COMBINATORICS AND PROBABILITY

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AGENDA

- **Sets**
- **Probability**
- **Permutation & Combinations**
- **Conditional probability**
- **Law of total probability**

SETS

- In probability theory, a set is denoted as a *well-defined collection of objects*.
- Define as '*A*'

- x∈A (x element belongs to A)
- x∉A (x does not belongs to A)

Eg.

SETS

Subsets

- If every element in set B is also in set A then we can say that is $B \subseteq A$.

Proper Subsets

- All proper subsets are subsets.

A = {1,2,3} , B = {1,2,3} ----- A is subset of B.

If C = {1,2} ----- C is both a subset and proper subset of A and also of B.

$C \subset A$

$C \subset B$

SETS

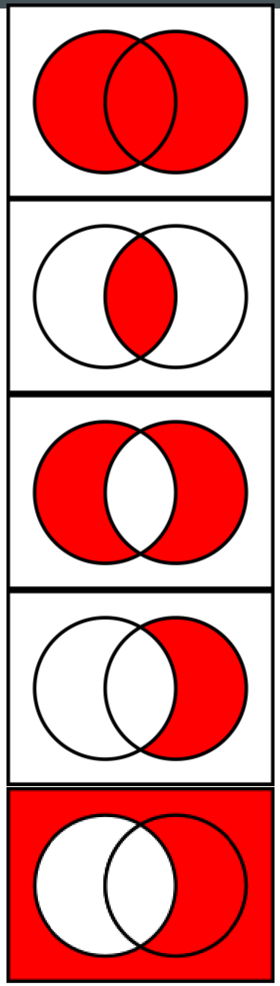
Universal Sets (Ω)

- The collection of all possible outcomes in a certain context or universe.
- Example of a universal set: all the possible outcomes when rolling a dice.

  $\Omega=\{1,2,3,4,5,6\}$
- Can have an infinite number of elements

SET OPERATIONS

- Union $A \cup B$
- Intersection $A \cap B$
- Symmetric difference $A \triangle B$
- Relative component $B \setminus A$ or $B - A$
- Absolute component A' or A^c

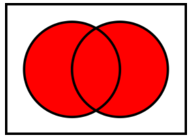


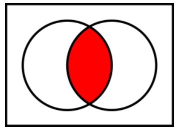
SET OPERATIONS

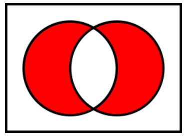
Ω = {1,2,3,4,5,6,7,8,9,10}

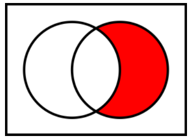
A = {1,2,3,4,5,6}

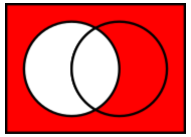
B = {5,6,7,8}

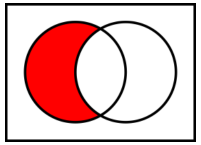
Union = {1,2,3,4,5,6,7,8} 

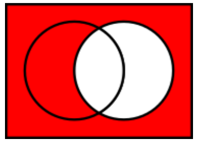
Intersection = {5,6} 

Symmetric difference = {1,2,3,4,7,8} 

Relative component of A = {7,8} 

Absolute Component of A = {7,8,9,10} 

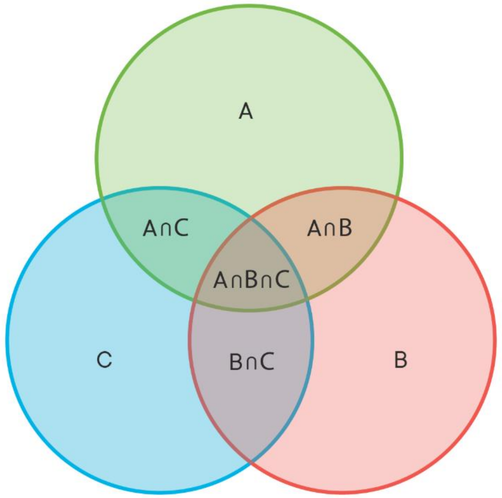
Relative component of B = {1,2,3,4} 

Absolute Component of B = {1,2,3,4,9,10} 

SET OPERATIONS

Inclusion Exclusion Principle

- It's a counting technique
- How many elements are in set A versus B
- Can we add all the elements?
- $|A \cup B| = |A| + |B| - |A \cap B|$ ---- horizontal line denotes cardinality of a set
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



