Lab 4 - Fourier Analysis - GPGN 404

23 October 2015

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1) Using the time series in the file *Lab4_t_xt.dat*, answer the following questions:

a) The sampling frequency is $\frac{(900-1)\ Samples}{18.16162\ sec} = 49.4999\ Hz$. However upon further inspection it can be deduced the amount of time per sampling interval is actually at least $0.02020202\ sec$. Every fifth sample in the data increases by at least $0.00021\ sec$ and every hundred samples this increase only takes four samples to occur. This means a more accurate sampling frequency could be $\frac{1}{0.02020202\ sec} = 49.50000\ Hz$.

b) The DC component of the signal is 2.

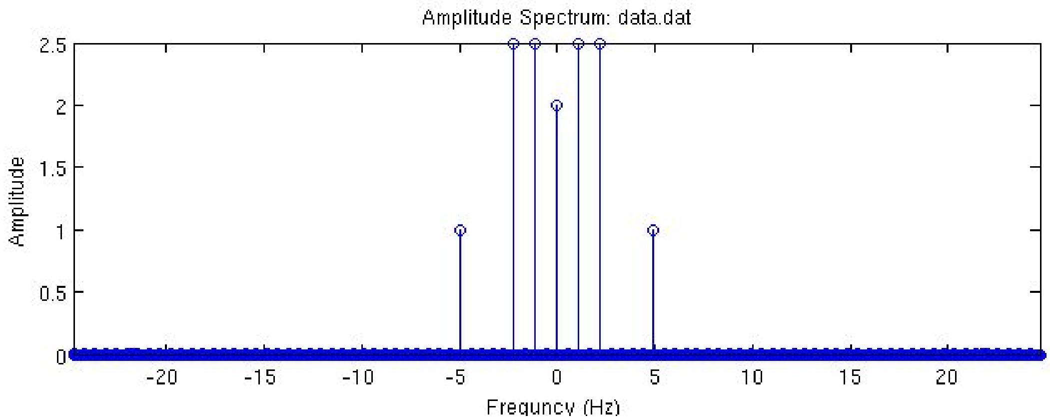

Figure 1: Amplitude Spectrum of the time signal in *data.dat*

c) By inspection the time period of this signal is 1.818 seconds.

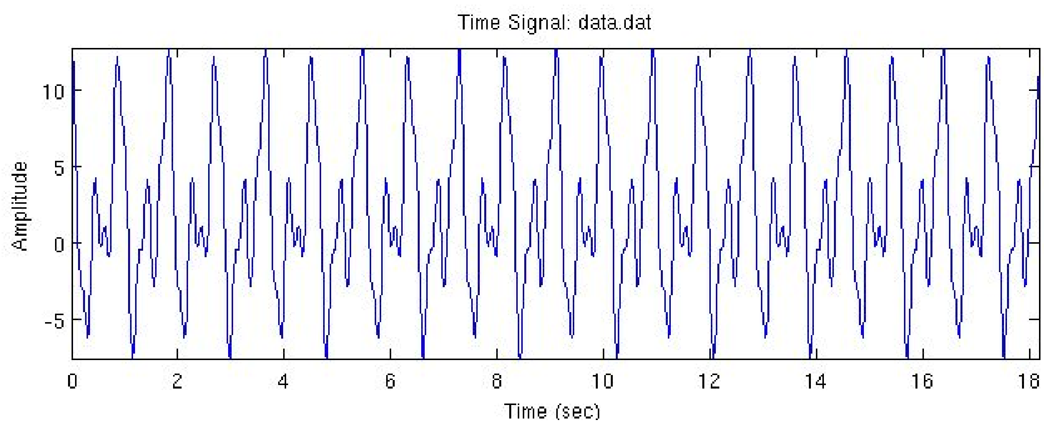

Figure 2: Plot of the time signal in *data.dat*

2) Aliasing and the Nyquist Frequency for the time signal $x(t) = \sin(10\pi t)$

a) Sketch the two-sided amplitude spectrum $F[x(t)]$

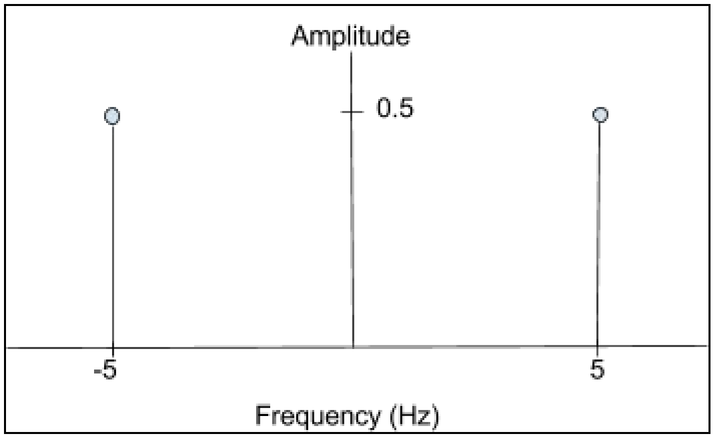

Figure 3: Sketch of the two sided amplitude spectrum for the time signal in number two

