Assignment 1: Comet Trajectory
GPGN 409

Garrett Sickles

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Introduction

You are making observations of a comet in space to figure out if it is on a collision course with the Earth. For this you need to estimate as accurately as possible the trajectory of the comet. Your observations consist of coordinates x and y of the comet relative to the Sun at angles α measured with respect to a fixed point centrally located inside the orbit (this information is not normally available, but it might make the solution to this assignment easier). The measurements have errors which you don't know, but you have a way to estimate the measurement uncertainties. The problem is 2D, i.e. the comet, the Earth and all other planets are in the same plane.

Summary of Assumptions and Definitions

1. Let the origin of our X-Y coordinate system also be the location of the Sun.

2. Let the center of the comet's trajectory, as in the center of a circle or ellipse, be defined by the point in the X-Y plane (x_o, x_o)

3. Let the trajectory of the comet be defined as a two dimensional ellipse not centered at the origin of the form

$$\frac{(x - x_o)^2}{a^2} + \frac{(y - y_o)^2}{b^2} = 1$$

where a and b are the major and minor radii in the direction of the X and Y axes and $\{x, y\}$ is the locus of all points that satisfy the above equation.

4. Let the angle of a point on the ellipse from the positive X axis be represented by the variable θ.

5. Let the formula for an inverse model be defined by the formulation of the least squares inversion where

$$\mathbf{m} = (\mathbf{G}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\,\mathbf{G})^{-1}\,\mathbf{G}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\,\mathbf{d}$$

as formulated and defined in homework 1.

Summary of Data

The data used in this problem was presented in the assignment and contains four types of data points, orthogonal measurements in astronomical units (AU) denoted by x_i and y_i which represent an observed location of the asteroid with respect to the Sun (the origin), an angle from the positive X axis to the recorded X-Y point denoted by θ, and uncertainty measurements denoted by s representing a radius of uncertainty around the observation point.

Formulation of the Forward Problem

Given the equation for an ellipse not centered at the origin

$$\frac{(x \; - \; x_o)^2}{a^2} + \frac{(y - y_o)^2}{b^2} \; = \; 1$$

we want to derive a formula where x and y are parameters of an angle θ. To do this, consider the parametric equations

$$\begin{aligned} x(\theta) &= x_o + a\cos(\theta) \\ y(\theta) &= y_o + b\sin(\theta) \end{aligned}$$

which define an ellipse in the X-Y plane as function of the variable θ. By substituting this set of equations into the general equation of an ellipse

$$\begin{aligned} \frac{(x(\theta) \; - \; x_o)^2}{a^2} + \frac{(y(\theta) - y_o)^2}{b^2} &= 1 \\ \frac{((x_o \; + \; a\cos(\theta)) \; - \; x_o)^2}{a^2} + \frac{((y_o \; + \; b\sin(\theta)) - y_o)^2}{b^2} &= 1 \\ \frac{(a\cos(\theta))^2}{a^2} + \frac{(b\sin(\theta))^2}{b^2} &= 1 \\ \cos^2(\theta) \; + \; \sin^2(\theta) &= 1 \end{aligned}$$

we find that equation of an ellipse is valid as long as a or b is nonzero.

Formulation of the Inverse Problem

Using our formulation of the forward problem and the data presented in the assignment we can formulate an inverse problem. In fact we can perform a least squares inversion by formulating two similar inverse problems differentiated by the x and y directions. By bringing the x and y solutions together the inversion can be performed in two parallel processes. From the equation for a model as defined in the least squares inversion we can define the quantities **W**, **d**, and **G** where

$$\mathbf{G}_x = \begin{bmatrix} 1 & \cos(\theta_1) \\ 1 & \cos(\theta_2) \\ ... & ... \\ 1 & \cos(\theta_N) \end{bmatrix}, \ \mathbf{W}_x = \begin{bmatrix} (w_1)^n & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & (w_N)^n \end{bmatrix}, \ \mathbf{d}_x = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_N \end{bmatrix}, \ \mathbf{m}_x = \begin{bmatrix} x_o \\ a \end{bmatrix}$$

