Assignment 5: Horizon Interpolation
GPGN 409

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April 21, 2016

1. *Define the horizon at all positions by solving a regularized least-squares inverse problem, solve for the mean and co-variance of the posterior marginal probability density. Make assumptions whenever necessary, but motivate your choices. Plot the horizon for three different values of the regularization parameter and discuss your results*

I am assuming a Gaussian distribution for the likelihood function any given model so that the following formulas may work. I am also assuming the range of model uncertainties, σ_m that will work to solve this problem is between 10^{-4} and 10^{4}.

Δx: Spacing in between each model point

m: Working Model

$\bar{m}$: Mean Starting Model

$\bar{d}$: Input Data

σ_m : Model Uncertainty

W_m: Model Shaping Matrix

$$W_{m_{ij}} \Delta x^2 \; = \begin{cases} -2 & \text{if } \; i = j \\ 1 & \text{if } \; |i-j| = 1 \end{cases}$$

W_D: Data Weighting

$$W_D \; = \; \mathbf{I} \, \vec{\sigma}_d$$

K: Operator matrix such that

$$z_{data} \; = d \; = \; K \, z_{model} \; = \; K \, m$$

r_d: Data residual

$$r_d \; = \; ||W_d(K\tilde{m} - \bar{d})||$$

r_d: Model residual

$$r_m = ||W_m(\tilde{m} - \bar{m})||$$

$\tilde{C}_m$: Model co-variance

$$\tilde{C}_m = (K^\intercal W_D^\intercal W_D K + \frac{1}{\sigma_m^2} W_m^\intercal W_m)^{-1}$$

$\tilde{\rho}_m$: Model correlation

$$\tilde{\rho}_{m_{ij}} = \frac{\tilde{C}_{m_{ij}}}{\sqrt{\tilde{C}_{m_{ii}}}\sqrt{\tilde{C}_{m_{jj}}}}$$

$\tilde{m}$: Objective Model

$$\tilde{m} = (K^\intercal W_D^\intercal W_D K + \frac{1}{\sigma_m^2} W_m^\intercal W_m)^{-1}(K^\intercal W_D^\intercal W_D \bar{d} + \frac{1}{\sigma_m^2} W_m^\intercal W_m \bar{m})$$

J(m): Objective Function

$$J(m) = \frac{1}{2}||W_D(Km - \bar{d})||^2 + \frac{1}{2\sigma_m^2}||W_m(m - \bar{m})||^2$$

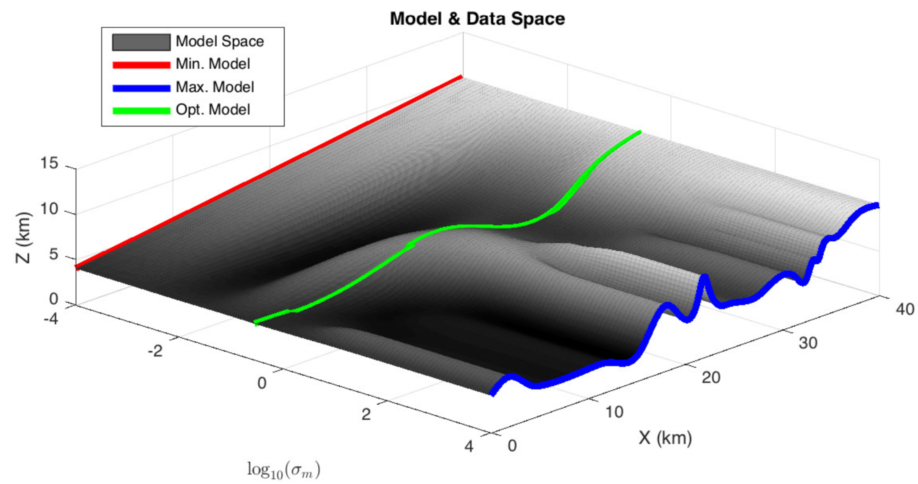

Figure 1: This figure shows the inverted model space for varying values of σ_m overlain with the optimal model (green), the minimum model (red), and the maximum model (blue) for a dipping model

Discussion The figure above shows three different models for different given values of σ_m. The red line shows a model for a minimum value of the model uncertainty, the blue line shows the model for a maximum value of model uncertainty, and the green line for an optimal value of model uncertainty. The surface shown shows how the model-data space change as a function of model uncertainty. As the model uncertainty increases the surface becomes less like the mean model we input and more like the data. However, as the model uncertainty increases, it also becomes less accurate because although it better fits our known data points it also diverges from the mean model more. That is why the green (optimal) model is between the minimum and maximum models.