b) Plot $x[n]$ obtained using a sampling frequency of $20\ Hz$

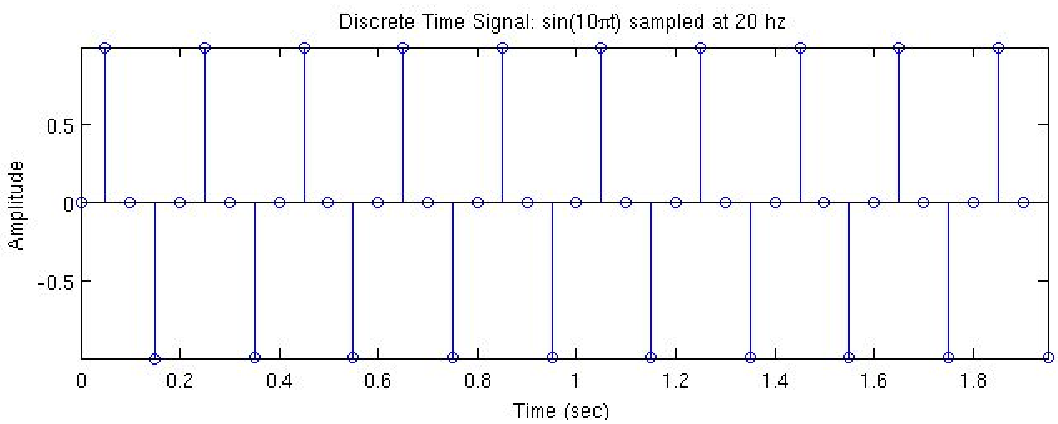

Figure 4: Time signal from number 2 sampled at $20\ Hz$

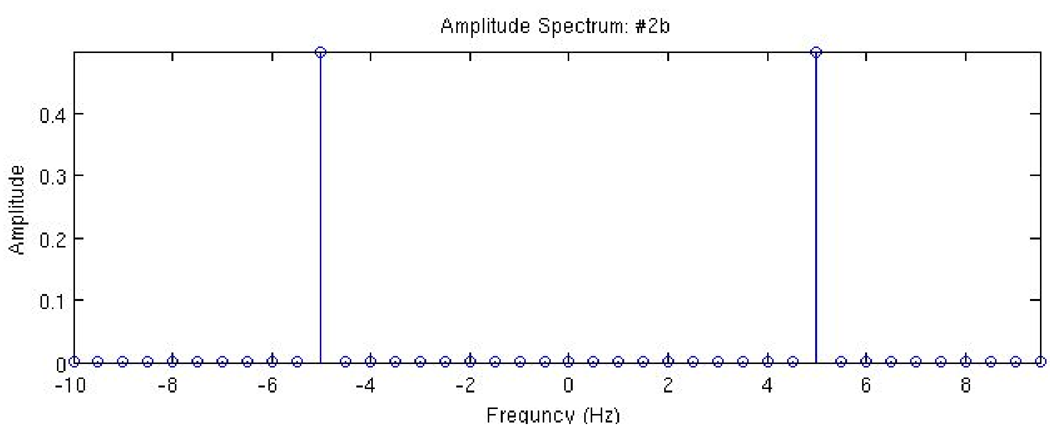

Figure 5: Amplitude spectrum of the discrete time signal sampled at $20\ Hz$

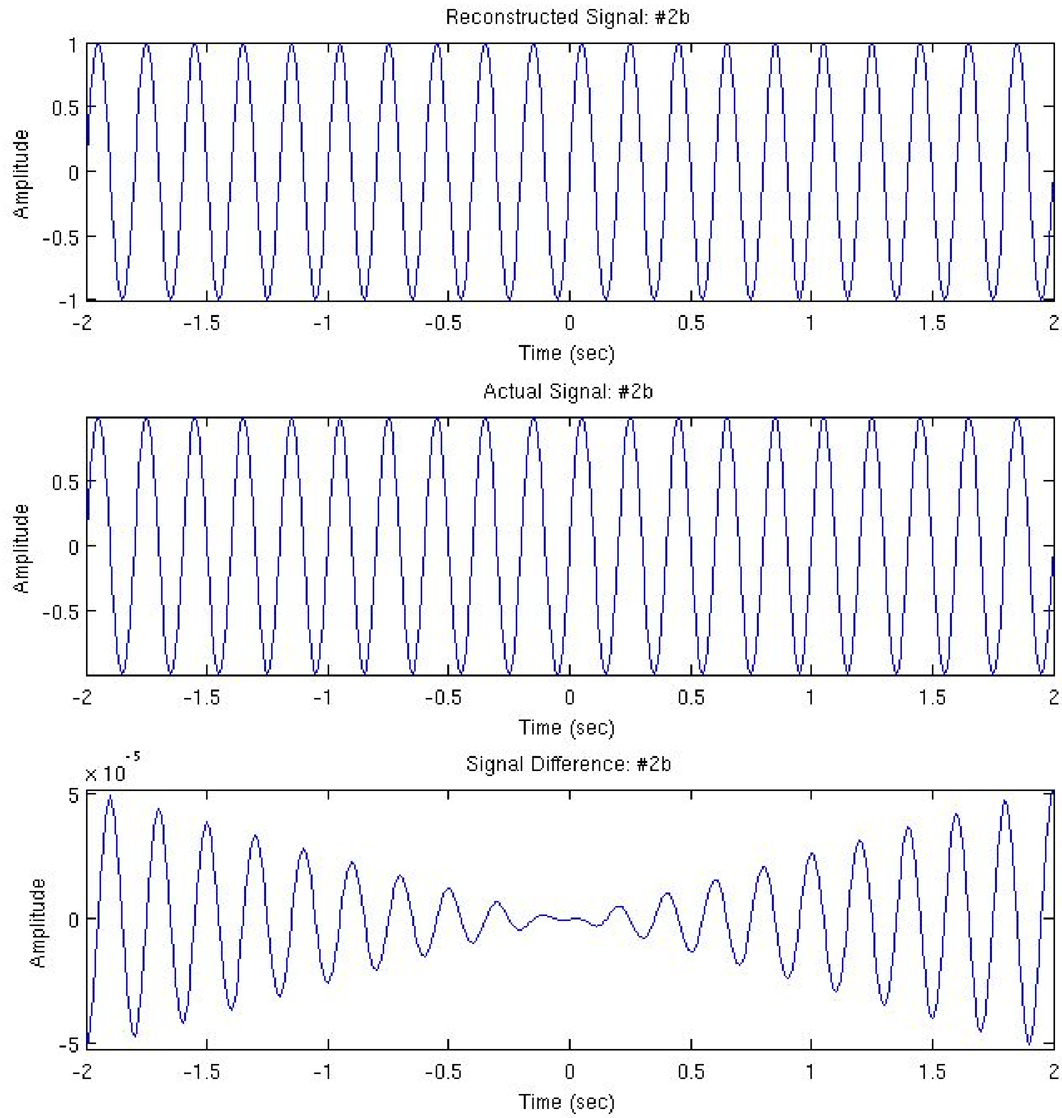

Figure 6: Comparison of the actual and reconstructed signal from number 2 sampled at $20\ Hz$

- The amplitude spectrum in 2b (figure 5) does match the amplitude spectrum sketch in 2a (figure 3).
- The reconstructed signal and the actual time signal do appear to be the same. Notice magnitude of difference between the two signals in figure 6 (bottom). The maximum difference in amplitude between the actual and reconstructed signal is of magnitude $5 * 10^{-5}$ on the interval $-2.0\ sec \leq t \leq 2.0\ sec$. This difference between the reconstructed and actual signal is negligible so the signals are essentially identical.

