



MCMC: An Introduction

Garrett J. Kepler
Washington State University
Math 420
November 19th, 2025

① Probability Practice

② History 1

③ Markov Chains

④ History 2

⑤ MCMC

⑥ Applications

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Coin Flipping

- Let's try flipping some coins!

WLLN

Theorem (Weak Law of Large Numbers)

If X_1, \dots, X_n are independent and identically distributed random samples, $\bar{X}_n \xrightarrow{P} \mu$ as $n \rightarrow \infty$. That is, the sample average converges to the mean of the distribution each X_i follows

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Pavel Nekrasov believed independence was necessary for this to hold and also claimed this had more metaphysical implications.



Figure 1: Pavel Nekrasov

Markov proved this is not the case using Markov Chains!



Figure 2: Andrey Markov

Markov's Example

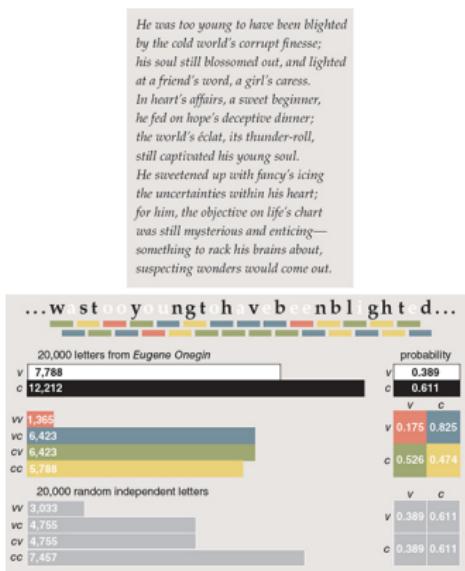


Figure 3: The distribution of consonants and vowels Markov found

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Definition

A Markov Chain is a sequence of events whose probability of transitioning to another event is only dependent on the current event.

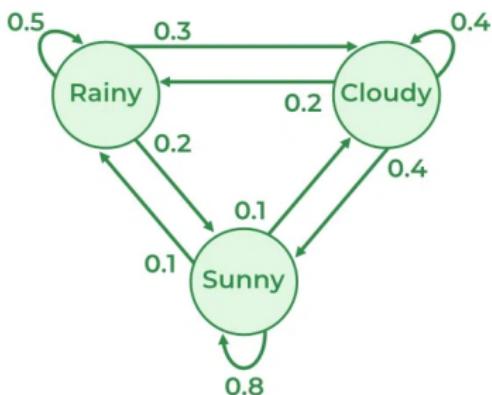


Figure 4: Basic Markov Chain

Matrix View

The matricial view aids in describing Markov Chains using stochastic matrices. Consider the weighted case of our coin flip game:

$$\begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix}$$

If we wish to assess the states of our heads or tails game at time step k , we merely have to raise this matrix to the power k .

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Inventor of Monte Carlo

Over a game of solitaire during a bed-ridden illness, Ulam invented what we call Monte Carlo.

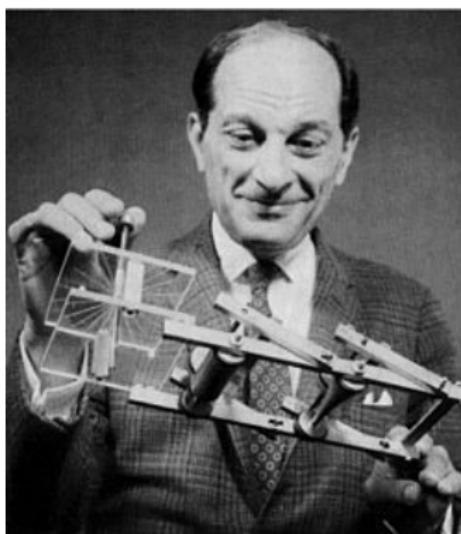


Figure 5: Stanislav Ulam, creator of Monte Carlo methods

Monte Carlo

The Monte Carlo method can be summarized as follows:

- Define a domain
- Generate samples from a distribution over domain
- Apply a function to determine outputs
- Accept sample or reject sample according to your desire

Example

We can use this to approximate π ! (Buffon's Needle)

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Combine the Two!