SETS

- Empty set $\emptyset$ or {} --- No element in the set

- In Python
  - Sets are unordered collections of unique elements.
  - Sets are iterable.
  - Sets are collections of lower level python objects (just like lists or dictionaries).

https://docs.python.org/2/library/sets.html

PROBABILITY

- Probability is the chance that a certain event will happen,
- In other words, how "likely" it is that some event will happen.

- Why it is important for data science?

PROBABILITY

- A = {1,2,3,4,5,6} possible outcome of throwing a dice

Throwing a dice once = **Random experiment**

Result of the experiment = **Outcome**

A contains all possible outcome = Universal set = **Sample space** **---** $A=\{x|x\in\mathbb{R}, 1\leq x\leq 6\}$

Subset of sample space = **Event space** --- $E\subseteq A$

throwing a number higher than 4 --- E = {5,6}

- The event space is a collection of events that we care about.

PROBABILITY

The law of relative frequency

- While conducting an endless stream of experiments, the relative frequency by which an event will happen becomes a fixed number.

$$P(E) = \lim_{n \to \infty} \frac{S(n)}{n}$$

- S(n) = count of successful experiment
- n = total number of experiment

PROBABILITY

- In the early 20th century, Kolmogorov and Von Mises came up with 3 axioms

1. **Positivity :** A probability is always bigger than or equal to 0, or $0 \leq P(E) \leq 1$
2. **Probability of a certain event:** If the event of interest is the sample space, we say that the outcome is a certain event, or $P(S)=1$
3. **Additivity:** The probability of the union of 2 exclusive events is equal to the sum of the probabilities of the individual events happening.

If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Addition law of probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

PROBABILITY

- Two events: (mutually exclusive)

event M means throwing a 6,

event N means that you throw an odd number N=1,3,5.

- What is the probability that your outcome will be a 6, or an odd number?

another event Q=4,5 (not mutually exclusive)

- What is the probability that N or Q will happen?

QUESTION?

Lab:

- A certain customer owns a Visa card (event A) or American express credit card (event b) and some has both

    $P(A) = 0.5$

    $P(B) = 0.4$

    both A and B = 0.25

1) Compute the probability that a selected customer has at least one credit card?

2) Compute the probability that a selected customer doesn't own any of the mentioned credit card?

3) Compute the probability that a customer only owns Visa card?

- We throw a dice twice, What is the probability of throwing a 5 at least once?

PERMUTATIONS AND FACTORIALS

- Why it is necessary? --- To create the sample space when it grows bigger
- Three favorite songs

Permutation: 3*2*1 = 6 ------ n! (What's a factorial?)

- Permutation of subset

Permutation: 13*12*11 = 1716

$$P_k^n = \frac{n!}{(n-k)!}$$

→ k-permutation of n

PERMUTATIONS AND FACTORIALS

Permutations with Replacement:

- We can play the song twice
- Permutations: 13 * 13 * 13 = 2197 ---- n^j

Permutations with Repetition:

"TENNESSEE"

How many different words can you create using these letters?

- 9 letters, 4 x E, 2 x N, 2 x S

$$\frac{9!}{4!2!2!} = 3780$$

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

QUESTION?

Lab

- You misplaced your iPhone and are afraid it was stolen. Luckily, your iPhone needs a 4-digit code in order to get in. Imagine that a potential thief can do five attempts at getting the code right before the phone is permanently locked, how big is the chance the thief unlocks the phone?

Hint: Think about the sample space and the event space separately.

- Right before you lost your phone you ate a pretzel, and you are pretty sure a grease pattern was left on the four crucial digits of your screen. The four letters in your access code are 3,4,7 and 8, and you realize that this information can increase the thief's chances massively. Assuming the thief interprets the smudgemarks in an intelligent way, what are the chances that the phone will be unlocked successfully?