c) Plot $x[n]$ obtained using a sampling frequency of $9\ Hz$

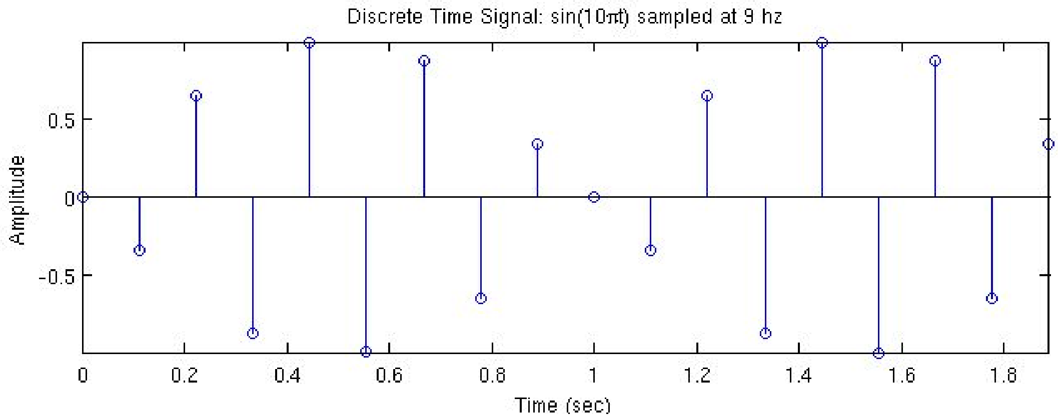

Figure 7: Time signal from number 2 sampled at $9\ Hz$

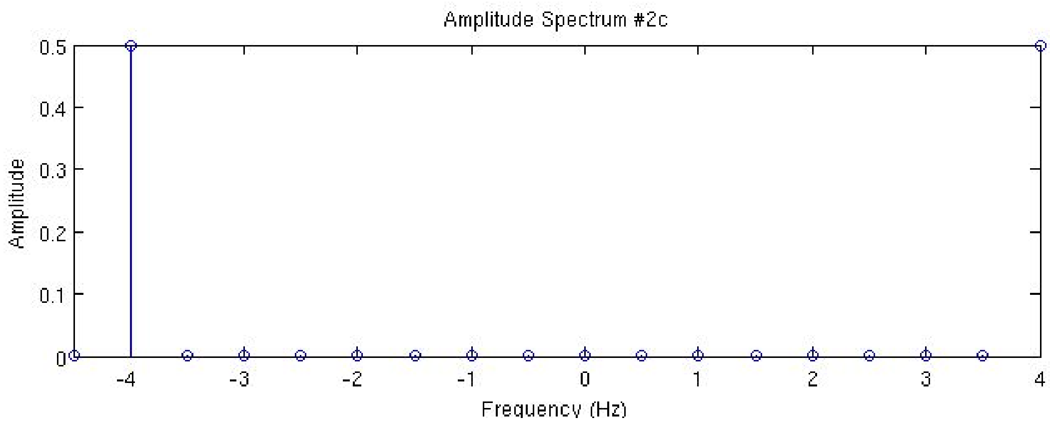

Figure 8: Amplitude spectrum of the discrete time signal sampled at $9\ Hz$

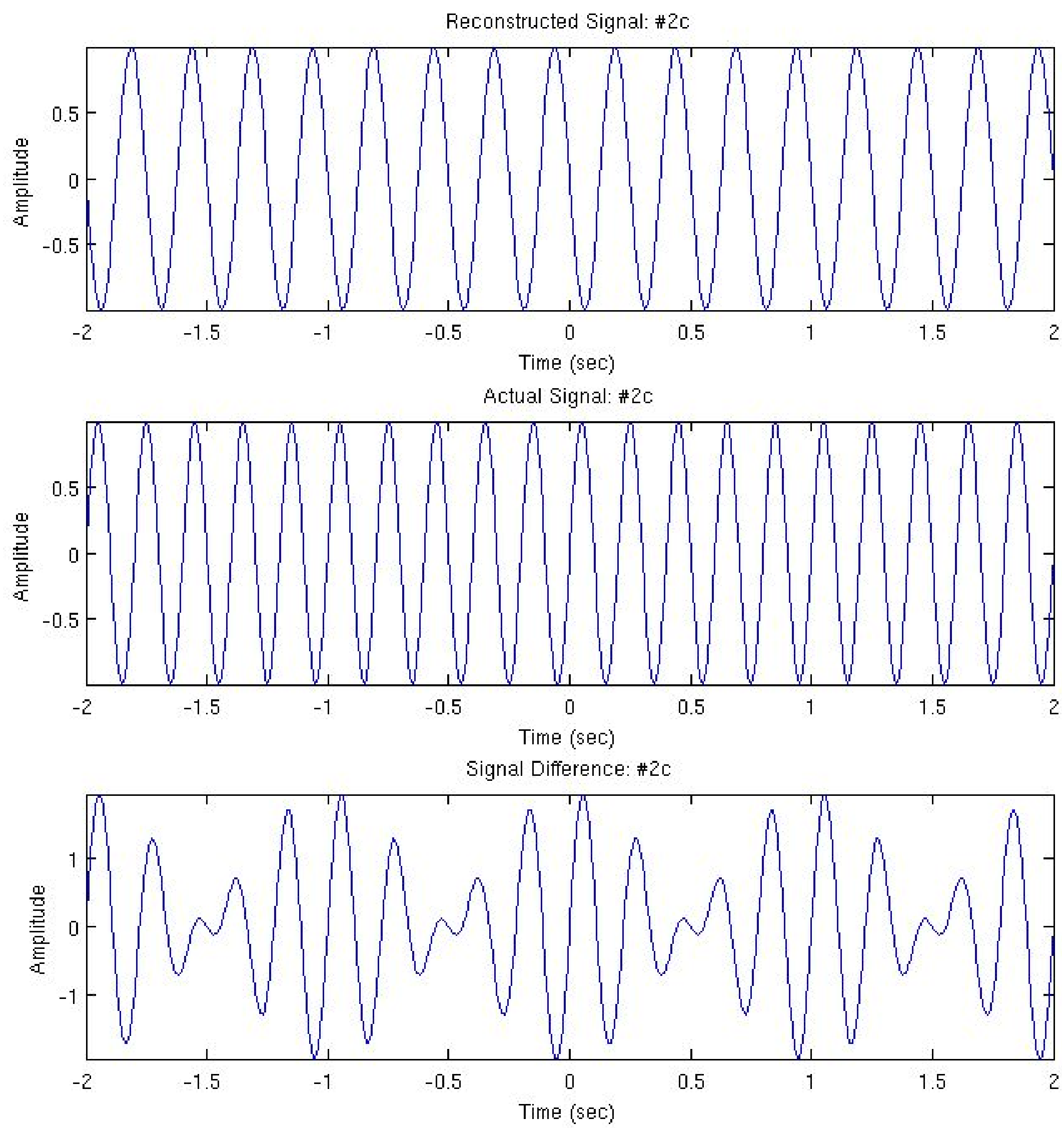

Figure 9: Comparison of the actual and reconstructed signal from number 2 sampled at $20\ Hz$

- The amplitude spectrum in 2c (figure 8) does not match the amplitude spectrum in 2a (figure 3). They do not match because the sampling frequency is below the nyquist frequency, $f_n = 2f_{max}$ where $f_{max} = 5\ Hz$ yields $f_n = 10\ Hz$. Since $f_s < f_n$ the signal is aliased as a sinusoid of frequency $4\ Hz$. This satisfies the aliased frequency equation where $f_{alias} = f_{sample} - lf_0 = (9\ Hz) - (1) * (5\ Hz) = 4\ Hz$.
- The reconstructed time signal and actual time signal are different because the fourier transform was calculated on a data set sampled below the nyquist frequency. Note the difference between the two signals plotted in figure 9 (bottom). The difference between the two signal is of magnitude $2 * 10^0$, five order of magnitude greater than in 2b and twice the magnitude of the signals maximum amplitude.