2. *Find the optimal value of the regularization parameter using the "L-curve" criterion. Plot the L-curve and the horizon obtained using the optimal parameter. Explain how you chose the optimal regularization parameter.*

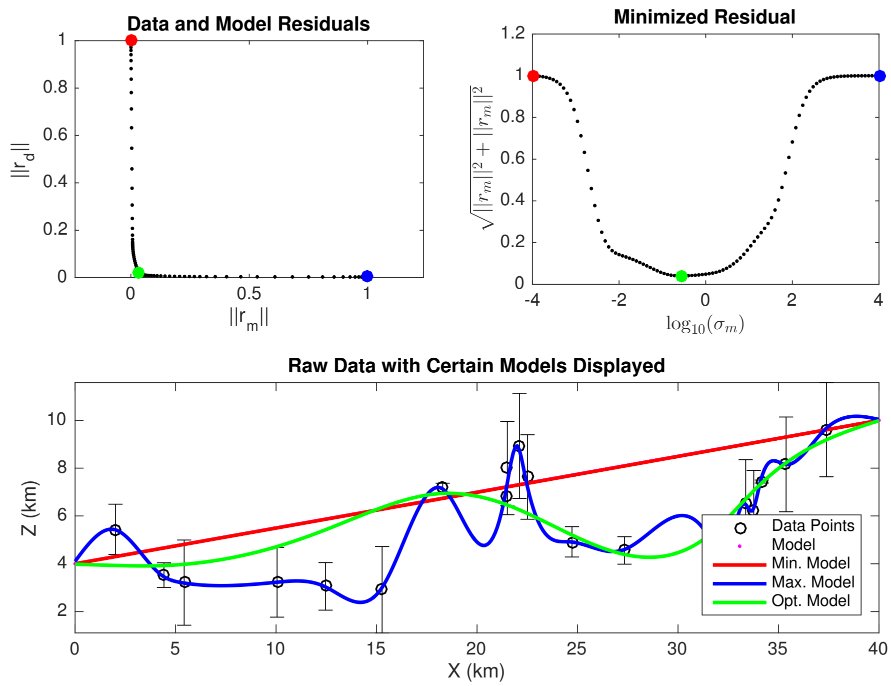

Figure 2: This figure shows the residual plot, the L-curve, and the data overlain with the optimal model (green), the minimum model (red), and the maximum model (blue) for a model with dipping beds and faults at 18 km and 27 km

Discussion I chose the optimal parameter by finding the value of model uncertainty, σ_m, which minimized the distance of the residuals from the origin (shown in green).

3. *Plot the posterior model co-variance and correlation matrices for the optimal value of the regularization parameter. Explain the meaning of this co-variance matrix*