COMBINATIONS

- When order is not important.
- Need 3 more people to play euchre

$$\binom{n}{k} = \frac{P_k^n}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!}$$

- Warmup: Assume one person is doing an individual project, and out of remaining 12 students, we want to make 4 project groups of 3 students each for Mod3. How many total combinations are possible?

CONDITIONAL PROBABILITY

- Extremely important in statistic
- A key component in most statistical machine learning algorithms
- Why is it important?

CONDITIONAL PROBABILITY

Independent Events:

Events A and B are independent when, the occurrence of A has no effect on whether B will occur (or not)

Eg: Getting heads after flipping a coin and getting a 5 on throw of a fair dice

Events A and B are independent if

$P(A \cap B) = P(A)P(B)$, and

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

CONDITIONAL PROBABILITY

Independent Events:

Three events A, B and C if

$P(A \cap B) = P(A)P(B)$

$P(A \cap C) = P(A)P(C)$

$P(B \cap C) = P(B)P(C)$

$P(A \cap B \cap C) = P(A)P(B)P(C)$

So you need both pairwise independence and three-way independence

CONDITIONAL PROBABILITY

Disjoint Events:

Events A and B are disjoint if A occurring means that B cannot occur.

Mutually exclusive

Eg: Heading East and West at the same time is impossible.

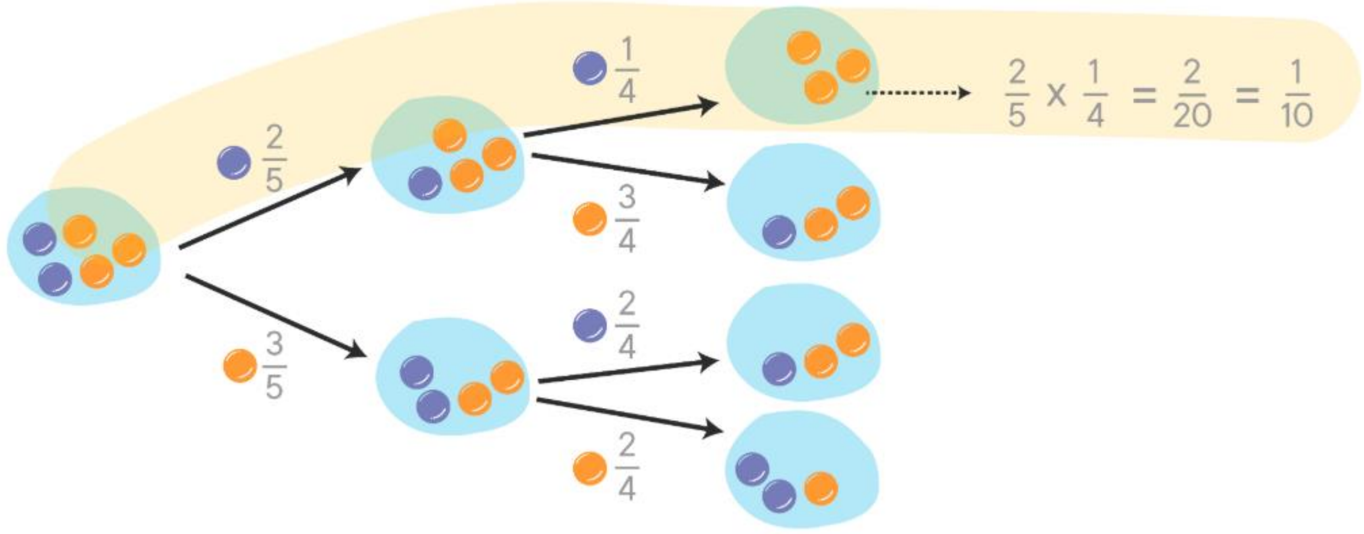
$P(A \cup B) = P(A) + P(B)$

As $P(A \cap B)$ is empty.