3) This is number 3
   a) Below is my derivation of the fourier coefficients c_k

Consider the boxcar function…

$$x(t) = \left\{1\ if\ 0 \le t < \tfrac{T_0}{2},\ 0\ if\ \tfrac{T_0}{2} \le t < T_0\right\}$$

Using the analytic solution for the complex fourier coefficient c_k we find…

$$c_k = \tfrac{1}{T_0}\int_0^{T_0} e^{-i\frac{2\pi kt}{T_0}}dt = \tfrac{1}{T_0}\left(\int_0^{\frac{1}{2}T_0} e^{-i\frac{2\pi kt}{T_0}}dt + \int_{\frac{1}{2}T_o}^{T_0} e^{-i\frac{2\pi kt}{T_0}}dt\right) = \tfrac{1}{T_0}\int_0^{\frac{1}{2}T_0} e^{-i\frac{2\pi kt}{T_0}}dt$$

Now we solve for the DC component of our signal, the 0^{th} harmonic c_0 …

$$c_0 = \tfrac{1}{T_0}\int_0^{\frac{1}{2}T_0} e^{-i\frac{2\pi(0)t}{T_0}}dt = \tfrac{1}{T_0}\int_0^{\frac{1}{2}T_0} dt = \tfrac{1}{T_0}\left(\tfrac{T_0}{2} - 0\right) = \tfrac{1}{2}$$

In general the complex fourier coefficient c_k is defined as…

$$c_k = \tfrac{1}{T_0}\int_0^{\frac{1}{2}T_0} e^{-i\frac{2\pi kt}{T_0}}dt \ = \tfrac{1}{-2\pi ik}(e^{-i\pi k} - 1) = \tfrac{1}{-2\pi ik}[(-1)^{\ k} - 1]$$

We can simplify this expression for even and odd numbers…

$$c_{even} = \tfrac{1}{-2\pi ik}[(-1)^{\ 2n} - 1] = \tfrac{1}{-2\pi ik}[1 - 1] = 0$$
$$c_{odd} \ = \tfrac{1}{-2\pi ik}[(-1)^{\ 2n+1} - 1] = \tfrac{1}{-2\pi ik}[1 * (-1)^1 - 1] = \tfrac{1}{-2\pi ik}[-2] = \tfrac{1}{\pi ik}$$

