Thesis

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0.1 Basics

0.1.1 Abstraction of the Workspace

In [3], Belta et al describe robot path planning as consisting of three parts: the specification level, execution level, and the implementation level. The first level, the specification level, involves creating a graph (Büchi automata, which will be defined later) that represents the robot motion. Next is the execution level, which involves finding a path through the graph that satisfies a specification. Lastly, in the implementation level robot controllers are constructed that satisfy the path found in the previous step.

We assume that we have one robot which is located in a given workspace denote $W_0 \subset \mathbb{R}^n$, which is bounded. To create a graph that represents the robot motion we need to consider the workspace along with the dynamics of the robot. To represent our workspace, which is a subspace of $\mathbb{R}^n$, in a finite graph we must partition it into a finite number of equivalence classes. A partition map is formally defined in definition 1. Any partition can be used as long as it satisfy the bisimulation property, which will be defined later once more notation has been introduced [4]. We denote the $\Pi = \pi_1, \pi_2, \ldots, \pi_w$ to be the set of equivalence classes the workspace has been partitioned into, and thus $\cup_{i=1}^{w} \pi_i = W_0$ and $\pi_i \cap \pi_j = \emptyset$, $\forall i, j = 1, 2, \ldots, w$ and $i \neq j$. We will henceforth refer to equivalence class π_i as region π_i for $i = 0, 1, \ldots, w$.

Definition 1. *A partition map, $T : W_0 \to \Pi$ sends each state $x \in W_0$ to the finite set of equivalence classes $\Pi = \pi_i$, $i = 1, 2, \ldots, w$. $T^{-1}(\pi_i)$ is then all the states $x \in W_0$ that are in the equivalence class π_i [7].*

We now introduce atomic propositions, which will be the building blocks for our task specification. Atomic propositions are boolean variables, and will be used to express properties about the state or the robot or the workspace. We define the following set of atomic propositions $AP_r = a_{r,i}$, $i = 1, 2, \ldots, w$ where

$$\alpha_{r,i} = \begin{cases} \top & \text{if the robot is in region } \pi_i \\ \bot & \text{else} \end{cases}$$

which represent the robot's location [9]. Other things we want to be able to express are potential tasks, denote $AP_p = \alpha_{p,i}$, $i = 1, 2, \ldots, m$. These can be statements such as "there is a ball in region π_I" or "the robot beeps" We now define the set of all propositions $AP = AP_r \cup AP_p$.

Definition 2. *The labelling function $L_C : W_0 \to 2^{AP}$ maps a point $x \in W_0$ to the set of atomic propositions satisfied by x [9].*

We also include a definition of the discrete counterpart

Definition 3. *The labelling function* $L_D : \Pi \to 2^{AP}$ *maps a region* $\pi_i \in \Pi$ *to the set of atomic propositions satisfied by* π_i*.*

Note: 2^{AP} is the powerset of AP, i.e. the set of all subsets of AP include the null set and AP.

For example, by definition, $a_{r,i} \in L_D(\pi_i)$.

To define a graph that represents our environment, we must also consider the dynamics of the robot. The dynamics are relevant because they define the relationship between the various regions. The relationship we refer to is known as a transition. We define a transition between two points in W_0 as follows

Definition 4. *There is a continuous transition,* $\to_C \subset W_0 \times W_0$ *from* x *to* x'*, denoted* $x \to_C x'$ *if it is possible to construct a trajectory* $x(t)$ *for* $0 \leq t \leq T$ *with* $x(0) = x$ *and* $x(T) = x'$ *and we have* $x(t) \in (T^{-1}(T(x)) \cup T^{-1}(T(x')))$ *[6]*

We then say that there is a transition between two regions if from any point in the first region there is a transition to a point in the second region. More formally

Definition 5. *There is a discrete transition,* $\to_D \subset \Pi \times \Pi$*, from* π_i *to* π_j*, denoted* $\pi_i \to_D \pi_j$ *if there exists* x *and* x' *such that* $T(x) = \pi_i$*,* $T(x') = \pi_j$ *and* $x \to_C x'$

We can now define bisimulations

Definition 6. *A partition* $T : W_0 \to \Pi$ *is called a bisimulation if the following properties hold for all* $x, y \in W_0$

1. *(Observation Preserving): If* $T(x) = T(y)$*, then* $L_C(x) = L_C(y)$*.*
2. *(Reachability Preserving): If* $T(x) = T(y)$*, then if* $x \to_C x'$ *then* $y \to_C y'$ *for some* y' *with* $T(x') = T(y')$

The Observation Preserving requirement makes sure we do not allow the situation where part of π_i fulfils $\alpha \in AP$ while part of π_i does not, and the Reachability Preserving requirement ensures that for every point in region π_i, there exists a trajectory to some point x', such that $T(x') = \pi_j$ if $\pi_i \to_D \pi_j$. These two requirements together guarantee that the discrete path we compute is feasible at the continuous level.

We can now define Finite-State Transition System (FTS), which is how we will represent our workspace.

Definition 7. *An FTS is a tuple* $\mathcal{T}_C = (\Pi, \rightarrow_D, \Pi_0, AP, L_C)$ *where* Π *is the set of states,* $\rightarrow_D \subseteq \Pi \times \Pi$ *is the transitions relation where* $(\pi_i, \pi_j) \in \rightarrow_C$ *iff there is a transition from* π_i *to* π_j *as defined in definition* **??**. *In adherence to common notation, we will write* $\pi_i \rightarrow_C \pi_j$. *Note:* $\pi_i \rightarrow_C \pi_i,\ \forall 1, 2, \ldots w.$ $\Pi_0 \subseteq \Pi$ *is the initial state(s),* $AP = AP_r \cup AP_p$ *is the set of atomic propositions, and* $L_C : \Pi \rightarrow 2^{AP}$ *is the labelling function defined in definition* **??**.

In this thesis, we will also consider the weighted FTS (WFTS)

Definition 8. *A WTFS is a tuple* $\mathcal{T}_C = (\Pi, \rightarrow_C, \Pi_0, AP, L_C, W_C)$ *where* Π, $\rightarrow_C$, Π_0, AP, *and* L_C *are defined as in definition 8 and* $W_C : \Pi \times \Pi \rightarrow \mathbb{R}^+$ *is the weight function i.e. the cost of a transition in* $\rightarrow_C$.

Note: Any FTS can be written can be converted to an WFTS with the weights of all transitions equalling one.

