Graph Theory and Gerrymandering: Computationally Assessing Fairness

Garrett J. Kepler
Washington State University
Math 398 Seminar
April 22nd, 2025

Epoch 1: Gary Isaacs

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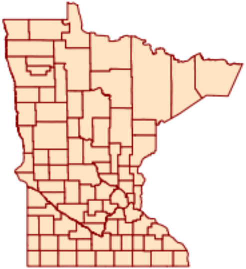

Epoch 2: Jaime Garcia

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Epoch 3: Andrea Arauza Rivera

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Epoch 3: Andrea Arauza Rivera (continued)

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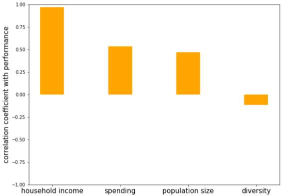

Figure 1: High School Standardized Test Data in RUMBA v1

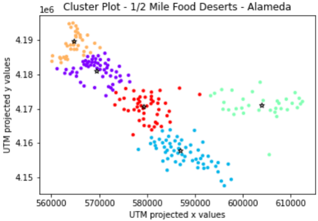

Figure 2: Food Desert Data Clustering in RUMBA v2

Epoch 4: Daryl DeFord

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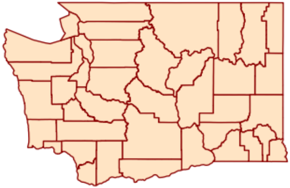

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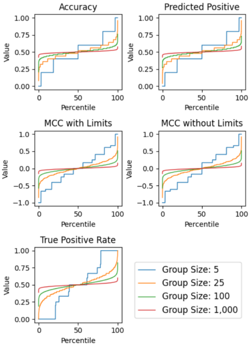

Figure 3: Machine Learning Metrics Bias

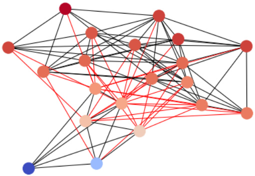

Figure 4: Resulting Spectral Partition from MCMC

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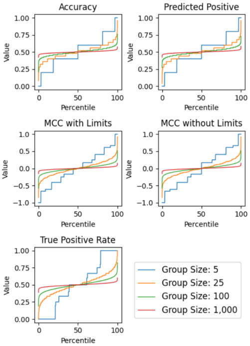

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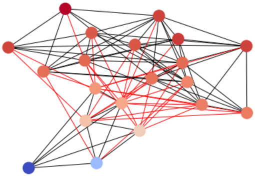

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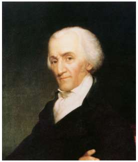

Figure 5: Elbridge Gerry Himself

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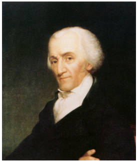

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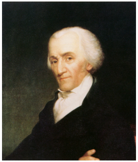

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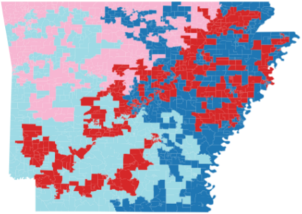

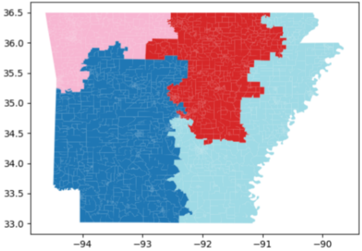

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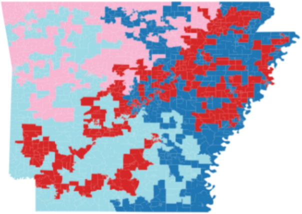

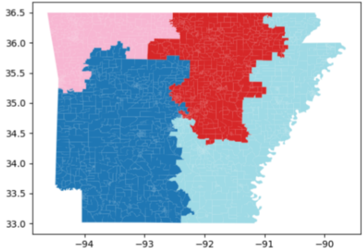

- "I mean. . . look at it!"