In conclusion…

$c_0 = \frac{1}{2}$

$c_{odd} = \frac{1}{\pi ik}$

$c_{even} \ = \ 0$

b) Plot of $x(t)$ on the interval $t = [0,\ 4]$ *sec*

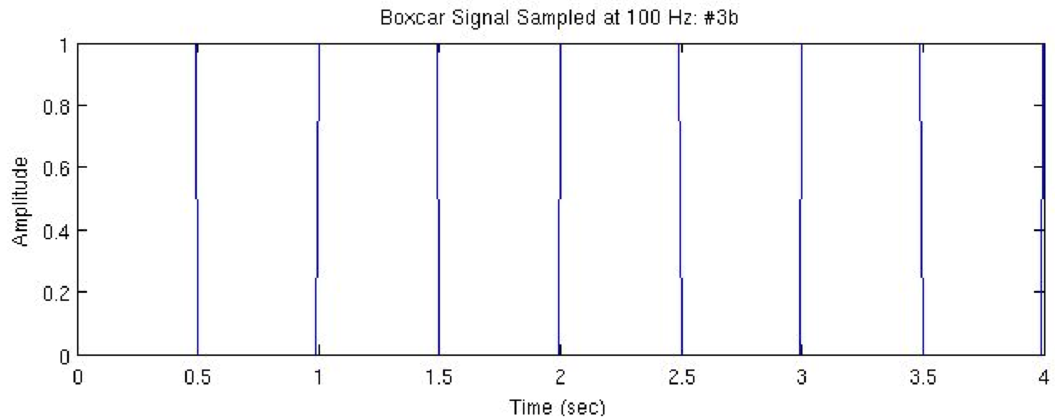

Figure 10: Boxcar function from 3a sampled at 100 *Hz*

c) Two sided amplitude spectrum for the first hundred harmonics of $x(t)$

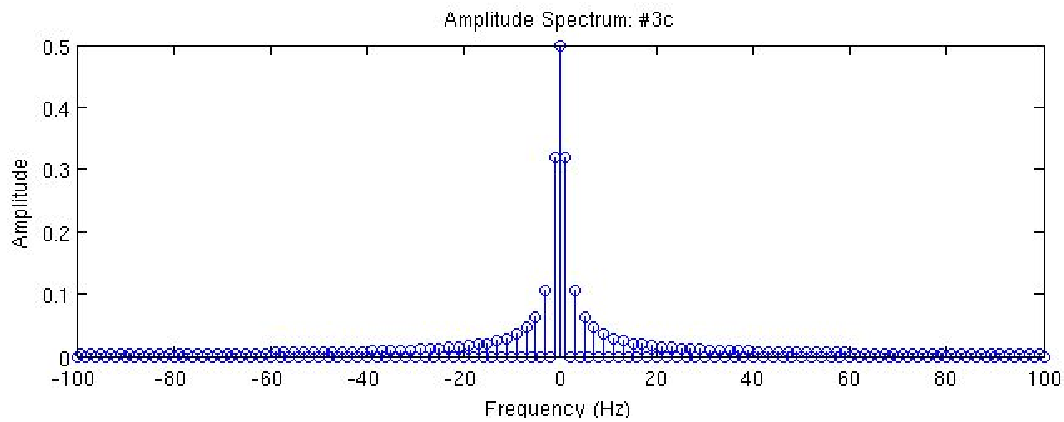

Figure 11: Two sided amplitude spectrum of the first 100 harmonics

d) Two sided phase spectrum for the first hundred harmonics of $x(t)$

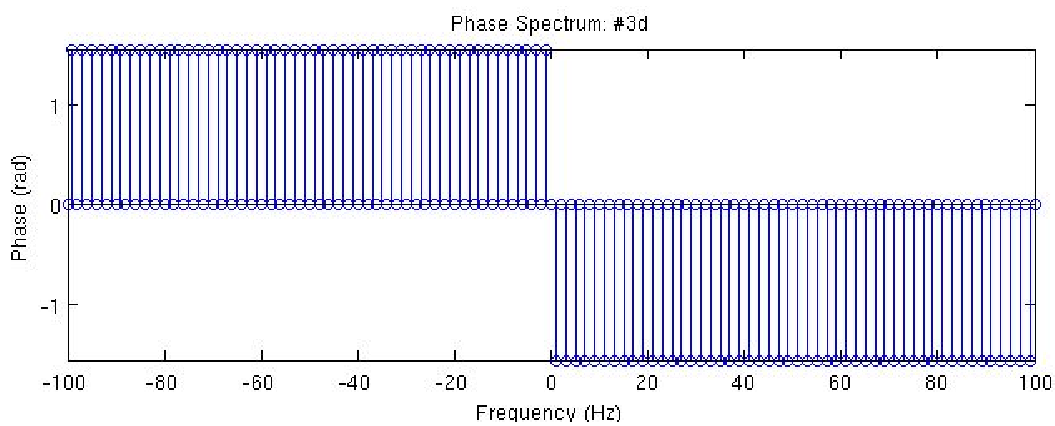

Figure 12: Two sided phase spectrum of the first 100 harmonics

e) Synthesized time signal using the first three harmonics of $x(t)$

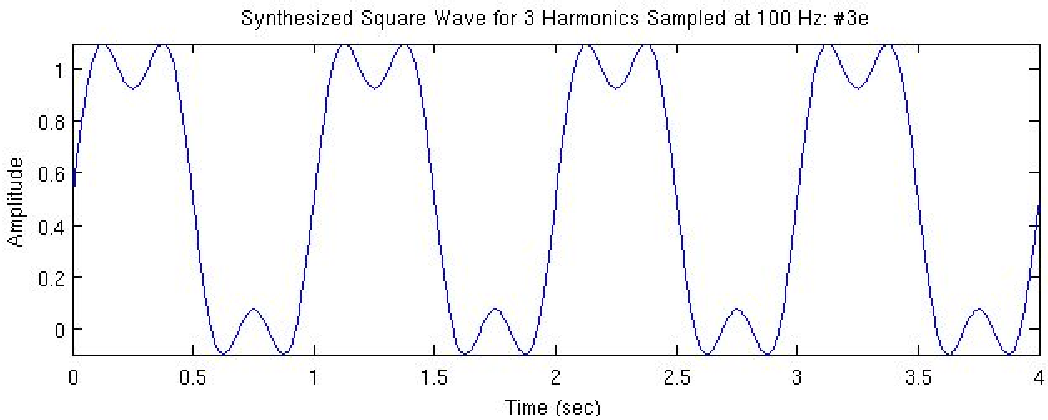

Figure 13: Synthesized boxcar function from 3a using the first three harmonics

f) Synthesized time signal using the first ten harmonics of $x(t)$

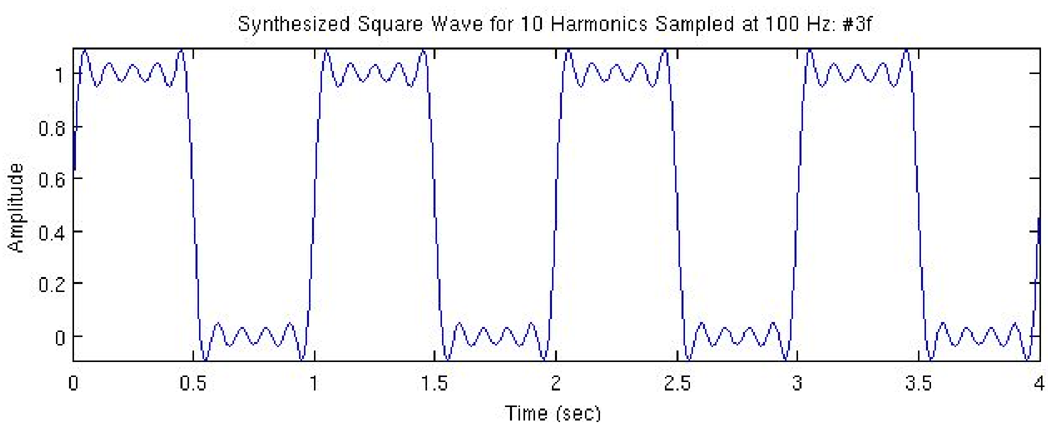

Figure 14: Synthesized boxcar function from 3a using the first ten harmonics

g) Synthesized time signal using the first twenty harmonics of $x(t)$

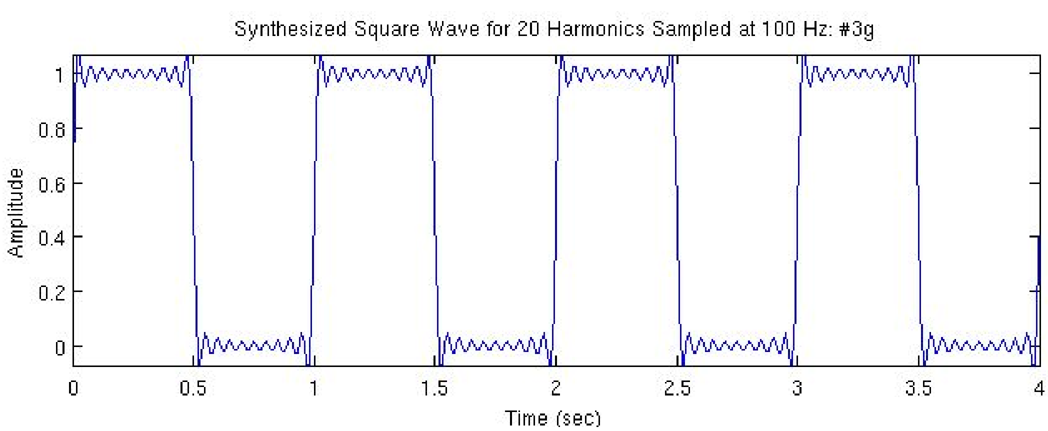

Figure 15: Synthesized boxcar function from 3a using the first twenty harmonics

Appendix I: The Code

```
clear;

filename = 'data.dat';
data = tblread(filename);
time = data(:,1);
value = data(:,2);

% ----------------- 1a -----------------
%Fs = length(time) / time(length(time));
Fs = 1 / 0.020202;

Xk = fftshift(fft(value)) / length(time);
Fk = -Fs/2:Fs/length(time):Fs/2;
Fk = Fk(1:length(time));

figure('position', [0, 0, 750, 250]);
plot(time, value);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Time Signal: ',filename]);

figure('position', [0, 0, 750, 250]);
stem(Fk, abs(Xk));
axis tight;
xlabel('Frequncy (Hz)');
ylabel('Amplitude');
title(['Amplitude Spectrum: ',filename]);

DC = 0.0;
for i = 1:length(Fk)
      if Fk(i) == 0.0 || (Fk(i+1) > 0 && Fk(i) < 0)
      DC = Xk(i+1);
      break;
      end
end

% ----------------- 2b -----------------
Fs = 20;
Ts = 1.0 / Fs;

t1 = 0.0;
t2 = 2.0;
time = t1:Ts:t2;
if time(length(time)) >= 2.0
      time = time(1:(length(time)-1));
end

Xt = sin(10*3.1415926*time);
```

```
figure('position', [0, 0, 750, 250]);
stem(time, Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Discrete Time Signal: sin(10\pit) sampled at ', num2str(Fs), ' hz']);