The MCMC method combines the two ideas:

- Start with an initial state X_0
- Propose a next step X'
- Compute acceptance probability $\alpha = \min\left(1, \frac{\mathbb{P}[X_0 \rightarrow X']}{\mathbb{P}[X' \rightarrow X_0]}\right)$
- Sample $U \sim \text{Uniform}(0, 1)$
- If $U \leq \alpha$, transition to X' . Else, stay put and try again!

1 Probability Practice

2 History 1

3 Markov Chains

4 History 2

5 MCMC

6 Applications

Gerrymandering

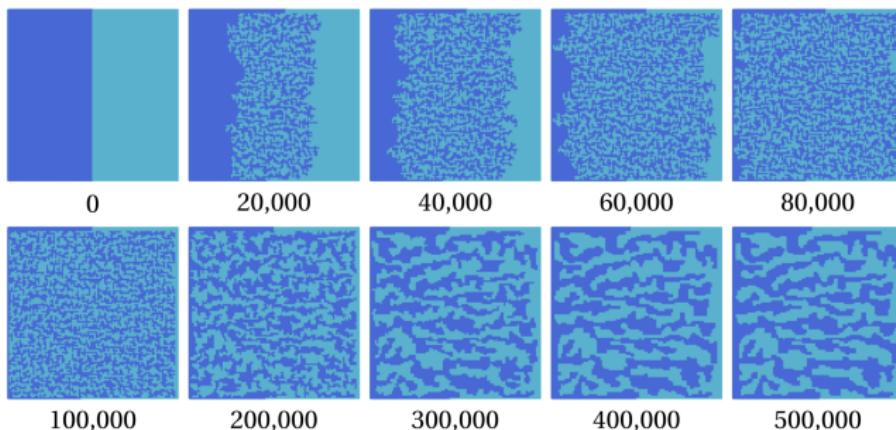


Figure 6: Generating sample of a state (guess which one?)

Gerrymandering

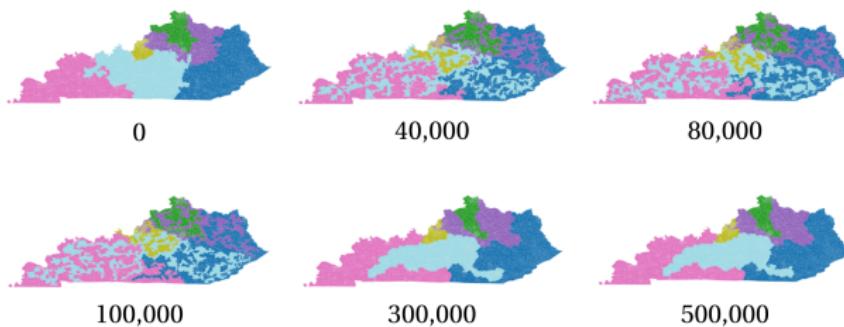


Figure 7: Generating sample of a state (guess which one?)