We use the FTS which represents our workspace to search for paths that are doable for our robot. When we search for a path, from one state we will only consider states which have a transition from our current state, because these are the only states the robot can move to. When talking about FTS, it can be helpful to use the notation $\texttt{Pre}(\pi_i) = \{\pi_j \in \Pi | \pi_j \rightarrow_D \pi_i\}$ to define the the predecessors of the state π_i and $\texttt{Post}(\pi_i) = \{\pi_j \in \Pi | \pi_j \rightarrow_D \pi_i\}$ to define the the successors of the state π_i. In this thesis, we will deal with infinite paths. An infinite path is an infinite sequence of states $\tau = \pi_1 \pi_2 \ldots$ such that $\pi_i \in \Pi_0$ and $\pi_i \in \text{Post}(\pi_{i-1})\ \forall i > 0$. The trace of a path is the sequence of sets of atomic propositions that are true in the states along a path i.e. $\text{trace}(\tau) = L_D(\pi_1) L_D(\pi_2) \ldots$.

0.1.2 Linear Temporal Logic (LTL)

To define tasks for our robot we must choose a high level language. Temporal logics are especially suited for defining robot tasks because of their ability to express not only fomulas constructed of atomic propositions and standard boolean connectives (conjunction, disjunction, and negation), but also temporal specifications e.g. α is true at some point of time. The particular temporal logic we will be using is known as linear temporal logic (LTL) [5]. LTL formulas are defined over a set of atomic propositions AP according to the following grammar:

$$\varphi ::= \top | \alpha | \neg \varphi_1 | \varphi_1 \vee \varphi_2 | \mathbf{X} \varphi_1 | \varphi_1 \boldsymbol{\mathcal{U}} \varphi_2$$

where $\top$ is the predicate true, $\alpha \in AP$ is an atomic proposition, φ_1 and φ_2 are LTL fomulas, $\neg$ and $\vee$ denote denote the standard Boolean connectives negation and disjunction respectively, X being the "Next" operator. $\boldsymbol{\mathcal{U}}$ is the temporal operator "Until", with $\varphi_1 \mathcal{U} \varphi_2$ meaning φ_1 is true until φ_2 becomes true. Given these operators, we can define the following additional prepositional operators:

$$\begin{aligned} \text{Conjunction: } & \varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2) \\ \text{Implication: } & \varphi_1 \Rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2 \\ \text{Equivalence: } & \varphi_1 \Leftrightarrow \varphi_2 = (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1) \end{aligned}$$

We note quickly that we have the false predicate, $\bot = \neg\top$. We are also able to derive the following additional temporal operators:

$$\begin{aligned} \text{Eventuality: } & \diamond \varphi_1 = TU\varphi_1 \\ \text{Always: } & \Box\varphi_1 = \neg \diamond \neg\varphi_1 \end{aligned}$$

There is a growing interest in path and mission planning in robots using temporal logic specifications given the easy extension from natural language to temporal logic [12]. We now give examples to illustrate this point and to introduce us to LTL formulas. First, the atomic operators generally capture properties of the robot or the environment i.e. "the robot is in region 1", "the ball is in region 2", "the robot is holding a ball". There are some common tasks converted to LTL formulas given in [6]

1. **Reachability while avoiding regions**: "Go to region π_{n+1} while avoiding regions $\pi_1, \pi_2, \ldots, \pi_n$"
   $\neg(\pi_1 \vee \pi_2 \ldots \pi_n)\mathcal{U}\pi_{n+1}$

2. **Sequencing**: "Visit regions π_1, π_2, π_3 in that order"
   $\diamond(\pi_1 \wedge \diamond(\pi_2 \wedge \diamond\pi_3))$

3. **Coverage**: "Visit regions $\pi_1, \pi_2, \ldots, \pi_n$ in any order"
   $\diamond\pi_1 \wedge \diamond\pi_2 \wedge \cdots \wedge \diamond\pi_n$

4. **Recurrence (Liveness)**: "Visit regions $\pi_1, \ldots, \pi_n$ in any order over and over again"
   $\Box(\diamond\pi_1 \wedge \diamond\pi_2 \wedge \cdots \wedge \diamond\pi_n)$

Of course more complicated tasks are also expressible in LTL, and atomic propositions need not only refer to the location of the robot. Here is an example given in [9]: "Pick up the red ball, drop it to one of the baskets and

then stay in room one"
$\diamond(rball \wedge \diamond basket) \wedge \diamond\Box r1$

We now look at what it means to satisfy an LTL formula. We will talk about *words* satisfying LTL formulas, in our case *infinite words*. An infinite word over the alphabet 2^{AP} is an infinite sequence $\sigma \in (2^{AP})^{\omega}$. That is, $\sigma = S_0 S_1 S_2 \ldots$, where $S_k \in 2^{AP}$ for $k = 1, 2, \ldots$ and S_k is the set of atomic propositions that are true at time step k [9]. An infinite word σ satisfies an LTL formula φ based on the LTL semantics.

Definition 9. *The semantics of LTL are defined as follows:*

$$\begin{aligned}
(\sigma, k) &\models \alpha \quad \textit{if} \quad \alpha \in S_k \\
(\sigma, k) &\models \neg\varphi \quad \textit{if} \quad (\sigma, k) \not\models \varphi \\
(\sigma, k) &\models \boldsymbol{X}\varphi \quad \textit{if} \quad (\sigma, k+1) \models \varphi \\
(\sigma, k) &\models \varphi_1 \vee \varphi_2 \quad \textit{if} \quad (\sigma, k) \models \varphi_1 \textit{ or } (\sigma, k) \models \varphi_2 \\
(\sigma, k) &\models \varphi_1 \mathcal{U} \varphi_2 \quad \textit{if} \quad \exists k' \in [k, +\inf],\ (\sigma, k') \models \varphi_2 \textit{ and} \\
&\qquad\qquad\quad \forall k'' \in (k, k'),\ (\sigma, k'') \models \varphi_1
\end{aligned}$$

Where (σ, k) refers to σ at time step k. So an infinite word σ is said to said to satisfy formula φ if $(\sigma, 0) \models \varphi$. For the ease of reading we will refer to $(\sigma, 0)$ as σ.

There is a connection between these infinite words and the FTS described earlier that is crucial in motion planning technique. Given an infinite path τ of an FTS, we have that the trace of the path, $\texttt{trace}(\tau)$, is an infinite word over the alphabet 2^{AP}. Given the LTL semantics, we now have the ability to verify if a path satisfies an LTL formula! We will say an infinite path τ *satisfies* φ if its trace satisfies φ, i.e. $\tau \models \varphi$ if $trace(\tau) \models \varphi$. A path satisfying φ will be referred to as a *plan* for φ.

0.1.3 Büchi Automata

We now know if a path of an FTS satisfies a given LTL formula, however we are interested in *generating* paths that satisfy a given formula, which requires a significantly more amout of work! We are going to need a finite representation of a given LTL formula, that we can search. This representation is a Nondeterministic Büchi automaton (NBA).

