SF2521: Homework Assignment 4

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1 Shallow Water with Non-Horizontal Bottom

We now return to the shallow water model from Homework 2. However, this time we study a modified version of the problem, namely a model with a non-horizontal bottom "bathymetry" $B(x)$. This is represented in the following model

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + hu_x \\ huu_x + gh(h_x + B_x) \end{pmatrix} = 0$$

1.1 Still Water Implies Horizontal Water Level

From the diagram, we know that the water level, lets call it $w(x,t)$, is $h(x,t) + B(x)$. We wish to show that for still water,i.e. $u = 0$, we the water level is horizontal, i.e. $w_x = 0$. We know that $u = 0 \implies u_x = u_t = 0$, so we simply plug these values into our model to obtain

$$\begin{pmatrix} h_t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 + 0 \\ 0 + gh(h_x + B_x) \end{pmatrix} = 0$$

We have that $h_t = 0$, which in combination with the fact that B is a function of only x, gives us that $w_t = h_t + B_t = 0$. Thus we know that the water level does not change with time. Now the second line of the system gives us that $gh(h_x + B_x) = 0 \implies h_x = -B_x$. This immediately gives us that $w_x = h_x + B_x = -B_x + b_x = 0$. Therefore with still water, we have a constant, horizontal water level.

1.2 Conservation Form

We now wish to write our equation in conservation form as

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h,m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h,m,x) \end{pmatrix}$$

where $m = hu$ and s is independent of the derivatives of h and m. We now set the equations equal to eachother and solve for f_2 and s

$$\begin{aligned} hu_t + huu_x + gh(h_x + B_x) &= m_t + (f_2(h,m))_x - s(h,m,x) \\ hu_t + huu_x + gh(h_x + B_x) &= h_t u + hu_t + (f_2(h,m))_x - s(h,m,x) \\ -h_t u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \end{aligned}$$

Now, using what we know from the first equation in the system, we get

$$\begin{aligned} h_t u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \\ (h_x u + hu_x)u + huu_x + gh(h_x + B_x) &= (f_2(h,m))_x - s(h,m,x) \\ h_x u^2 + 2huu_x + ghh_x + ghB_x &= (f_2(h,m))_x - s(h,m,x) \end{aligned}$$

From this it is obvious that

$$f_2(h,m) = hu^2 + \frac{gh^2}{2} = \frac{m^2}{h} + \frac{1}{2}gh^2$$

and

$$s(h,m,x) = -ghB_x(x)$$

Thus our conservation form is

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{gh^2}{2} \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghB_x(x) \end{pmatrix}$$

2 First Order Roe Scheme with Flat Bottom

2.1 Scheme

We first implement the first order Roe scheme to simulate the shallow water equations for a flat bottom. The algorithm works as follows. We have our equation set up as a conservation law i.e.,

$$q_t + f(q)_x = 0$$

so we need a method in conservation form, i.e.

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

We use the following numerical flux formula

$$F_{i+1/2} = \frac{1}{2}(f(Q_i^n) + f(Q_{i+1}^n)) - \frac{1}{2}\sum_{p=1}^{2}|\hat{\lambda}_{j+1/2}^p|W_{j+1/2}^p$$

From the previous section, we have

$$f(q) = \begin{pmatrix} q_2 \\ \frac{q_2^2}{q_1} + \frac{gq_1^2}{2} \end{pmatrix}$$

From the course text book we have $\hat{\lambda}_{j+1/2}^1 = \hat{u} - \hat{c}$ and $\hat{\lambda}_{j+1/2}^2 = \hat{u} + \hat{c}$ where $\hat{c} = \sqrt{0.5g(h_{i-1} + h_i)}$,

$$\hat{u} = \frac{\sqrt{h_{i-1}}u_{i-1} + \sqrt{h_i}u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}}$$

and letting $\delta = Q_i - Q_{i-1}$ we get $W_{i-1/2}^1 = \frac{(\hat{u}+\hat{c})\delta^1 - \delta^2}{2\hat{c}}\begin{pmatrix} 1 \\ \hat{u} - \hat{c} \end{pmatrix}$ and $W_{i-1/2}^2 = \frac{-(\hat{u}-\hat{c})\delta^1 + \delta^2}{2\hat{c}}\begin{pmatrix} 1 \\ \hat{u} + \hat{c} \end{pmatrix}$.

2.2 Single pulse

We are now going to use this scheme a produce a single pulse, with "wall" boundary condition. We were given the following initial condition for the height :

$$h(x,0) = H + ae^{-(x-L/2)^2/w^2}$$

With $H = 1$, $w = 0.1L$ and $a = H/5$. The goal is now to find $m(x,0)$ so that we only have one wave traveling. To find this initial condition, let us diagonalize the PDE using the chain rule. Because we have a flat bottom ($B_x = 0$), the chain rule gives :

$$q_t + (f(q))_x = q_t + f'(q)q_x = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -\frac{q_2^2}{q_1^2} + gq_1 & 2\frac{q_2}{q_1} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3 Steady Solutions with Non-Horizontal Bottom

We now implement the roe solver to approximate the solution of a problem which has a non-horizontal bottom. Having a non-horizontal bottom now introduces a source term, so we must edit our scheme. Our new conservative first order scheme is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) + \Delta t S(Q^n)$$

where

$$S(Q^n) = \begin{pmatrix} 0 \\ -gh^n B_x(x) \end{pmatrix}$$

From the problem, we have that

$$B(x) = \begin{cases} B_0 \cos^2(\frac{\pi(x-L/2)}{2r}) & |x - L/2| < r \\ 0 & |x - L/2| \geq r \end{cases}$$

from which we are able to get

$$B'(x) = \begin{cases} -\pi B_0 \sin(\frac{\pi(x-L/2)}{r}) & |x - L/2| < r \\ 0 & |x - L/2| \geq r \end{cases}$$