler vector is pretty good at approximating where to cut a graph in two with little work. Consider the following graph

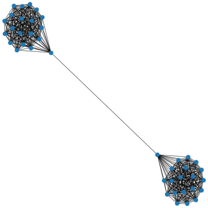


Figure 2: Barbell Graph

If I have edge cutting scissors, what is the least number of edges I need to split the graph in two? Which edges do I cut?

Example

Looking at the Fiedler vector entries, we can figure out which edge to cut!

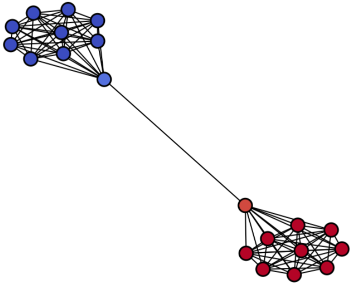


Figure 3: Barbell Graph Colored According to the Non-negative Entries (Blue) and Negative Entries (Red) of Corresponding Fiedler Vector

Example 2

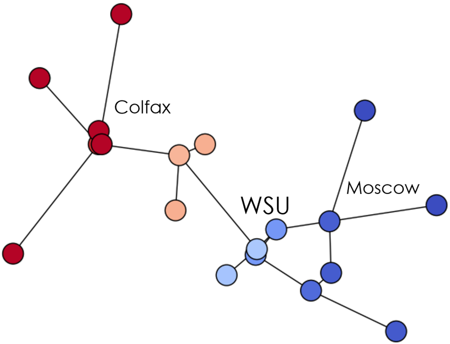
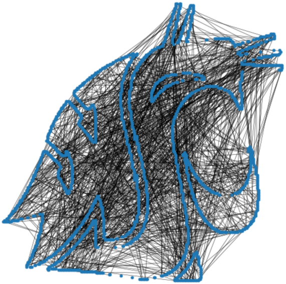


Figure 4: Palouse Natural Gas Pipelines Colored by its Fiedler Vector

Example 3



Example 3

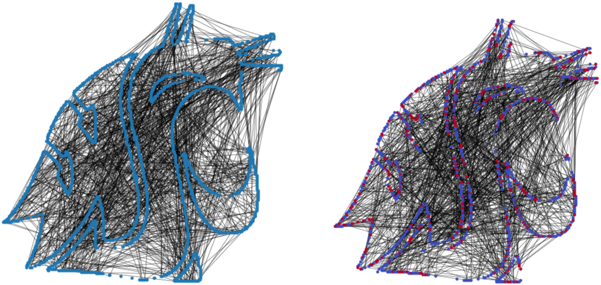


Figure 5: WSU Logo colored by its Fiedler Vector

Example 3

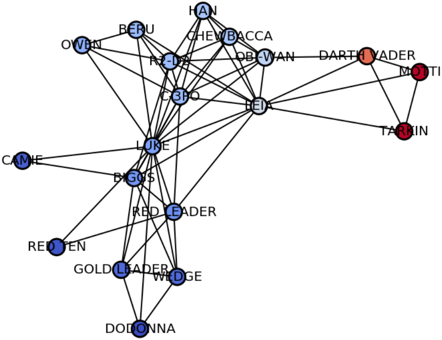
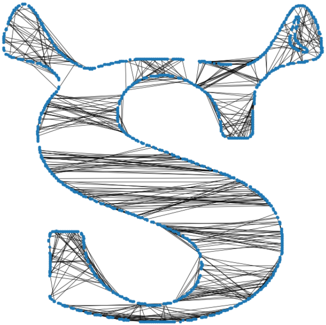
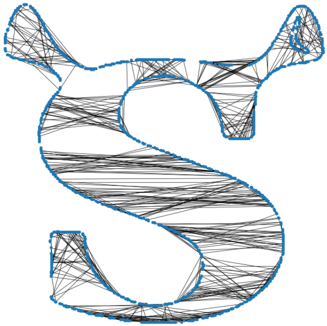
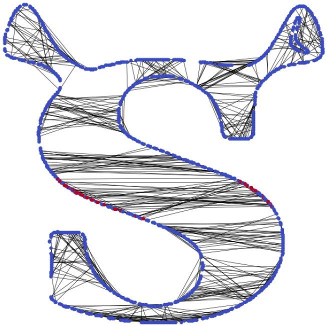