Definition 10. *An NBA $\mathcal{A}_\varphi$ is defined by a five-tuple:*

$$\mathcal{A}_\varphi = (\mathcal{Q}, 2^A P, \delta, \mathcal{Q}_0, \mathcal{F})$$

where $\mathcal{Q}$ is a finite set of states, $\mathcal{Q}_0 \subseteq \mathcal{Q}$ is the set of initial states, 2^{AP} is the alphabet, $\delta : \mathcal{Q} \times 2^{AP} \to 2^{\mathcal{Q}}$ is a transition relation, and $\mathcal{F} \subseteq \mathcal{Q}$ is the set of accepting states

An infinite run of an NBA is an infinite sequence or states, $r = q_0 q_1 \ldots$, where that starts from an initial state i.e. $q_0 \in Q_0$ and $q_{k+1} \in \delta(q_k, S)$ for some $S \in 2^{AP}$, for $k = 0, 1, \ldots$. The requirements for a run r to be accepting is $\texttt{Inf}(r) \cap \mathcal{F} \neq \emptyset$, where $\texttt{Inf}(r)$ is the set of states that appear in r infinitely often [9].

Now to tie together the concept of words and runs on an NBA. An infinite word $\sigma = S_0 S_1 \ldots$ corresponds to $r_\sigma = q_0 q_1 \ldots$ where $q_0 \in Q_0$ and $q_i + 1 \in \delta(q_i, S_i)$

It has been shown that given an LTL formula φ over AP, there exists a NBA over 2^{AP} corresponding to φ, denoted A_φ [2]. When we say an NBA corresponds to an LTL formula, we mean that the set of words that correspond to accepting runs of the NBA is the same as the set of words accepted by the LTL formula.

0.1.4 Product Automata

These two structures are then combined to create the product automaton. The product automata is also a Büchi automata and is defined as follows:

Definition 11. *The weighted product Büchi automaton is defined by $\mathcal{A}_p = \mathcal{T} \otimes \mathcal{A}_\varphi = (Q', \delta', Q'_0, \mathcal{F}', W_p)$, where $Q' = \Pi \times Q = \{\langle \pi, q \rangle \in Q' | \forall \pi \in \Pi,\ \forall q \in Q\}$; $\delta' : Q' \to 2^{Q'}$. $\langle \pi_j, q_n \rangle \in \delta'(\langle \pi_i, q_m \rangle)$ iff $(\pi_i, \pi_j) \in \to_c$ and $q_n \in \delta(q_m, L_c(pi))$; $Q'_0 = \{\langle \pi, q \rangle | \pi \in \Pi_0,\ q_0 \in Q_0\}$, the set of initial states: $\mathcal{F}' = \{\langle \pi, q \rangle | \pi \in \Pi, q \in \mathcal{F}$, the set of accepting states; $W_p : Q' \times Q' \to \mathbb{R}^+$ is the weight function: $W_p(\langle \pi_i, q_m \rangle, \langle \pi_j, q_n \rangle) = W_c(\pi_i, \pi_j)$, where $\langle \pi_j, q_n \rangle \in \delta'(\langle \pi_i, q_m \rangle)$*

Given a state q' = $\langle \pi, q \rangle \in Q'$, its projection on Π is denoted by $q'|_\Pi = \pi$ and its projection on Q is denoted by $q'|_Q = q$. Given an infinite run $R = q'_0 q'_1 a'_2 \ldots$ of $\mathcal{A}_p$, its projection on Π is denoted by $R|_\Pi = q'_0|_\Pi q'_1|_\Pi q'_2|_\Pi \ldots$ and its projection on Q is denoted by $R|_Q = q'_0|_Q q'_1|_Q q'_2|_Q \ldots$ [9].

Given that $\mathcal{A}_p$ is a Büchi automaton, the requirements of an accepting run, r, is the same as before i.e. $\texttt{Inf}(r) \cap \mathcal{F} \neq \emptyset$

Our problem is now to find an accepting run of $\mathcal{A}_p$. This can be a difficult task, given that an accepting run is a infinite sequence of states, and there are infinitely many possibilities. We also want to have some sort of measure of optimality, making the problem harder. To accomplish this, we are going to restrict our search to plans with a finite representation. This limits the plans

that we can calculate, however it is much easier to deal paths that admit a finite representation. Specifically, we are going to be looking for paths in the prefix-suffix structure i.e.

$$\tau = \langle \tau_{pre}, \tau_{suf} \rangle = \tau_{pre}[\tau_{suf}]^{\omega}$$

The prefix, τ_{pre}, is the path from an initial node to an accepting node. The suffix, τ_{suf}, is going to be a path from the same accepting node back to itself. So the full path is going to be the prefix and then the suffix repeated infinitely many times (which is the meaning of the ω superscript). Thus, the accepting node appears infinitely many times in τ which makes τ accepting. Plans of this form are much easier to deal with because, while they are still infinite plans, they have a finite representation which is easier to deal with.

0.1.5 Cost of a Run

As we said before, we want to have a way to measure the optimality of a run. We introduce the concept of the *cost* of a run to satisfy this requirement. We are focusing on the accepting runs of $\mathcal{A}_p$ with the prefix-suffix structure

$$\begin{aligned} R &= \langle R_{pre}, R_{suf} \rangle = q'_0 q'_1 \dots q'_f [q'_f q'_{f+1} \dots q'_n]^{\omega} \\ &= \langle \pi_0, q_0 \rangle \dots \langle \pi_{f-1}, q_{f-1} \rangle [\langle \pi_f, q_f \rangle \langle \pi_f, q_f \rangle \dots \langle \pi_n, q_n \rangle]^{\omega} \end{aligned}$$

where $q'_0 = \langle \pi_0, q_0 \rangle \in \mathcal{Q}'_0$ and $q'_f = \langle \pi_f, q_f \rangle \in \mathcal{F}'$.

As we can see, our path is a sequence of states, $q'_0, q'_1, \dots, q'_n$ in $\mathcal{A}_p$, where $q'_{i+1} \in \delta'(q'_i)$ for all $i = 0, 1, \dots, n-1$. Each of these transitions has a weight or cost associated with it, given by $W_p(q'_i, q'_{i+1}) = W_c(q'_i|_{\Pi}, q'_{i+1}|_{\Pi})$. We simply define the cost of our path as the sum of the cost of the transitions in the path, with the cost of the suffix being weighted.

$$\begin{aligned} \texttt{Cost}(R, \mathcal{A}_p) &= \sum_{i=0}^{f-1} W_p(q_i, q_{i+1}) + \gamma \sum_{i=f}^{n-1} W_p(q_i, q_{i+1}) \\ &= \sum_{i=0}^{f-1} W_c(\pi_i, \pi_{i+1}) + \gamma \sum_{i=f}^{n-1} W_c(\pi_i, \pi_{i+1}) \end{aligned}$$

where $\gamma \geq 0$ is the relative weighting of the transient response (prefix) cost and steady response (suffix) cost [9]. In [6] they say that they search for the path with the least amount of transitions and say this is the optimal path. This is an example converting a FTS to a WFTS by setting the weight of every transition to one.