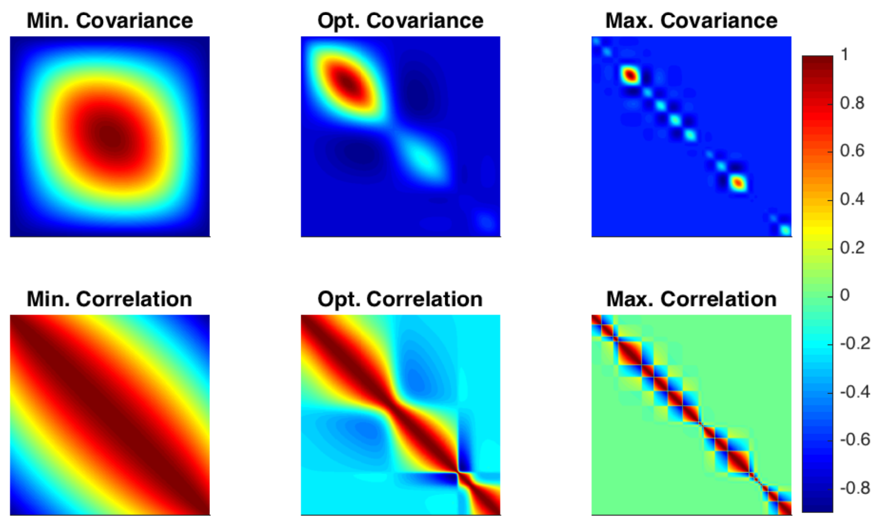

Figure 3: This figure shows the co-variance (top row) and correlation (bottom row) matrices for the minimum, optimal, and maximum values of σ_m

Discussion The co-variance matrices above (first row) show uncertainty and correlation information for the various models corresponding to minimum, optimal, and maximum values of model uncertainty, respectively. The second row shows the correlation matrices for various models corresponding to minimum, optimal, and maximum values of model uncertainty, respectively. If the values in the correlation matrix are large the then the model correlates well to points around it. If it is small then it means it does not correlate well.

4. *Insert discontinuities (i.e. faults) in the horizon at coordinates $x = 18$ km and $x = 27$ km. Explain how you obtain the fractured horizon.*

After inserting discontinuities into the starting model at $x = 18$ km and $x = 27$ km I changed the K matrix such that the two faults were represented by a discontinuity in the interpolation discrete signal in the following manner. I also altered the mean (starting) model such that there were three dipping beds surrounding the two inserted faults. These beds do not dip in the same direction or at the same angle and begin at different elevations.

$$K_{faulted} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Although this is not the exact K matrix, the significant difference is illustrated by the bold zeros. If faults were to be inserted in between the second and third entries in the model and the fourth and fifth entries in the model we could use $K_{faulted}$ in our inversion. In the case of the actual inversion if a fault had an index of i in the model then we would alter the K by following the following algorithm.

$$\begin{aligned} K_{i+1,\, j} &= 0 \\ K_{i,\, j+1} &= 0 \end{aligned}$$

Using The revised K matrix, $K_{faulted}$, we obtain the following inversion results.

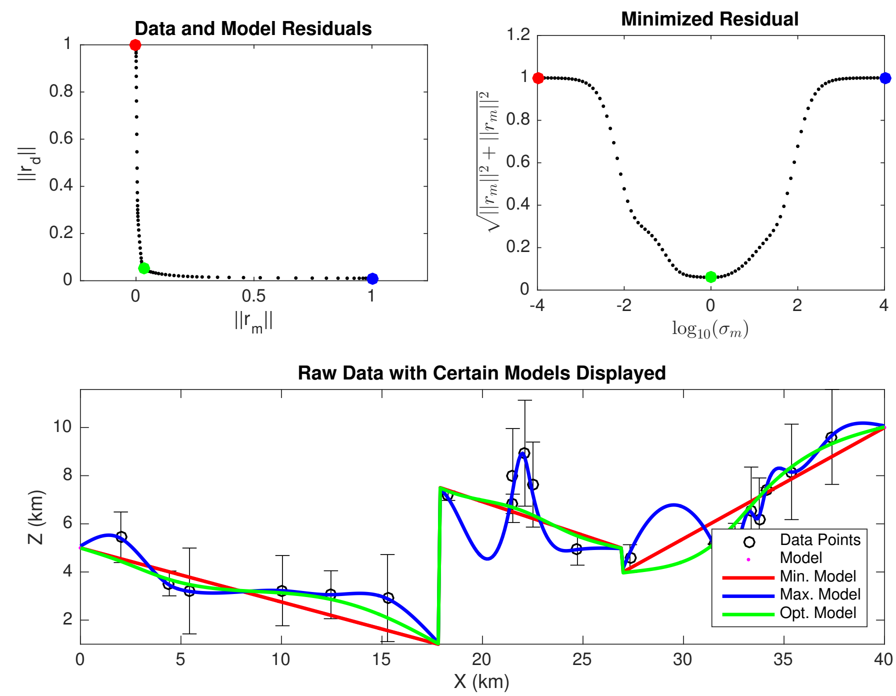

Figure 4: This figure shows the residual plot, the L-curve, and the data overlain with the optimal model (green), the minimum model (red), and the maximum model (blue) for a model with dipping beds and faults at 18 km and 27 km