Figure 6: Star Wars On-Screen Social Network Colored by its Fiedler Vector

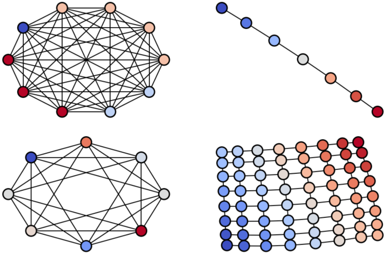






Question

A big question we have is how do we relate these eigenvectors of graphs? What happens to graph eigenvectors when we perturb the graph?



Google Collab interlude!

Math 325

If any of these methods pique your interest, try Math 325: Combinatorics in the Spring!

Cat 2

Cat 2

Figure 7: Nezuko: Princess of Chaos

Weak Law of Large Numbers

Let's talk about some probabilistic approaches we may take to these problems.

- Google Collab interlude!

Weak Law of Large Numbers

So, we can safely find the mean of a distribution by repeated sampling. But, the mean of a distribution does not fully describe the distribution.

- What if we wanted to sample from the distribution and we had no way of knowing how to? MCMC!

Data Collection for MCMC

What is the best potato dish? French fries, mashed potatoes, or baked/oven-roasted potatoes? (A vote for educational purposes)



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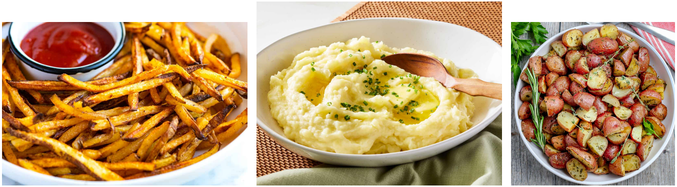

Figure 8: Mashed Potatoes and Two Wrong Answers

Google Collab interlude!

Markov Chain Monte Carlo (MCMC) Conclusion

So, in essence, we do not know exactly *what* graphs have similar eigenvectors under perturbation of some graph. But, we can hope that we can approximate the subset of graphs that do! This is not the full story though and much work is to be done to fully understand these mechanisms.

Data Science

We often use spectral methods in data analysis without clear understanding of the underlying mechanisms. Clearly understanding the properties of graph eigenvectors may help formalize these methods.

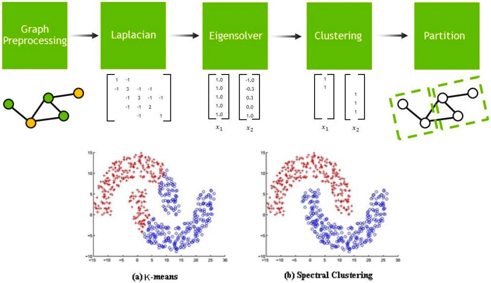

Figure 9: Example of how spectral clustering is used in Data Analysis

Gerrymandering

When analyzing state redistricting plans, we utilize graph theoretic methods to generate and analyze potential maps. While in practice they generate nice districts and allow for nice analysis of potential maps, explicitly describing how this is a product of the eigenvectors would help formalize this.

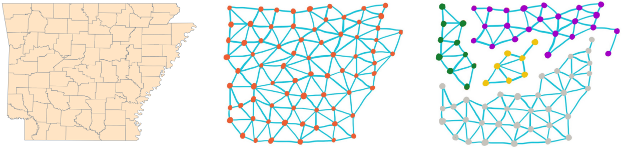

Figure 10: Dual graph of Arkansas