We will denote the accepting run with prefix-suffix structure that minimizes the total cost as R_{opt}, with the corresponding plan $\tau_{opt} = R_{opt}|_{\Pi}$. We note however that this plan may not actually be the true optimal plan with prefix-suffix structure. In [16] we see that simplifications in the translation from LTL formulas to NBA can result in a loss of optimality. This will come up again when we analyse the paths our algorithm generates.

0.2 Search Algorithms

The task is now to compute a path that satisfies our LTL formula. The current accepted algorithm does an exhaustive search of the product automaton to find the optimal path (again this may not actually be the optimal path [16]. This however is a computationally intensive task. We present an approximation algorithm that gives a *good* path, but not necessarily the optimal path. This can be attractive if the cost of the path is not of dire importance. We first present the current standard algorithm and then our algorithm.

0.2.1 Accepted Algorithm

The search algorithm used in many recent works on the specific type of control planning synthesis comes from this prefix-suffix structure. The basic idea is to find a path from an initial node, q_0 to an accepting node, q_f, and then find a path from the q_f back to itself. The first part from q_0 to q_f is the prefix and the second part q_f back to q_f is the suffix. Then the resulting path, τ, will be the prefix, followed by the suffix repeated infinitely many times. This path is thus accepting because the suffix finds the path from an initial state back to itself, and thus contains the initial state, and is repeated infinitely many times $q_f \in \texttt{Inf}\tau \Rightarrow \texttt{Inf}\tau \cap \mathcal{F} \neq \emptyset$. This algorithm, or simple variations of it, are used in many works on motion planning synthesis [6], add more, so we will refer to it as the *accepted* algorithm.

Algorithm 1, from [9], gives pseudocode of how to compute R_{opt}.

Procedure 1 OptRun()

Input: Input $\mathcal{A}_p, S' = \mathcal{Q}'_0$ by default
Output: R_{opt}

1: For each accepting state $q'_f \in \mathcal{F}'$, calculate the optimal path back to q'_f.
2: For initial state $q'_0 \in S'$, find the optimal path to each $q'_f \in \mathcal{F}$.
3: Find the pair of $(q'_{0,opt}, q'_{f,opt})$ that minimizes the total cost
4: Optimal accepting run R_{opt}, prefix: shortest path from q'_{0*} to q_{f*}; suffix: the shortest cycle from q'_{f*} and back to itself.

Meng Guo has created a public github repository, P-MAS-TG (Planner for Multiple Agent System with Temporal Goals) [10]. The function `dijkstra_plan_networkX` in `P_MAS_TG`
`discrete_plan.py` is approximately equvialent to Algorithm 1. The work of finding the optimal path from q'_f back to q'_f and from q'_0 to all q'_f is done by `dijkstra_predecessor_and_distance` from the NetworkX python package

[15]. `dijkstra_predecessor_and_distance`$(\mathcal{A}_p, q_0)$ returns two dictionaries; one containing a list of all the nodes q_0 is a predecessor of and one containing the distances to each of these nodes. When we provide computational examples for the accepted algorithm, we will be using this repository. The script run to create the examples is included in the appendix.

The worst case computational complexity of this algorithm $\mathcal{O}(|\delta'| \cdot \log|\mathcal{Q}'| \cdot (|\mathcal{Q}'_0| + |\mathcal{F}'|))$ because the worst case complexity for a Dijkstra search is $\mathcal{O}(|\delta'| \cdot \log|\mathcal{Q}'|)$ and Algorithm 1 does $(|\mathcal{Q}'_0| + |\mathcal{F}'|)$ Dijkstra searches (one for each initial node and one for each accepting node).

0.2.2 Our Algorithm

As we can see, the current algorithm has to do a lot of work. First it has to do Dijkstra's search for each initial state, and then one for each accepting state (the number of accepting states is at least the size of the FTS). The state space that is being searched can also become very big, which is known as the state explosion problem [5]. The size of the product automaton, $|\mathcal{A}_p|$ is the size of the Büchi automaton corresponding to the LTL formula times the size of the FTS i.e. $|\Pi|\dot{|}\mathcal{Q}|$. The size of the Büchi automaton corresponding to the LTL formula is then usually exponential in the size of the formula. We can imagine how much searching is needed if we have and FTS and Büchi that are both fairly large. To solve this problem, we suggest an algorithm that sacrifices optimality but preforms much faster than the current accepted algorithm.

The idea stems from the fact that $q' = \langle \pi, q \rangle \in \mathcal{Q}'$ is an accepting state of $\mathcal{A}_p$ iff $q \in Q$ is an accepting state of $\mathcal{A}_\varphi$. Thus finding an accepting state in the product automaton is essentially finding an accepting state of the LTL Büchi automaton. We therefore suggest assigning a distance measure in the LTL Büchi automaton that carries over to the product automaton. To do this, we first define a Büchi automaton that includes information on the distance to an accepting state.

Definition 12. *An NBA with distance, NBAD, is defined by a six-tuple:*

$$\mathcal{A}_{\varphi,d} = (\mathcal{Q}, 2^{AP}, \delta, \mathcal{Q}_0, \mathcal{F}, d)$$

where $\mathcal{Q}, 2^AP, \delta, \mathcal{Q}_0, \mathcal{F}$ *are defined as in definition 10 and* $d : \mathcal{Q} \to \mathbb{Z}$ *is defined as*

$$d(q_n) = \min_x \{x | q_x \in \mathcal{F} \text{ and } q_k \in \delta(q_{k-1}, S_{k-1}) \text{ for some } S_k \in 2^{AP} \text{ and } k = 0, 1, \ldots, x-1\}$$

which is the length of the number of transitions in the shortest path from q_n *to an accepting state.*

Then we also have a product automaton with distance, $\mathcal{A}_{p,d} = \mathcal{T} \otimes \mathcal{A}_\varphi = (Q', \delta', Q'_0, \mathcal{F}', W_p, d_p)$, defined similarly, with $d_p(q') = d(q'|\mathcal{Q})$. We will refer to q' as being on level n if $d_p(q') = n$.