Xk = fftshift(fft(Xt)) / length(time);
Fk = -Fs/2:Fs/length(time):Fs/2;
Fk = Fk(1:length(time));

figure('position', [0, 0, 750, 250]);
stem(Fk, abs(Xk));
axis tight;
xlabel('Frequncy (Hz)');
ylabel('Amplitude');
title('Amplitude Spectrum: #2b');

time = -2.0:0.01:2.0;
Xr = zeros(1,length(time));
Xt = sin(10*3.1415926*time);
for i=length(Fk)/2:length(Fk)
      Xr = Xr + Xk(i) * exp(2*1i*3.14159*Fk(i)*time);
      Xr = Xr + conj(Xk(i)) * exp(-2*1i*3.14159*Fk(i)*time);
end

figure('position', [0, 0, 750, 750]);
subplot(3,1,1);
plot(time, Xr);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Reconstructed Signal: #2b');
subplot(3,1,2);
plot(time, Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Actual Signal: #2b');
subplot(3,1,3);
plot(time, Xt-Xr);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Signal Difference: #2b');

% ---------------- 2c ----------------
Fs = 9;
Ts = 1.0 / Fs;
```

```
t1 = 0.0;
t2 = 2.0;
time = t1:Ts:t2;
if time(length(time)) >= 2.0
      time = time(1:(length(time)-1));
end

Xt = sin(10*3.1415926*time);

figure('position', [0, 0, 750, 250]);
stem(time, Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Discrete Time Signal: sin(10\pit) sampled at ', num2str(Fs), ' hz']);

Xk = fftshift(fft(Xt)) / length(time);
Fk = -Fs/2:Fs/length(time):Fs/2;
Fk = Fk(1:length(time));

figure('position', [0, 0, 750, 250]);
stem(Fk, abs(Xk));
axis tight;
xlabel('Frequency (Hz)');
ylabel('Amplitude');
title('Amplitude Spectrum #2c');

time = -2.0:0.01:2.0;
Xr = zeros(1,length(time));
Xt = sin(10*3.1415926*time);

for i=length(Fk)/2:length(Fk)
      Xr = Xr + Xk(i) * exp(2*1i*3.14159*Fk(i)*time);
      Xr = Xr + conj(Xk(i)) * exp(-2*1i*3.14159*Fk(i)*time);
end

figure('position', [0, 0, 750, 750]);
subplot(3,1,1);
plot(time, Xr);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Reconstructed Signal: #2c');
subplot(3,1,2);
plot(time, Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Actual Signal: #2c');
```

```
subplot(3,1,3);
plot(time, Xt-Xr);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title('Signal Difference: #2c');

% ----------------- 3b -----------------
Fs = 1000;
period = 1.0;
time = 0.0:1/Fs:4.0;

figure('position', [0, 0, 750, 250]);
plot(time, mod(time, period) < period*0.5);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Boxcar Signal Sampled at ', num2str(Fs), ' Hz: #3b']);