Gerrymandering

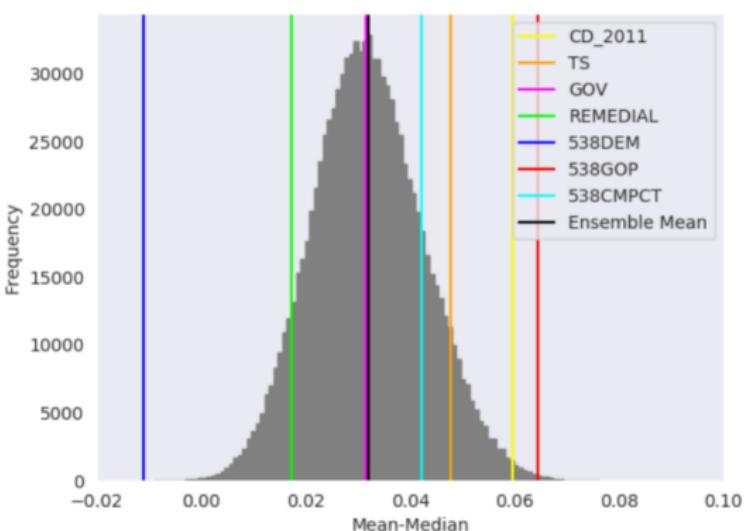


Figure 8: Ensemble approach in Gerrymandering

Cryptography

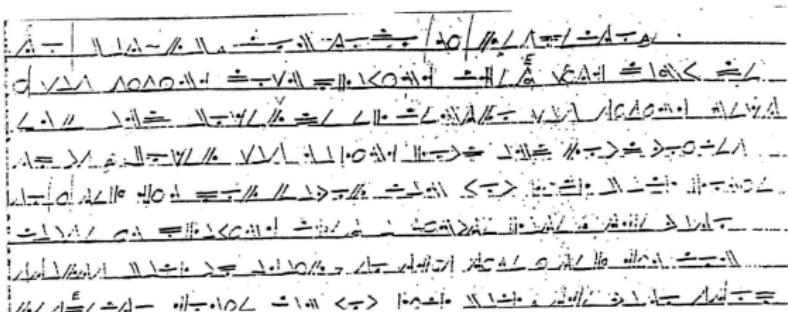


Figure 9: An encrypted message from inside a prison in the Bay Area, California

Cryptography

ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS
NOBLER IN THE MIND TO SUFFER THE SLINGS AND ARROWS OF OUTRAGEOUS
FORTUNE OR TO TAKE ARMS AGAINST A SEA OF TROUBLES AND BY OPPOSING END

Figure 10: Hamlet as we know and love it

Cryptography

100 ER ENOHDIAE OHDLO UOZEOUNORU O UOZEO HD OITO HEOQSET IUROFHE HENO ITORUZAEN
200 ES ELOHRNDE OHRNO UOVEOULOSU O UOVEO HR OITO HEOQAET IUSOPHE HELO ITOSUVDEL
300 ES ELOHANDE OHANO UOVEOULOSU O UOVEO HA OITO HEOQRET IUSOFHE HELO ITOSUVDEL
400 ES ELOHINME OHINO UOVEOULOSU O UOVEO HI OATO HEOQRET AUSOWHE HELO ATOSUVTEL
500 ES ELOHINME OHINO UODEOULOSU O UODEO HI OATO HEOQRET AUSOWHE HELO ATOSUDTEL
600 ES ELOHINME OHINO UODEOULOSU O UODEO HI OATO HEOQRET AUSOWHE HELO ATOSUDTEL
900 ES ELOHANME OHANO UODEOULOSU O UODEO HA OITO HEOQRET IUSOWHE HELO ITOSUDTEL
1000 IS ILCHANMI OHANO RODIORLORS O RODIO HA OETO HIOQUIT ERSOWHI HILo ETOSRDMIL
1100 ISTILOHANMITOHANOT ODIO LOS TOT ODIOTHATOEROATIOQUIRTE SOWHITHILOTEROS DMIL
1200 ISTILOHANMITOHANOT ODIO LOS TOT ODIOTHATOEROATIOQUIRTE SOWHITHILOTEROS DMIL
1300 ISTILOHARMITOHAROT ODIO LOS TOT ODIOTHATOENOTHOQUINTE SOWHITHILOTENOS DMIL
1400 ISTILOHAMRITOHAMOT OFIO LOS TOT OFIOTHATOENOTHOQUINTE SOWHITHILOTENOS FRIL
1600 ESTEL HAMRET HAM TO CE OL SOT TO CE THAT IN THE QUENTIOS WHETHER TIN SOCREL
1700 ESTEL HAMRET HAM TO BE OL SOT TO BE THAT IN THE QUENTIOS WHETHER TIN SOBREL
1800 ESTER HAMLET HAM TO BE OR SOT TO BE THAT IN THE QUENTIOS WHETHER TIN SOBLER
1900 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER
2000 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER

Figure 11: Recovering Hamlet using MCMC!