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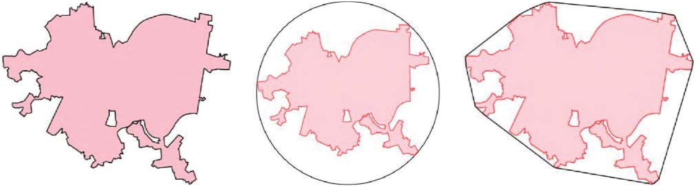

(a) City Boundary (b) Bounding Circle (c) Convex Hull

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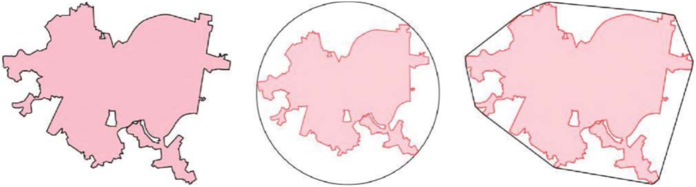

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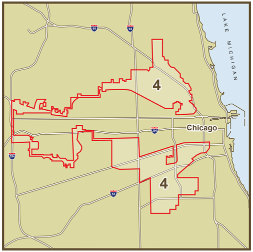

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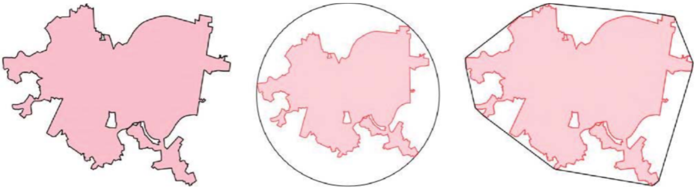

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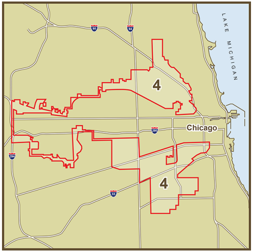

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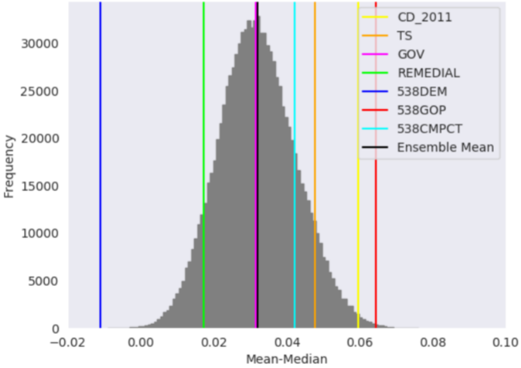
Ensemble Analysis



- "If the map you gave me is soooo fair, it better be fairer than lots of random maps!"

Gerrymandering Resources

https://mggg.org/districtr

https://redistrictingdatahub.org/

RDH

Check-In

1. My Life & I
2. Gerrymandering Goodies
3. Gerrymandering Goodies
4. **Graph Theory Essentials**
5. Spectral Graph Theory
6. Invitation for Exploration

Graph Theory Basics

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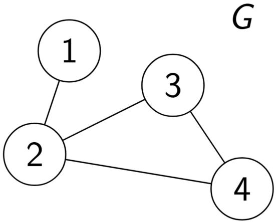

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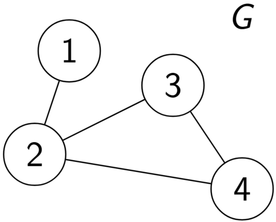

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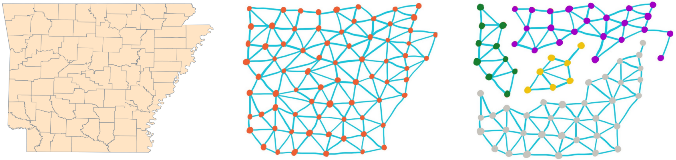

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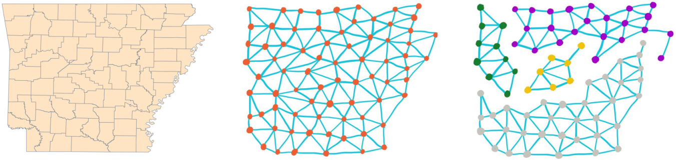