$$\mathbf{G}_y = \begin{bmatrix} 1 & \sin(\theta_1) \\ 1 & \sin(\theta_2) \\ ... & ... \\ 1 & \sin(\theta_N) \end{bmatrix}, \ \mathbf{W}_y = \begin{bmatrix} (w_1)^n & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & (w_N)^n \end{bmatrix}, \ \mathbf{d}_y = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_N \end{bmatrix}, \ \mathbf{m}_y = \begin{bmatrix} y_o \\ b \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{m}_x &= (\mathbf{G}_x^\intercal \mathbf{W}_x^\intercal \mathbf{W}_x \ \mathbf{G}_x)^{-1} \ \mathbf{G}_x^\intercal \mathbf{W}_x^\intercal \mathbf{W}_x \ \mathbf{d}_x \\ \mathbf{m}_y &= (\mathbf{G}_y^\intercal \mathbf{W}_y^\intercal \mathbf{W}_y \ \mathbf{G}_y)^{-1} \ \mathbf{G}_y^\intercal \mathbf{W}_y^\intercal \mathbf{W}_y \ \mathbf{d}_y \end{aligned}$$

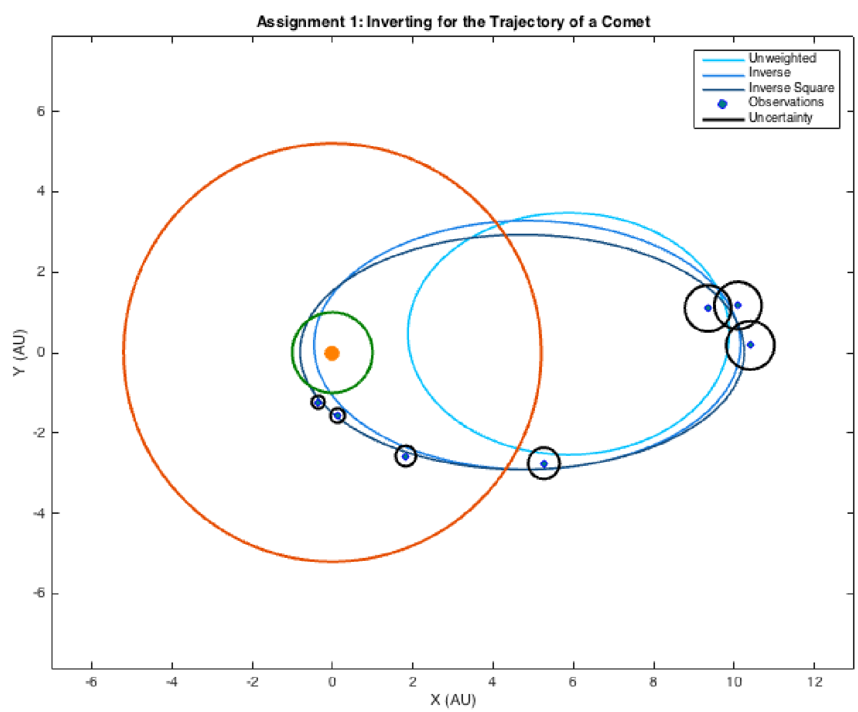
Using these two models, $\mathbf{m}_x$ and $\mathbf{m}_y$, we see that the constant quantities x_o, y_o, a, and b can be defined corresponding to the quantities in the models $\mathbf{m}_x$ and $\mathbf{m}_y$ such that the equation

$$\frac{(x - x_o)^2}{a^2} + \frac{(y - y_o)^2}{b^2} = 1$$

is in fact the equation of a known ellipse in the X-Y plane as defined in this problem.

Inverting the Data

Based on the data, the formulation of the forward problem, and the formulation of the inverse problem we can perform the inversion. This inversion is implemented in MatLab based on the formulations performed in previous sections of this assignment. Performing this operation yields the following plot containing the raw data, an equally weighted inversion, and two alternative weighting schemes. The following table contains the model parameters in terms of x_o, y_o, a, and b and the n value corresponds to the exponent in the $(w_i)^n$ term from the weighting matrix.