The idea of our algorithm is to start from $q'_0 \in \mathcal{Q}'$, say $d_p(q'_0) = n$ and then use a Dijkstra search to find the closest node that is on next smallest level, $n-1$. Then we will do another Dijkstra search on the next level down to find the closest node that has a transition down, and so on. This ensures that we will approach the accepting states i.e. those states on level 0. Once we reach an accepting state, we use a either a Dijkstra search or a decreasing levels search to find the fastest way from the accepting state back to itself. Sometimes we have to use a Dijkstra search instead of use the idea of decreasing levels because, although it would be faster, in general this procedure cannot be use to find a specific accepting state. We will refer to the run generated by this algorithm as R_{nn} in which nn stands for nearest neighbour. We choose this name because in some situations this search is equivalent to the nearest neighbour search algorithm for the travelling salesperson problem [11]. Pseudocode is given in Algorithm 2

Procedure 2 NearestNeighborRun()

Input: Input $\mathcal{A}_{p,d}, S' = \mathcal{Q}'_0$ by default
Output: R_{nn}

```
Level = d_p(q'_0 ∈ Q'_0), Prefix = [q'_0]
while Level > 0 do
    find the closest node, NextNode, that is on level one less than Level
    add path to NextNode onto Path
    if d_p(NextNode) == 0 then
        break
    Level = Level - 1
Suffix = use Dijkstra search to find the optimal path from NexNode back to itself
R_nn, prefix + suffix.
```

Algorithm 2 is equivalent to the function `Garrett_search` which is provided in the appendix. This code was based on `dijkstra_plan_networkX` from [10] and still shares some of the structure. Finding the closest node on the level below the current, i.e. NextNode is done using the function `adapted_dijkstra_multisource` which is also included in the appendix. This code was based on the function `_dijkstra_multisource` in [15]. When we provide computation runs of our algorithm in the following text, we will be referring to runs done with this algorithm. Again, the script used to do these runs is also provided.

As we can see, assuming that we reach an accepting state in a strongly connected component, we will do $n + 1$ searches. This still may seem like a lot, however the searches are done on much smaller graphs. The first n searches only look at graphs with $|\Pi|$ nodes. These smaller graphs have a number of edges less than or equal to $|\delta|$ i.e. the number of edges $\mathcal{T}$ has. This is because $\langle \pi_j, q_n \rangle \in \delta'(\langle \pi_i, q_m \rangle)$ iff $(\pi_i, \pi_j) \in \rightarrow_c$ and $q_n \in \delta(q_m, L_c(pi))$, which implies the number of edges on one level is less than or equal to $|\delta|$. Therefore our worst case complexity will be $\mathcal{O}(|\delta| \cdot \log|\mathcal{T}| \cdot n) + \mathcal{O}(|\delta'| \cdot \log|\mathcal{Q}'|) = \mathcal{O}(|\delta| \cdot \log|\mathcal{T}| \cdot n + |\delta'| \cdot \log|\mathcal{Q}'|)$ where n is the level of the initial node. This complexity applies in the situation that the accepting node we find has a path back to itself and that we do not have transfers on the same level of the Büchi automaton i.e. if there is a transfer from q_i to $q_{i+1} \rightarrow d(q_i) \neq d(q_{i+1})$. However, when we get to examples in the complex formulas chapter this is not the case. If this distance requirement is not fulfilled, the worst case complexity of our algorithm is the same as the worst case complexity of the accepted algorithm i.e. $\mathcal{O}(|\delta'| \cdot \log|\mathcal{Q}'|)$.

We now analyse how this algorithm performs in when the LTL formula expresses certain behaviours.

0.3 Algorithm Performance with Specific Behaviours

To show how our algorithm differs with the current accepted algorithm, we illustrate examples using the FTS in figure 1

This FTS will be used in figures unless otherwise stated. We chose it to be very simple because the state explosion problem applies even to this report. If we chose a more complex FTS we would not be able to include illustrations of the product automaton because it gets very big very quickly. We will henceforth refer to this FTS as simple FTS. When providing computational results, we will use an FTS that is much larger, to bring out the difference between our algorithm and the accepted algorithm.

0.3.1 Reachability while avoiding regions

Reachability while avoiding regions is a property in which we wish to not cross over certain areas, say $\pi_1, \pi_2, \ldots, \pi_n$, until we get to a specified region, say π_{n+1}. After reaching π_{n+1} we are free to do what we want. This behaviour is expressed by the formula $\neg(\pi_1 \vee \pi_2 \vee \ldots \pi_n)\boldsymbol{\mathcal{U}}\pi_{n+1}$.

The Büchi automaton corresponding to this formula is given in figure 2

As we can see $d_p(q_1) = 1$ and $d_p(q_1) = 0$. For our example, we will look at the specific formula $\neg\pi_4\boldsymbol{\mathcal{U}}\pi_5$ The product automaton is shown in figure 3

Note: in figure 3 all nodes have a self loop, which are not included for the sake of the reader. Our algorithm does $n+1$ Dijkstra searches where n is the maximum level of a state in the Büchi automaton. As we can see in 2, which is the Büchi automaton corresponding to the general from of reachability while avoiding regions, n is 1 for all formulas of this form. Therefore our algorithm does one Dijkstra search starting from (π_1, q_1) which ends at (π_5, q_2). This is exactly what the accepted algorithm does, so we do not gain anything when using our algorithm on a formula of this type.

0.3.2 Sequencing

Sequencing is the behaviour of visiting regions $\pi_1, \pi_2, \ldots, \pi_n$ in that order. A formula that describes this behaviour for $n = 3$ is $\diamond(\pi_1 \wedge \diamond(\pi_2 \wedge \diamond\pi_3))$ and is shown in figure 4. This behaviour is ideal for our algorithm.

We show why in an example using the formula $\diamond(\pi_2 \wedge \diamond\pi_5))$ the simple FTS as before. The product automaton is shown in figure 5

Our algorithm finds (π_4, q_2), then starts another Dijkstra search and finds (π_5, q_3). Will search through extraneous nodes, for example (π_5, q_1). This

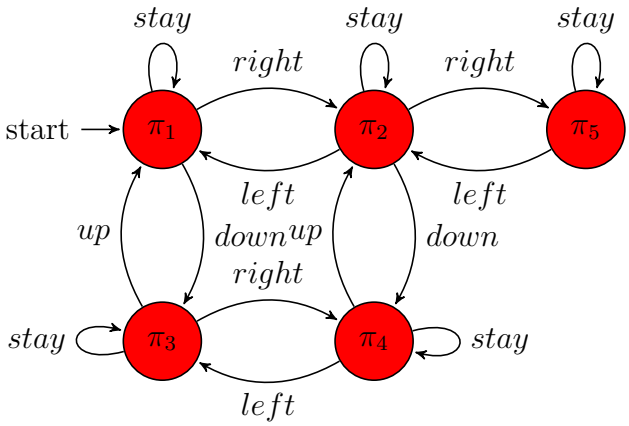


Figure 1: Simple Finite Transition System

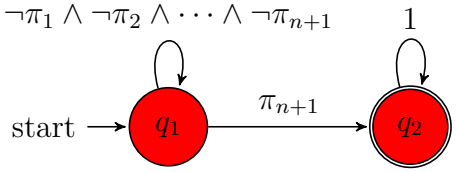


Figure 2: Büchi automaton corresponding to $\neg(\pi_1 \vee \pi_2 \vee \ldots \pi_n)\boldsymbol{\mathcal{U}}\pi_{n+1}$