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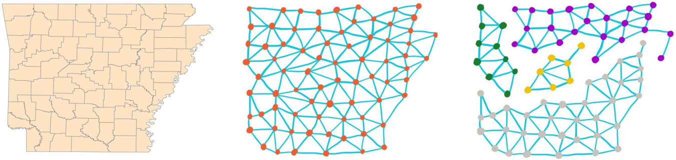

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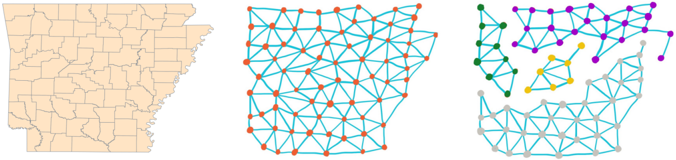

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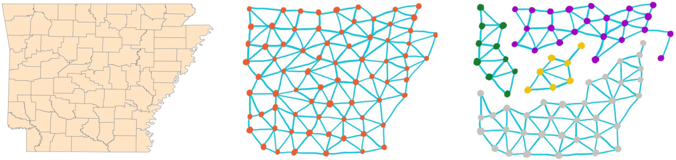

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Rabbit hole: this is called Markov Chain Monte Carlo (MCMC)

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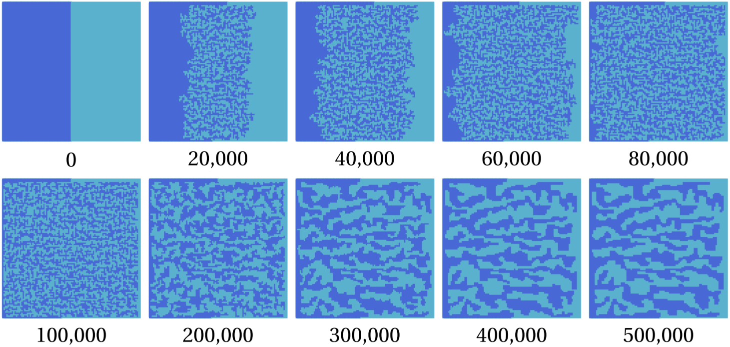

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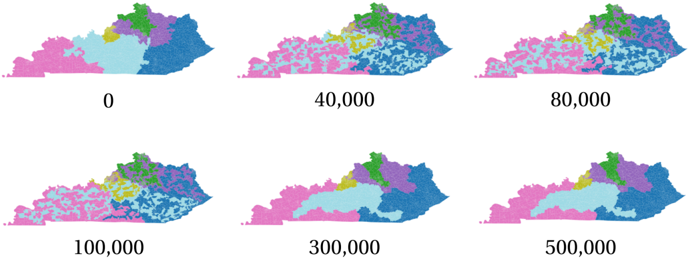

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The *Laplacian* of G is the matrix L defined such that

$$L_{ij} = \begin{cases} \text{degree of node } i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and node } i \sim \text{node } j \\ 0 & \text{otherwise} \end{cases}$$

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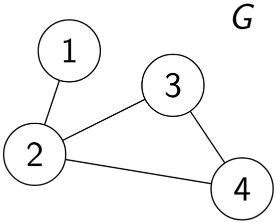

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

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- If my matrix A takes in a vector x and outputs the same vector times some number λ, then I have an eigenvector.

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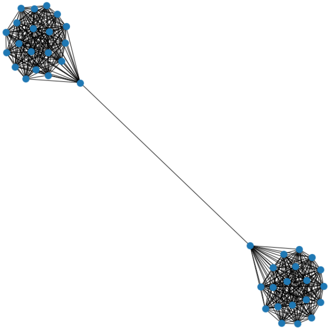

