Linear Algebra

Notes on various linear algebra topics that I've forgotten or never knew.

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1 The Moore-Penrose Pseudoinverse

A general matrix $M \in \mathbb{R}^{m\times n}$ will not have an inverse, but it is guaranteed to have a next-best-thing which is also unique: the **Moore-Penrose Pseudoinverse** [1] , denoted by $M^+ \in \mathbb{R}^{n\times m}$. Obviously, it cannot satisfy $MM^+ = \mathbf{1}$ in general (since the identity doesn't even exist for the general $n \neq m$ case we have in mind). However, it satisfies the closest analogues[2] :

$$\begin{aligned} M^+MM^+ &= M^+ \\ MM^+M &= M \\ (MM^+)^T &= MM^+ \\ (M^+M)^T &= M^+M \ . \end{aligned} \tag{1.1}$$

In the limit where the columns are linearly-independent (such that MM^T is invertible), the MP inverse has the explicit form $M^+(MM^T)^{-1}$. Similarly, if the rows are linearly-independent (such that M^TM is invertible), then $M^+ = (M^TM)^{-1}M^T$. By uniqueness (not proven here), M^+ reduces to M^{-1}, when it exists.

If solutions x of linear equations of the form $M \cdot x = b$ exist, they can be written in terms of the psueduoinverse:

$$x = M^+b + M\left(\mathbf{1} - M^+M\right)y \ , \quad y \text{ arbitrary} \tag{1.2}$$

which generalizes the form of the solution when M is invertible. Demonstrating that the above solves the desired equation follows from the properties in (1.1), and an additional result that $MM^+b = b$ necessarily holds if any solutions exist [3].

A Conventions and Notation

B TODO

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[1]Some concise notes on the properties of this matrixcan be found here.

[2]These properties do not imply one another.

[3]Proof: obviously for solutions to exist, b must live in the range of M. When this is true, there is a v for which $Mv = b$. Then, from (1.1), $b = Mv = MM^+Mv = MM^+b$.

References