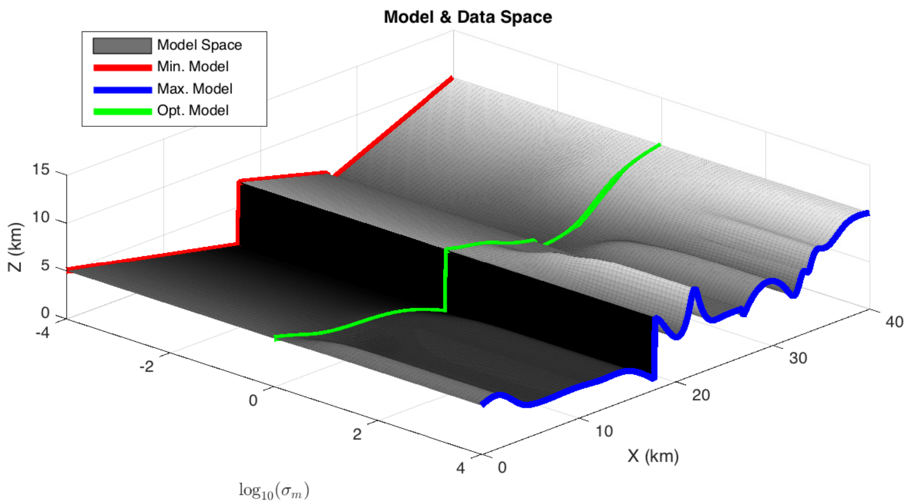

Figure 5: This figure shows the inverted model space for varying values of σ_m overlain with the optimal model (green), the minimum model (red), and the maximum model (blue) for a model with dipping beds and faults at 18 km and 27 km

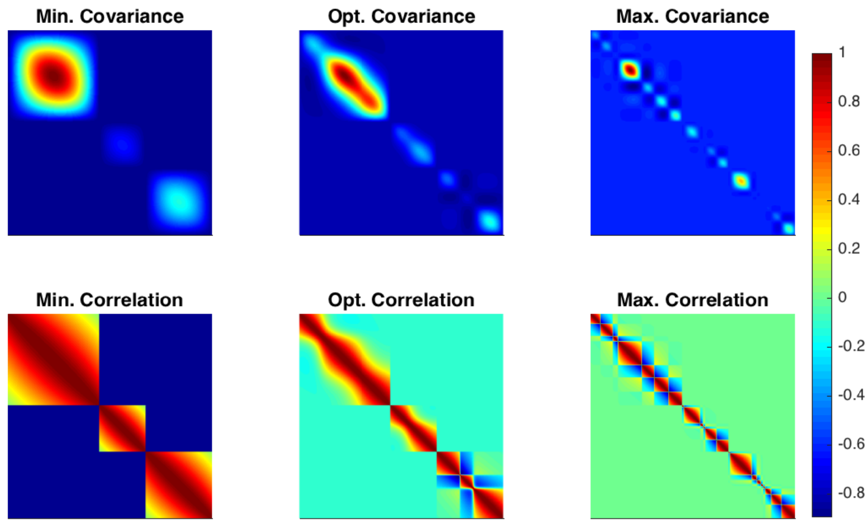

Figure 6: This figure shows the co-variance (top row) and correlation (bottom row) matrices for the minimum, optimal, and maximum values of σ_m with a dipping and faulted model.

