BRIEF ARTICLE

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We are studying the map

$$\Psi : x \mapsto \Phi(g(\Phi(x, \tau(x))), T - \tau(x))$$

where

$$h(\Phi(x, \tau(x))) = 0.$$

If x is the initial condition and T is the total time, then $\tau(x)$ is the time to the event surface, defined by the event function h and the jump function g. We assume that $T \neq \tau(x)$, since otherwise the function is ill-defined.

Let $\tau_0 := \tau(x_0)$, $\tau_1 := T - \tau_0$, $x_1 := g\Phi(x_0, \tau_0)$, $x_2 = g(x_1)$, and $x_3 = \Phi(x_2, \tau_1)$. It follows that

$$h_{,x}(x_1) \cdot \big(\Phi_{,x}(x_0, \tau_0) + \Phi_{,t}(x_0, \tau_0)\tau_{,x}(x_0)\big) = 0 \Rightarrow \tau_{,x}(x_0) = -\frac{h_{,x}(x_1)}{h_{,x}(x_1) \cdot f(x_1)} \cdot \Phi_{,x}(x_0, \tau_0)$$

since $\Phi_{,t}(x_0, \tau_0) = f(x_1)$. Consequently,

$$\Psi_{,x}(x_0) = \Phi_{,x}(x_2, \tau_1) \cdot g_{,x}(x_1) \cdot \big(\Phi_{,x}(x_0, \tau_0) + \Phi_{,t}(x_0, \tau_0)\tau_{,x}(x_0)\big) - \Phi_{,t}(x_2, \tau_1)\tau_{,x}(x_0)$$

or, in other words,

$$\begin{aligned} \Psi_{,x}(x_0) = \Phi_{,x}(x_2, \tau_1) \cdot g_{,x}(x_1) \cdot \left(I - \frac{f(x_1) \cdot h_{,x}(x_1)}{h_{,x}(x_1) \cdot f(x_1)}\right) \cdot \Phi_{,x}(x_0, \tau_0) \\ + \frac{f(x_3) \cdot h_{,x}(x_1)}{h_{,x}(x_1) \cdot f(x_1)} \cdot \Phi_{,x}(x_0, \tau_0) \end{aligned}$$

since $f(x_3) = \Phi_{,t}(x_2, \tau_1)$. The first term corresponds to a flow that differs from T, so the second term is required to ensure that the derivative reflects flow of total time T.

It is possible to show that $f(x_3) = \Phi_{,x}(x_2, \tau_1) \cdot f(x_2)$. Substitution then yields

$$\Psi_{,x}(x_0) = \Phi_{,x}(x_2, \tau_1) \cdot \left(g_{,x}(x_1) + \frac{\big(f(x_2) - g_{,x}(x_1) \cdot f(x_1)\big) \cdot h_{,x}(x_1)}{h_{,x}(x_1) \cdot f(x_1)} \right) \cdot \Phi_{,x}(x_0, \tau_0)$$