Color	x_0 (AU)	y_0 (AU)	a (AU)	b (AU)	n
Light Blue (Unweighted)	5.8801	0.4672	3.9996	3.0073	0
Blue ($\propto$ Radius)	4.8514	0.1881	5.3109	3.0921	1.0
Dark Blue ($\propto$ Area)	4.7257	0.0142	5.5292	2.9192	2.0

This plot shows that weighting the data differently significantly affects the trajectory of the comet. The inversion that equally weights the data points (light blue) does not accurately reflect the shape of the comets trajectory. This is obvious because the most well observed points, that is the points recorded with the lowest uncertainty, are not near the proposed trajectory. The inversion that uses weights proportional to the radius of uncertainty (Blue), that is the uncertainty as a radial distance surrounding each point, comes closer to an reasonable comet trajectory. However, the trajectory of the comet still does not come within one radius of uncertainty of the most accurately observed point. The final inversion which uses weights proportional to the area of uncertainty, that is the area inscribed by the circle of uncertainty surrounding each data point, appears to be the most accurate inversion because the trajectory intersects every area of uncertainty at each point.

Appendix I: The Code

```
function Comet_Trajectory(filename)
    % Import the data
    data = importdata(filename);
    x = data.data(:,1);
    y = data.data(:,2);
    angle = data.data(:,3);
    sigma = data.data(:,4);
    t = 0:0.001:(2*3.1415926);
    % Setup the Operator
    Gx = ones(length(x), 2);
    Gx(:,2) = cosd(angle);
    Gy = ones(length(y), 2);
    Gy(:,2) = sind(angle);
    % Unweighted Inversion
    mx = Least_Squares_Inversion(x, Gx, diag(sigma.^0));
    my = Least_Squares_Inversion(y, Gy, diag(sigma.^0));
    % Setup and run the inversion (^-1)
    mx1 = Least_Squares_Inversion(x, Gx, diag(sigma.^-1));
    my1 = Least_Squares_Inversion(y, Gy, diag(sigma.^-1));
    % Setup and run the inversion (^-2)
    mx2 = Least_Squares_Inversion(x, Gx, diag(sigma.^-2));
    my2 = Least_Squares_Inversion(y, Gy, diag(sigma.^-2));
    % Plot the trajectories
    figure;
    plot(mx(1)+mx(2)*cos(t), my(1)+my(2)*sin(t),...
    'Color', [0 .76 1],'LineWidth',1.5);
    hold on;
    plot(mx1(1)+mx1(2)*cos(t), my1(1)+my1(2)*sin(t),...
    'Color', [.1 .5 .9], 'LineWidth',1.5);
    hold on;
    plot(mx2(1)+mx2(2)*cos(t), my2(1)+my2(2)*sin(t),...
    'Color', [.08 .3 .5],'LineWidth',1.5);
    hold on;
    % Plot the data points
    scatter(x,y,20,'MarkerEdgeColor','b',...
        'MarkerFaceColor',[0 .5 .5],...
```

```
        'LineWidth',1.0);
    hold on;
    % Plot the error
    for i=1:length(sigma)
       plot(x(i)+sigma(i)*cos(t), y(i)+sigma(i)*sin(t),'k',...
           'LineWidth',2.0);
       hold on;
    end
    % Plot the sun
    plot(0, 0,'-ko',...
        'LineWidth',1,...
        'MarkerEdgeColor',[1 .5 0],...
        'MarkerFaceColor',[1 .5 0],...
        'MarkerSize',10);
    hold on;
    % Plot the Earth's Orbit
    plot(cos(t), sin(t),...
        'Color', [0 0.5 0],...
        'LineWidth',2);
    % Plot Jupiter's Orbit
    Jd = 5.20;
    plot(Jd*cos(t), Jd*sin(t),...
        'Color', [.9 .3 0],...
        'LineWidth',2);
    title('Assignment 1: Inverting for the Trajectory of a Comet');
    xlabel('X (AU)');
    ylabel('Y (AU)');
    axis([-7,13,-7,7])
    axis equal;
    legend('Unweighted', 'Inverse',...
    'Inverse Square', 'Observations',...
    'Uncertainty');
end

function [ m ] = Least_Squares_Inversion(d, G, W)
    m = pinv((G.')*(W.')*(W)*(G))*(G.')*(W.')*(W)*(d);
end
```