% ----------------- 3c + 3d -----------------
harmonic = 20;
Fs = 100;
period = 1.0;
time = 0.0:1/Fs:4.0;

Xt = zeros(1,length(time));

for k=-harmonic:harmonic
      if mod(k,2)
      Xt = Xt + ((1i*3.14159*k).^(-1)*exp((2i*3.14159*k*time)/period));
      end
end

Xt = Xt + 0.5;

Xk = fftshift(fft(Xt)) / length(time);
Fk = -Fs/2:Fs/length(time):Fs/2;
Fk = Fk(1:length(time));

% ----------------- 3e -----------------
harmonic = 100;
Fs = 100;
period = 1.0;
time = 0.0:1/Fs:4.0;

harmonic_matrix = zeros(2*harmonic+1, length(time));
c = zeros(2*harmonic+1,1);
for k=-harmonic:harmonic
      row = k + 1 + harmonic;
      if mod(k,2)
```

```
      c(row) = (1i*3.14159*k).^(-1);
      harmonic_matrix(row,:) =
(1i*3.14159*k).^(-1)*exp((2i*3.14159*k*time)/period);
      elseif k == 0
      c(row) = 0.5;
      harmonic_matrix(row,:) = harmonic_matrix(row) + 0.5;
      else
      c(row) = 0.0;
      harmonic_matrix(row,:) = 0.0;
      end
end

figure('position', [0, 0, 750, 250]);
stem(-100:1:100, abs(c));
axis tight;
xlabel('Frequency (Hz)');
ylabel('Amplitude');
title('Amplitude Spectrum: #3c');

figure('position', [0, 0, 750, 250]);
stem(-100:1:100, angle(c));
axis tight;
xlabel('Frequency (Hz)');
ylabel('Phase (rad)');
title('Phase Spectrum: #3d');

h = 3;
Xt = zeros(1,length(time));
for k=-h:h
      row = k + 1 + harmonic;
      Xt = Xt + harmonic_matrix(row,:);
end
figure('position', [0, 0, 750, 250]);
plot(time,Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Synthesized Square Wave for ', num2str(h),' Harmonics Sampled at ',
num2str(Fs), ' Hz: #3e']);

h = 10;
Xt = zeros(1,length(time));
for k=-h:h
      row = k + 1 + harmonic;
      Xt = Xt + harmonic_matrix(row,:);
end
figure('position', [0, 0, 750, 250]);
plot(time,Xt);
axis tight;
xlabel('Time (sec)');
```

```
ylabel('Amplitude');
title(['Synthesized Square Wave for ', num2str(h),' Harmonics Sampled at ',
num2str(Fs), ' Hz: #3f']);

h = 20;
Xt = zeros(1,length(time));
for k=-h:h
      row = k + 1 + harmonic;
      Xt = Xt + harmonic_matrix(row,:);
end
figure('position', [0, 0, 750, 250]);
plot(time,Xt);
axis tight;
xlabel('Time (sec)');
ylabel('Amplitude');
title(['Synthesized Square Wave for ', num2str(h),' Harmonics Sampled at ',
num2str(Fs), ' Hz: #3g']);
```