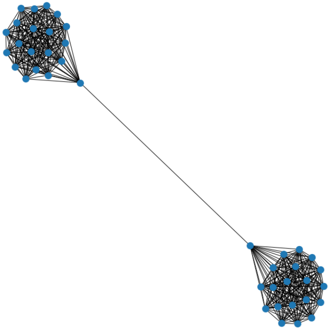

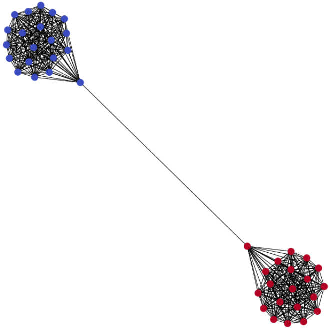

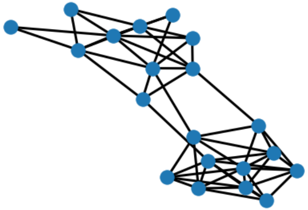

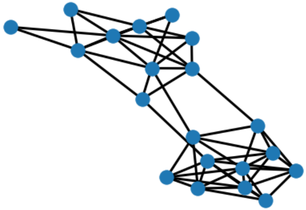

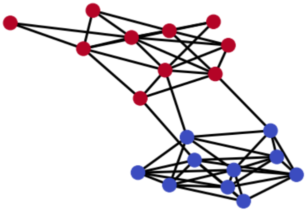

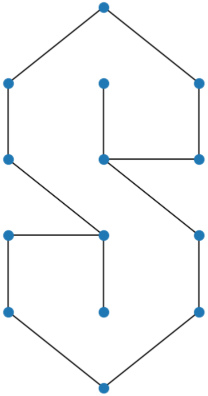

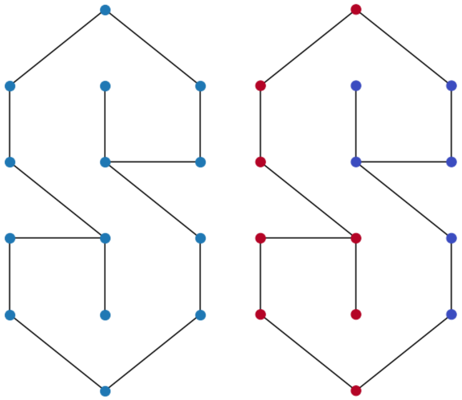

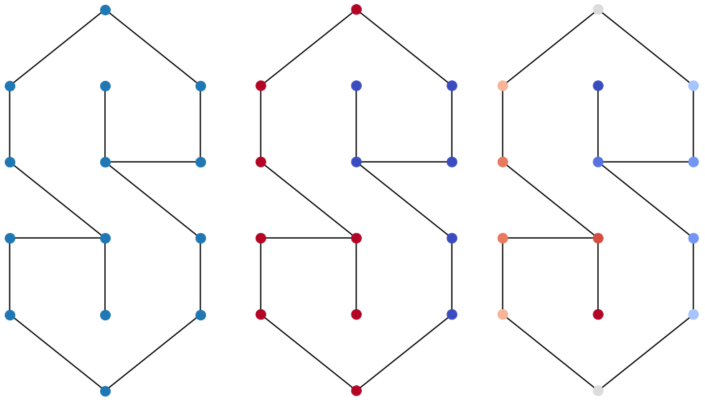

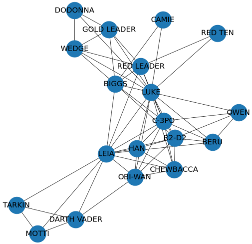
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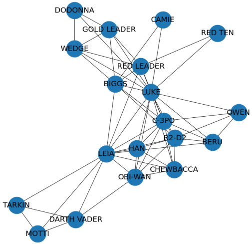
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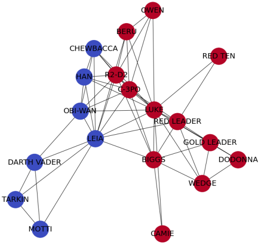
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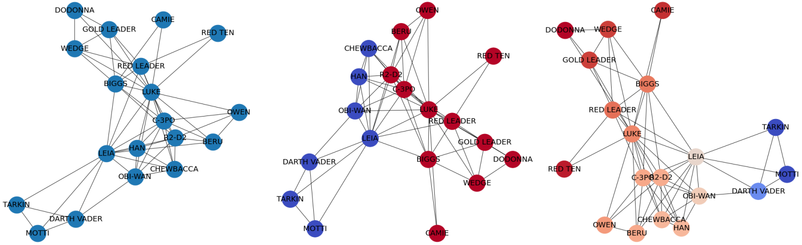
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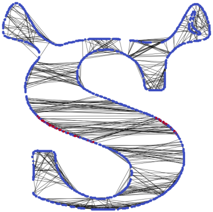