Appendix I: The Code

```
function HorizonInterpolation()
    % Import the data
    filename = 'data.txt';
    data = importdata(filename);
    data = data.data;
    x_data = data(:, 1); % X data
    z_data = data(:, 2); % Z data
    s_data = data(:, 3); % Std. Dev. in each Z measurement

    % Setup the K matrix
    x_delta = 0.1; % Distance between discrete X measurements
    x_min = 0.0; % Min X
    x_max = 40.0; % Max X
    x = x_min:x_delta:x_max; % Create X vector
    K = zeros(length(x_data), length(x)); % Initialize the K matrix
    for r=1:length(x_data)
        index = round((x_data(r)-x_min)./(x_delta)); % Find closest point
        K(r, index) = 1; % Set entry to 1
    end

    % Setup the inverse problem
    W_d = diag(1./(s_data)); % Diagonalize the data uncertainties
    W_mk = diag(ones(length(x), 1)).*(-2)...
        + diag(ones(length(x)-1, 1), 1)...
        + diag(ones(length(x)-1, 1), -1);
        % Setup the interpolation matrix

    W_mk = W_mk./(x_delta.^2);

    iter = 101;
    z = zeros(length(x), iter);
    z_model = ones(length(x), 1);
    z_model(1:length(z_model)) = linspace(4.0,10.0,length(x));
    r_d = zeros(iter, 1);
    r_m = zeros(iter, 1);
    S_mv = logspace(-4,4,iter);
```

```
% Uncomment the below section to add in a fault
x1 = 18;
x2 = 27;
ix1 = round((x1-x_min)./(x_delta));
ix2 = round((x2-x_min)./(x_delta));
W_mk(ix1, ix1-1) = 0;
W_mk(ix1-1, ix1) = 0;
W_mk(ix2, ix2+1) = 0;
W_mk(ix2+1, ix2) = 0;
z_model(1:(ix1-1)) = linspace(5,1,ix1-1);
z_model(ix1:ix2) = linspace(7.5,5,ix2-ix1+1);
z_model((ix2+1):length(z_model)) =...
 linspace(4,10,length(z_model)-ix2);

for i=1:iter
    W_m = W_mk./(S_mv(i));
    z(:,i) = pinv(K'*(W_d'*W_d)*K+W_m'*W_m)*...
     (K'*(W_d'*W_d)*z_data+W_m'*W_m*z_model);
    r_d(i) = (norm(W_d*(K*z(:,i)-z_data)));
    r_m(i) = (norm(W_mk*(z(:,i)-z_model)));
end

r_d = r_d./max(r_d(:));
r_m = r_m./max(r_m(:));
r_norm = (r_d.^2+r_m.^2).^(0.5);
[junk, index] = min(r_norm);

W_m = W_mk./(S_mv(1));
C_mt = pinv((K'*(W_d'*W_d)*K)+(W_m'*W_m));
Rho = zeros(size(C_mt));
for i=1:size(C_mt,1)
    for j=1:size(C_mt,2)
        Rho(i,j) = C_mt(i,j)/(C_mt(i,i).^(0.5).*C_mt(j,j).^(0.5));
    end
end

figure('Position',[400 400 600 400]);
```

```
subplot(2,3,1);
surf(C_mt,'EdgeAlpha',0.0);
colormap jet;
view([0 90])
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Min. Covariance');
axis tight;
axis equal;

subplot(2,3,4);
surf(Rho,'EdgeAlpha',0.0);
colormap jet;
view([0 90])
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Min. Correlation');
axis tight;
axis equal;

W_m = W_mk./(S_mv(index));
C_mt = pinv((K'*(W_d'*W_d)*K)+(W_m'*W_m));
Rho = zeros(size(C_mt));
for i=1:size(C_mt,1)
    for j=1:size(C_mt,2)
        Rho(i,j) = C_mt(i,j)/...
         (C_mt(i,i).^(0.5).*C_mt(j,j).^(0.5));
    end
end

subplot(2,3,2);
surf(C_mt,'EdgeAlpha',0.0);
```

```
colormap jet;
view([0 90])
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Opt. Covariance');
axis tight;
axis equal;

subplot(2,3,5);
surf(Rho,'EdgeAlpha',0.0);
colormap jet;
view([0 90])
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Opt. Correlation');
axis tight;