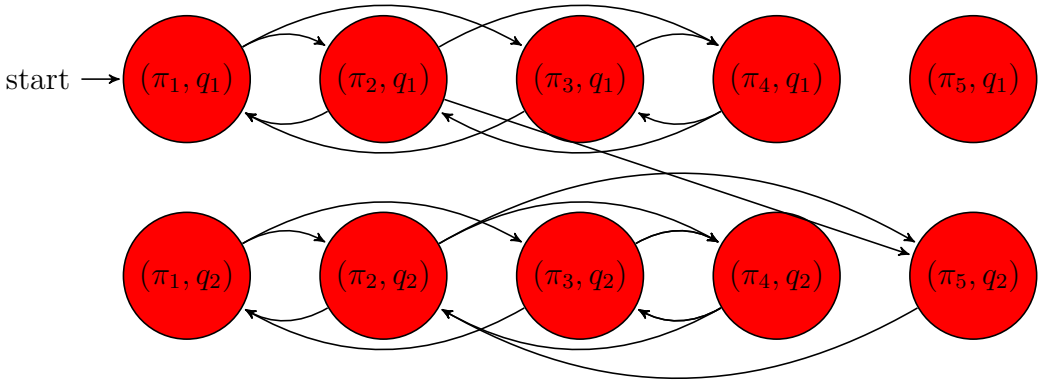


Figure 3: Product Automaton for $\neg\pi_4\boldsymbol{\mathcal{U}}\pi_5$ with Simple FTS

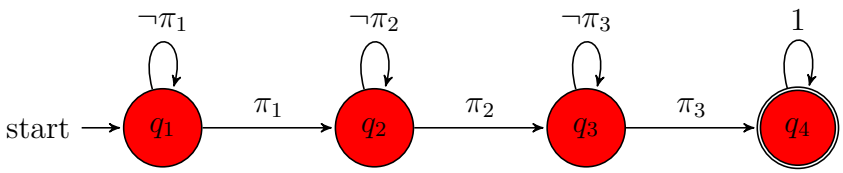


Figure 4: Büchi Automaton Corresponding to $\diamond(\pi_1 \wedge \diamond(\pi_2 \wedge \diamond\pi_3))$

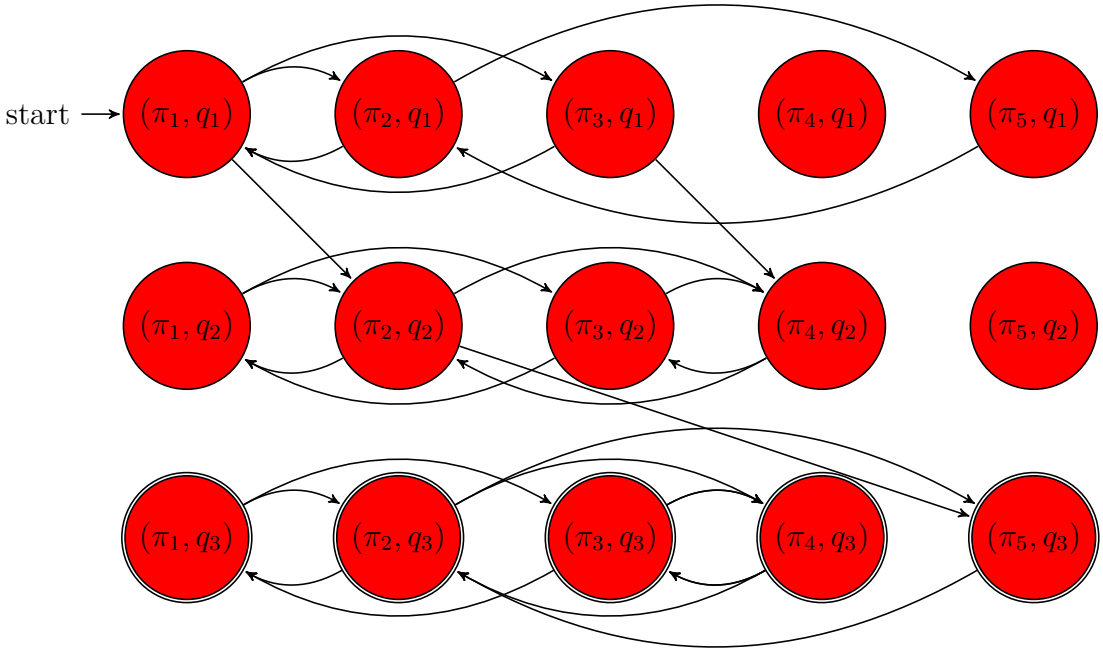


Figure 5: Product Automaton for $\diamond(\pi_2 \wedge \diamond\pi_5))$ with Simple FTS

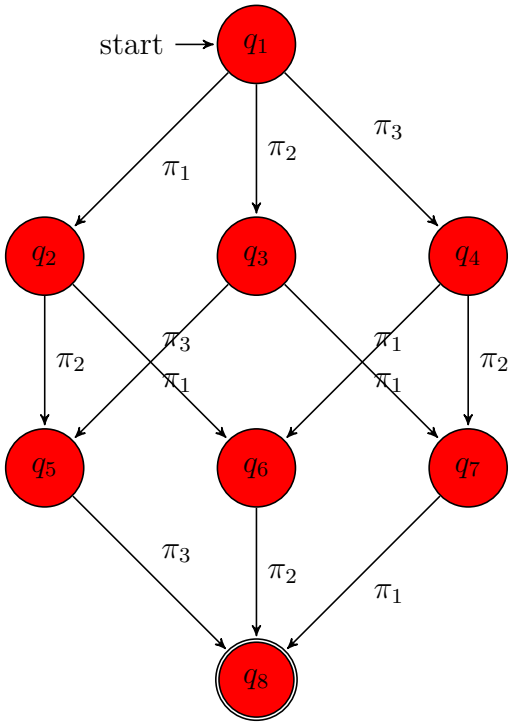


Figure 6: Büchi Automaton Corresponding to $\diamond\pi_1 \wedge \diamond\pi_2 \wedge \pi_3$

may not seem like a lot in this example, but when we expand to larger problems the difference becomes significant. Check how many nodes are searched with both algorithms, and show time difference.