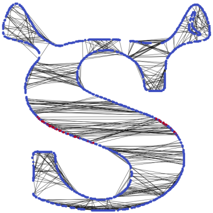

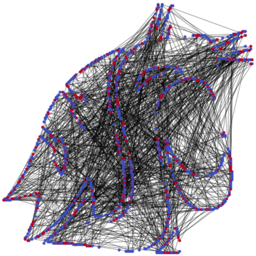

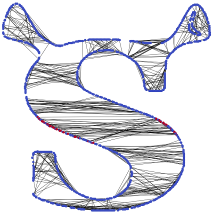

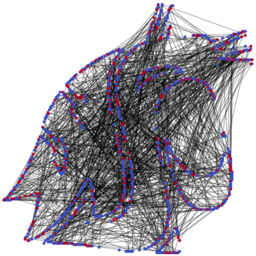

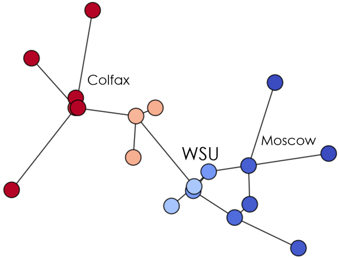
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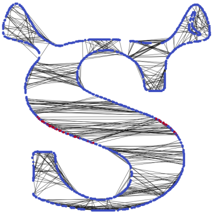

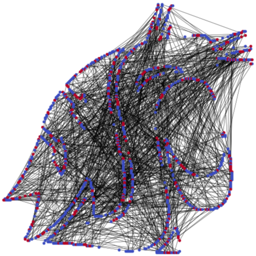

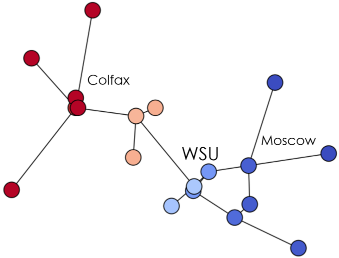
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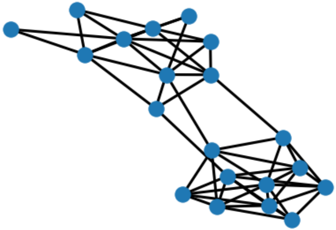