axis equal;

W_m = W_mk./(S_mv(iter));
C_mt = pinv((K'*(W_d'*W_d)*K)+(W_m'*W_m));
Rho = zeros(size(C_mt));
for i=1:size(C_mt,1)
    for j=1:size(C_mt,2)
        Rho(i,j) = C_mt(i,j)/...
         (C_mt(i,i).^(0.5).*C_mt(j,j).^(0.5));
    end
end

subplot(2,3,3);
surf(C_mt,'EdgeAlpha',0.0);
colormap jet;
view([0 90])
```

```
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Max. Covariance');
axis tight;
axis equal;

subplot(2,3,6);
surf(Rho,'EdgeAlpha',0.0);
colormap jet;
view([0 90])
set(gca,'xtick',[]);
set(gca,'xticklabel',[]);
set(gca,'ytick',[]);
set(gca,'yticklabel',[]);
set(gca,'YDir','reverse');
title('Max. Correlation');
axis tight;
axis equal;

figure('Position',[400 400 600 400]);
subplot(2,2,1);
plot(r_m, r_d,...
    '.', 'MarkerFaceColor', 'k', 'color', 'k'); hold on;
scatter(r_m(index), r_d(index),...
    'o', 'MarkerFaceColor', 'g', 'MarkerEdgeColor', 'g'); hold on;
scatter(r_m(1), r_d(1),...
    'o', 'MarkerFaceColor', 'r', 'MarkerEdgeColor', 'r'); hold on;
scatter(r_m(iter), r_d(iter),...
    'o', 'MarkerFaceColor', 'b', 'MarkerEdgeColor', 'b'); hold on;
xlabel('||r_{m}||');
ylabel('||r_{d}||');
xlim([0.0 1.0]);
ylim([0.0 1.0]);
axis equal;
title('Data and Model Residuals');
```

```
subplot(2,2,2);
plot(log10(S_mv), sqrt(r_d.^2+r_m.^2),...
    '.', 'MarkerFaceColor', 'k', 'MarkerEdgeColor', 'k'); hold on;
plot(log10(S_mv(index)), r_norm(index),...
    'o', 'MarkerFaceColor', 'g', 'MarkerEdgeColor', 'g'); hold on;
plot(log10(S_mv(1)), r_norm(1),...
    'o', 'MarkerFaceColor', 'r', 'MarkerEdgeColor', 'r'); hold on;
plot(log10(S_mv(iter)), r_norm(iter),...
    'o', 'MarkerFaceColor', 'b', 'MarkerEdgeColor', 'b'); hold on;
xlabel('\sigma_{m}');
ylabel('$\sqrt{||r_{m}||^{2}+||r_{m}||^{2}}$','Interpreter','Latex');
title('Minimized Residual');

subplot(2,2,[3,4]);
errorbar(x_data, z_data, s_data, 'ro', 'color', 'k'); hold on;
scatter(x, z_model, '.',...
    'MarkerEdgeColor','m',...
    'MarkerFaceColor','m'); hold on;
plot(x, z(:,1), 'color', 'r',...
 'LineWidth', 2); hold on;
plot(x, z(:,iter), 'color', 'b',...
 'LineWidth', 2); hold on;
plot(x, z(:,index), 'color', 'g',...
 'LineWidth', 2); hold on;
xlabel('X (km)');
ylabel('Z (km)');
axis tight;
legend('Data Points','Model','Min. Model',...
 'Max. Model','Opt. Model',...
    'Location','southeast');
title('Raw Data with Certain Models Displayed');

figure('Position',[400 400 600 400]);
[X, Y] = meshgrid(log10(S_mv), x);
surf(X, Y, z, 'EdgeColor', [0 0 0], 'EdgeAlpha', 0.1);
colormap('gray');
hold on;
```

```
    plot3(log10(S_mv(1)).*ones(length(x),1),x,z(:,1),...
        'Color', 'r', 'LineWidth', 4); hold on;
    plot3(log10(S_mv(iter)).*ones(length(x),1),x,z(:,iter),...
        'Color', 'b', 'LineWidth', 4); hold on;
    plot3(log10(S_mv(index)).*ones(length(x),1),x,z(:,index),...
        'Color', 'g', 'LineWidth', 4); hold on;
    view([43, 54]);
    title('Model & Data Space');
    xlabel('$\log_{10}(\sigma_{m})$','Interpreter','Latex');
    ylabel('X (km)');
    zlabel('Z (km)');
    legend('Model Space','Min. Model','Max. Model','Opt. Model',...
        'Location','northwest');
end
```