CONDITIONAL PROBABILITY

Dependent Events:

Events A and B are dependent when, the occurrence of A somehow has an effect on whether B will occur (or not)



In simple terms, the probability of seeing an event B in the second trial depends on the outcome A of the first trial. We say that $P(B)$ is conditional on $P(A)$.

CONDITIONAL PROBABILITY

- Conditional probability emerges when the outcome a trial may influence the results of the upcoming trials.

- Let's say that $P(A)$ is the event we are interested in, and this event depends on a certain event B that has happened.

(Probability of A **given** B) can be written as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

CONDITIONAL PROBABILITY

- **Theorem 1 - Product Rule:** the conditional probability is easy to compute than intersection

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

if A and B are independent $P(A|B)=P(A)$, and $P(A \cap B)=P(A)P(B)$

- **Theorem 2 - Chain Rule:** also called the general product rule

$$P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A \mid B \cap C)P(B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$$

- **Theorem 3 - Bayes Theorem:**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

QUESTION?

- If he picked a day at random from the 50 days on record, what is the probability that he was in a good mood on that day, $P(G)$?
- What is the probability that the day chosen was a Sunny day, $P(S)$?
- What is probability of having a good mood given its a sunny day $P(G|S)$?
- What is probability that it will be all nice and sunny given that he is in a good mood P(S|G) ?

	Sunny weather	Cloudy weather
Good mood	14	11
Bad mood	2	23

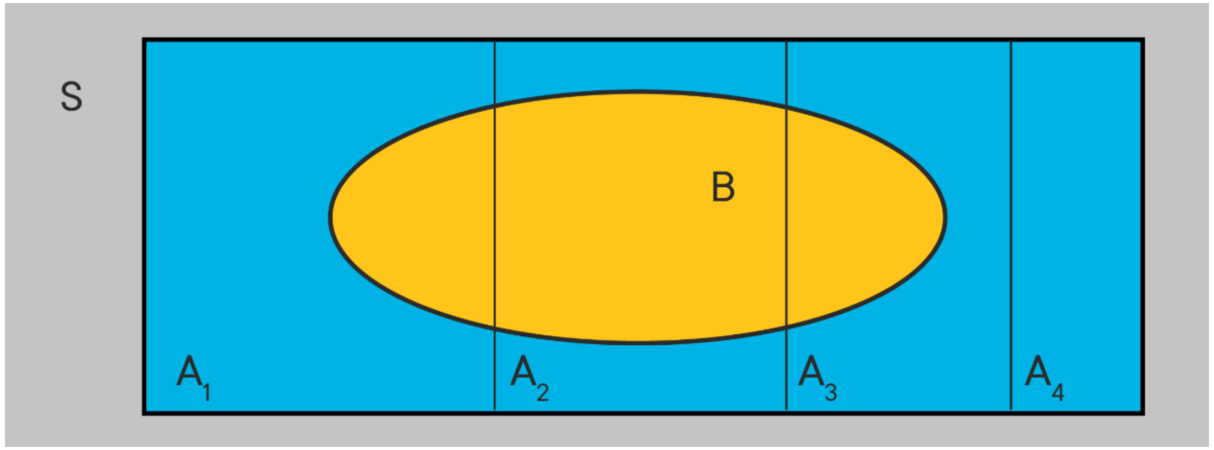
LAW OF TOTAL PROBABILITY

- A_1, A_2, A_3, A_4 are four events in sample space and they are mutually exclusive and collectively exhaustive
- Also called partition of sample space

$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup (B \cap A_4)$

$P(B \cap A_4) = \emptyset$

$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4)$



LAW OF TOTAL PROBABILITY

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4)$$

$$P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) + P(B \mid A_4)P(A_4)$$

If B1,B2,B3,⋯ is a partition of the sample space S, then for any event A we have

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A \mid B_i)P(B_i)$$

QUESTION?

- In a certain county, 60% of registered voters are Republicans, 30% are Democrats and 10% are Independents.

  When those voters were asked about increasing military spending

  40% of Republicans opposed it

  65% of the Democrats opposed it

  55% of the Independents opposed it.

  What is the probability that a randomly selected voter in this county opposes increased military spending?

DISCUSSION

THANK YOU!!