SF2521: Homework Assignment 4

Thomas Garrett & Weicker David

6th November 2015

1 Shallow Water with Non-Horizontal Bottom

We now return to the shallow water model from Homework 2. However, this time we study a modified version of the problem, namely a model with a non-horizontal bottom "bathymetry" $B(x)$. This is represented in the following model

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + hu_x \\ huu_x + gh(h_x + B_x) \end{pmatrix} = 0$$

1.1 Still Water Implies Horizontal Water Level

From the diagram, we know that the water level, lets call it $w(x,t)$, is $h(x,t) + B(x)$. We wish to show that for still water,i.e. $u = 0$, we the water level is horizontal, i.e. $w_x = 0$. We know that $u = 0 \implies u_x = u_t = 0$, so we simply plug these values into our model to obtain

$$\begin{pmatrix} h_t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 + 0 \\ 0 + gh(h_x + B_x) \end{pmatrix} = 0$$

We have that $h_t = 0$, which in combination with the fact that B is a function of only x, gives us that $w_t = h_t + B_t = 0$. Thus we know that the water level does not change with time. Now the second line of the system gives us that $gh(h_x + B_x) = 0 \implies h_x = -B_x$. This immediately gives us that $w_x = h_x + B_x = -B_x + b_x = 0$. Therefore with still water, we have a constant, horizontal water level.

1.2 Conservation Form

We now wish to write our equation in conservation form as

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h,m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h,m,x) \end{pmatrix}$$

where $m = hu$ and s is independent of the derivatives of h and m. We now set the equations equal to eachother and solve for f_2 and s

$$\begin{aligned} hu_t + huu_x + gh(h_x + B_x) &= m_t + (f_2(h,m))_x - s(h,m,x) \\ hu_t + huu_x + gh(h_x + B_x) &= h_t u + hu_t + (f_2(h,m))_x - s(h,m,x) \\ -h_t u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \end{aligned}$$

Now, using what we know from the first equation in the system, we get

$$\begin{aligned} h_t u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \\ (h_x u + hu_x)u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \\ h_x u^2 + 2huu_x + ghh_x + ghB_x &= (f_2(h,m))_x - s(h,m,x) \end{aligned}$$

From this it is obvious that

$$f_2(h,m) = hu^2 + \frac{gh^2}{2}$$

and

$$s(h,m,x) = -ghB_x(x)$$

Thus our conservation form is

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ hu^2 + \frac{gh^2}{2} \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghB_x(x) \end{pmatrix}$$