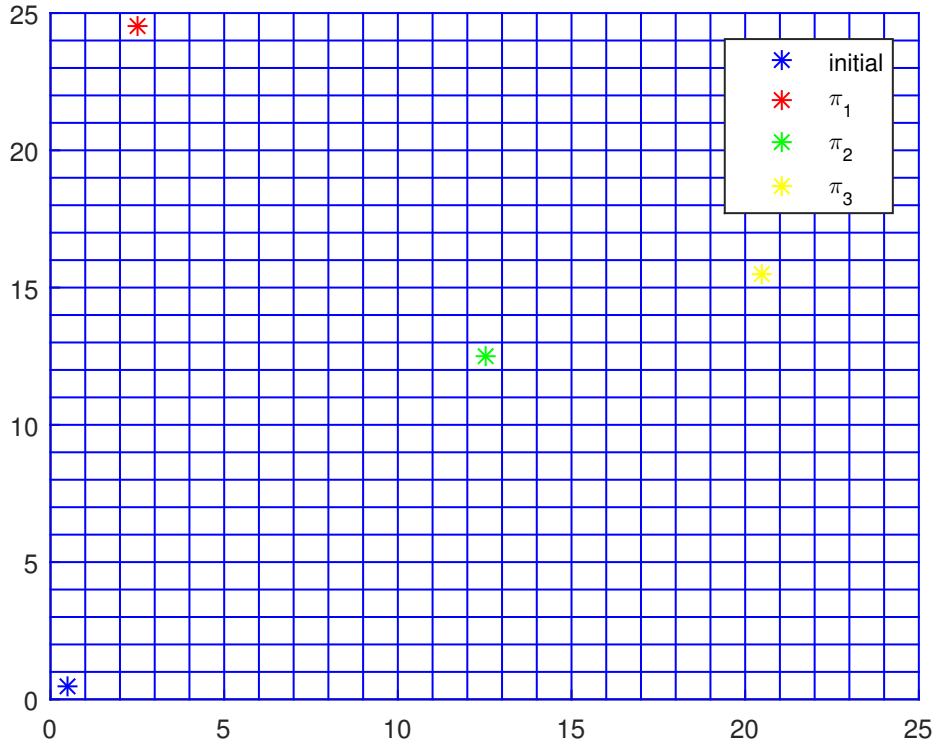
0.3.3 Coverage

A coverage formula represents the statement visit $\pi_1, \pi_2, \ldots, \pi_n$ in that order, and is of the form $\varphi = \diamond\pi_1 \wedge \diamond\pi_2 \wedge \cdots \wedge \pi_n$. We show the Büchi automaton corresponding to the formula $\diamond\pi_1 \wedge \diamond\pi_2 \wedge \pi_3$ in figure 6

So, we can see that to get to the accepting node, we have to choose which node to go to first, and which node to go to second (the third node we then have to visit is already decided). So, there are 6 possible paths to take from the initial node, q_1 to accepting state q_8. This is true in the product automaton too, if we only consider the option of taking the optimal path between nodes. The order that our algorithm will pick is it will pick first pick π_i which is the closest to it. From then, it will pick the next closest π_j out of the two that have not been visited yet.

We now define a workspace to use with our problem. Our workspace is a grid, 25 units across and 25 units up, a total of 625 regions. We say our robot

can can move horizontally and vertically, however it cannot move diagonally. Additionally we say that the unit cost of going from any adjacent to another region is 1. The initial position is located at $(0, 0)$, region π_1 is located at (2,24), region π_2 is located at (12,12), and region π_3 is located at (20,15) . See figure 0.3.3



When we use our algorithm on a coverage formula, we may not get the optimal path. We will however get an accepting path, and we now show that this path corresponds to the one generated by the nearest neighbour approach to the travelling salesperson problem. We also provide a provide a bound on the distance of our path based on the worst case ratio of the nearest neighbour tour to the optimal tour given by Rosendrantz, Stearns, and Lewis [14]. The travelling salesperson problem is stated in layman's terms as finding the shortest path for a salesperson to take such that he passes through a given set of cities and then returns back home at the end. More formally, it can be stated as finding the minimum Hamiltonian circuit with the lowest sum of distances between the nodes (cities). This problem has

been studied extensively and "give quote about importance". This problem is NP-hard, and thus many algorithms and heuristics exist for finding an approximate solution. One very simple algorithm to do this is called the nearest neighbour algorithm. It says from the starting city, pick the closest city to be the next stop. From there, pick the next closest city not including the starting city, and so on. If there is a tie in the next closest neighbour, we assume that the next node can be decided arbitrarily. This is exactly what our algorithm does in this situation, the first Dijkstra search finds the closest node, and the we start another search.

To formulate our problem as a travelling salesman problem we use the idea of a dummy node from Lenstra and Rinnooy Kan's computer wiring example in [13]. In their example, they are designing a computer interface at the Institute for Nuclear Physical Research in Amsterdam. An interface is made up of several modules, with multiple pins on each module. A given subset of pins has to be interconnnected by wires, and at most two wires can be connected to any pin. For obvious regions, it is desirable to minimize the amount of wire used. They show that this is actually a travelling salesperson problem in disguise. The only difference between this problem and a travelling salesman problem is that in the travelling salesman problem, the salesman must return home at the end. This is not true in this problem. It is also not true in our problem, we only need to pass through π_1, π_2 and π_3, there is no need to return to the starting state after we do this. To formulate this seemingly unrelated problem into a travelling salesperson problem, they set P to be the set of pins to be interconnected, c_{ij} to be the distance between pin i and pin j. They then introduce a dummy node $*$ that is a distance 0 from all the other nodes i.e. $c_{i*} = c_{*i} = 0$ for all i. Then the corresponding problem is solving the travelling salesperson problem on the set of nodes $N = P \cup \{*\}$.

For our problem, we set $c_{ij} = d(\pi_i, \pi_j)$, for $i, j = 0, 1, 2, 3$ where where the initial state is from now on known as π_0, to be the shortest path our robot can take from π_i to π_j, insuring that the triangle inequality is satisfied for all i and j. We must preserve the the triangle inequality for a proof of a worst case scenario bound we will provide later on. We use this same idea as above of adding a dummy node, however to preserve the triangle inequality we cannot have the dummy node be distance 0 from the other nodes. Indeed, if $c_{i*} = c_{*i} = 0$ the triangle inequality would be violated because $c_{i*} + c_{*j} = 0 \geq c_{ij}$ which would make the cost from getting to any point 0, thus rendering the problem extremely trivial.

We can represent the relationship between the regions in our graph with the following *complete* subgraph, shown in figure. A complete graph is an undirected graph in which every pair of vertices is connected by an edge.

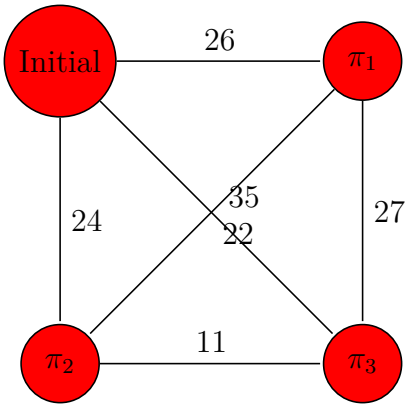


Figure 7: Complete Graph between Regions of Interest

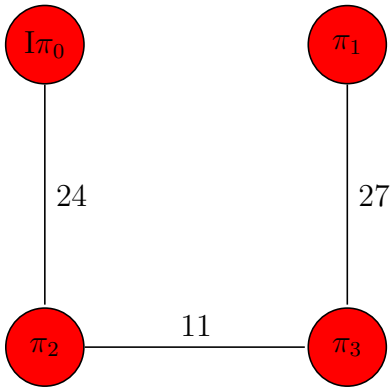


Figure 8: Nearest Neighbour Path

For the distances, we use the so called *Manhattan distance*, i.e. $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ because our robot can only move horizontally and vertically, not diagonally. Given the weights between the vertices, we easily see that the path that our algorithm, and the nearest neighbour, will take is shown in figure 8. The cost of this path is 62.