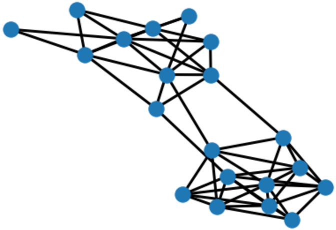

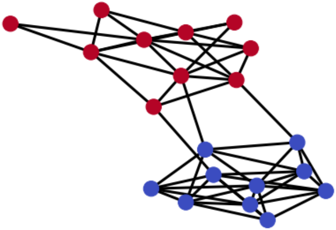

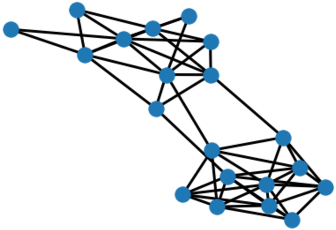

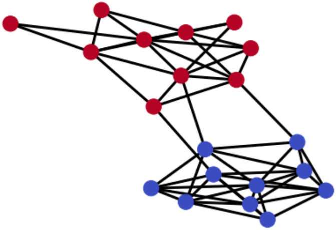

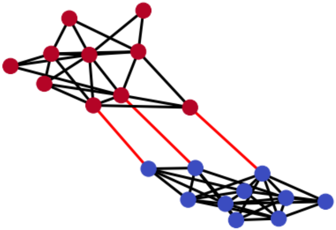

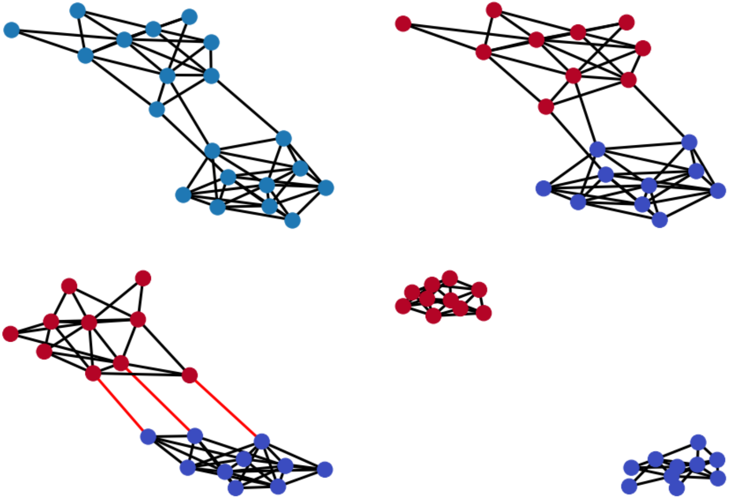

Question

Given the two connected components of a Fiedler-partitioned graph, can we determine what set of edges are missing using the entries of the Fiedler vector?

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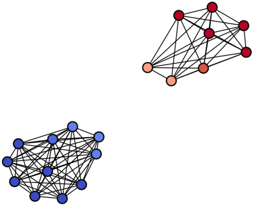

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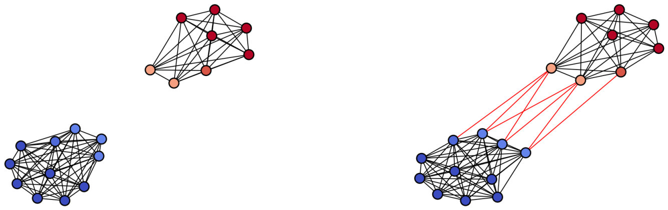

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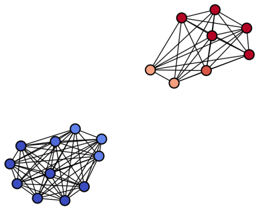

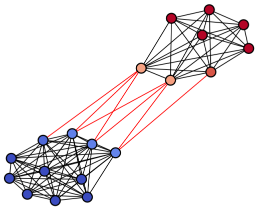

Progress & Future Progress

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Equitable Spectral Partitions

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- Using the Fiedler vector, finding edges between oppositely signed nodes helps us find bottlenecks
- Let's use this to our advantage!

Spectral Partitioning Algorithm

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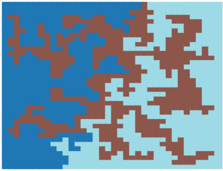

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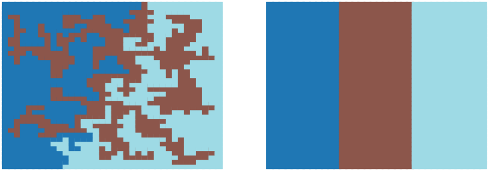

Figure 7: Convergence to a "Nice" Partition of the Algorithm After 10,000 Steps on a 36 by 36 Grid Graph

Main Questions

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Main Questions

Question

- Does this always converge?
- What happens if we choose different classes of graphs?
- Does this also converge using other spectral clustering techniques?
- In what context is it appropriate to use these types of algorithms while drawing maps?

Conclusion

Conclusion

- Math is wherever we put it!

Rabbit Holes

Rabbit Holes

1. Networks & Data Science:

Rabbit Holes

1. Networks & Data Science:
   1. https://networkrepository.com/
   2. "Network Science" by Barabási
      https://networksciencebook.com/
2. Gerrymandering:

Rabbit Holes

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2. Gerrymandering:
   1. https://redistrictingdatahub.org/
   2. https://mggg.org/districtr
   3. "Political Geometry" by Duchin & Walch https://mggg.org/gerrybook.html

Questions?

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Thanks!

garrett.kepler@wsu.edu