This is not the optimal path though, which is shown in figure 9 and has a cost of 59.

Because we have to make sure that the dummy node does not change the order that our algorithm and the nearest neighbour algorithm takes we have to set the distance the dummy node is away from every other node to be $max_{i,j}c_{ij}$ where c_{ij} is the distance between the nodes in the complete subgraph in figure 7. In our case, this is 35, the path between π_0 and π_3. This insures that the path taken is the same as the accepted neighbour because the dummy node will be the last node to be visited. This is because in the

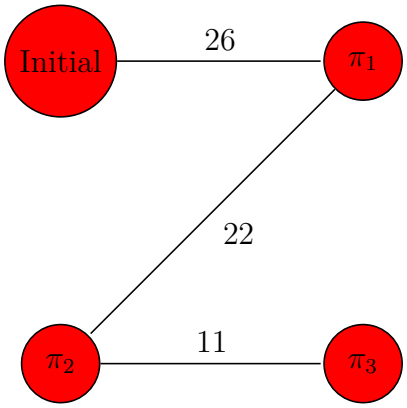


Figure 9: Optimal Path

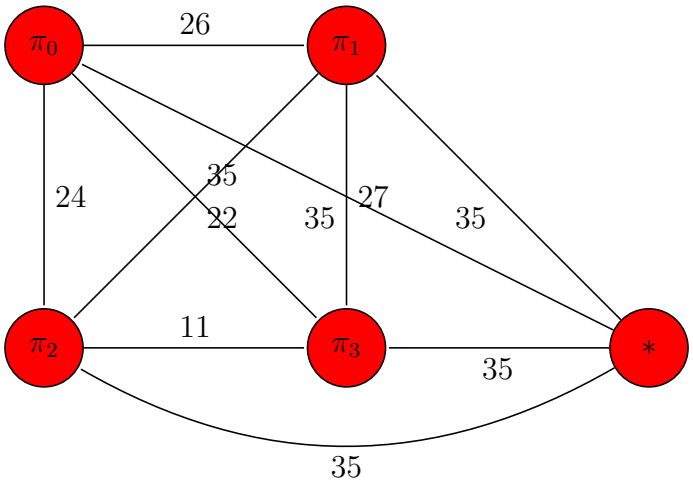


Figure 10: Complete Subgraph with Dummy Node

nearest neighbour algorithm, ties are broken arbitrarily. Thus, the only time where it is a possibility that the nearest neighbour algorithm goes to the dummy node i.e. when the next nodes are $\max_{i,j} c_{ij}$ from the current node is when and if we are faced with the only choice being take the maximum path $\max_{i,j} c_{i,j}$ to π_j or to go to the dummy node, and we can choose to go to π_j because the ties can be broken arbitrarily. In any other case, the nearest neighbour path will choose a to go to a node where the cost is $c_{i',j'} < c_{i,j}$.

We show the new subgraph in figure 10

The path that the nearest neighbour algorithm takes in this situation, the complete Hamiltonian circuit, is given in figure 11, which gives a total cost of 132.

We node however that this is not the optimal solution. This optimal

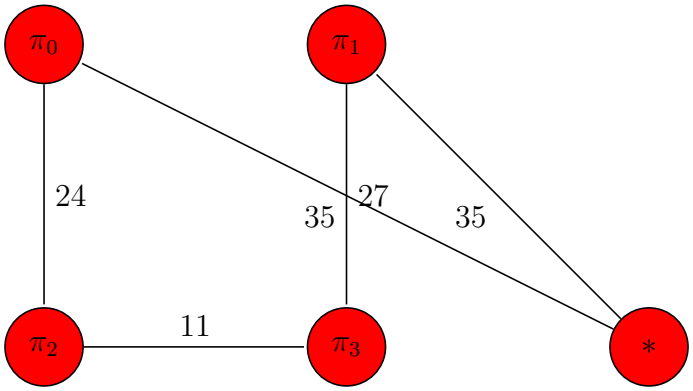


Figure 11: Nearest Neighbour Path with Dummy Node

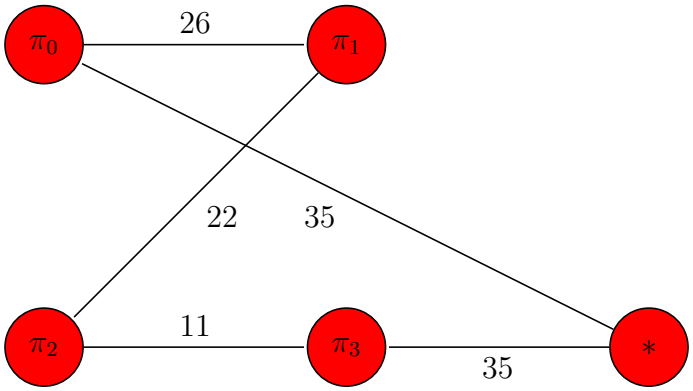


Figure 12: Optimal Path with Dummy Node

solution is shown in figure 12 and has a cost of 129.

It has been shown that for an n-node travelling salesperson problem which satisfies the triangle inequality i.e. $d(i,j) = d(j,k) \geq d(i,k)$ for all i, j, and k where $d(i,j)$ is the nonnegative distance between nodes i and j,

$$\text{NEARNEIBR} \leq (\frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2})\text{OPTIMAL}$$

where NEARNEIBR is the cost of the path generated by the nearest neighbour algorithm and OPTIMAL is the cost of the optimal path.

Our values do indeed satisfy this inequality

$$\begin{aligned} \text{NEARNEIBR} &\leq (\frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2})\text{OPTIMAL} \\ 132 &\leq (\frac{1}{2}\lceil \log(5) \rceil + \frac{1}{2})129 \\ 132 &\leq (2)129 \\ 132 &\leq 258 \end{aligned}$$

We also see that it is very conservative worst case bound and we will likely do much better.

We provide a proof of

$$\frac{\text{NEARNEIBR}}{\text{OPTIMAL}} \leq \frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2} \tag{1}$$

which can be found in [14]. Proof: We begin by proving

$$\text{OPTIMAL} \geq 2 \sum_{i=k+1}^{\min(2k,n)} l_i \tag{2}$$

for all k, $0 \leq k \leq n$. Let l_i be the length of the i^{th} largest edge in the tour obtained by the nearest neighbour algorithm. For each i, $0 \leq i \leq n$, let a_i be the node *onto which* the i^{th} largest edge is added to (that would be the edge with length l_i). Let H be the complete subgraph defined on the set of nodes $\{a_i | 1 \leq i \leq min(2k, n)\}$.

Now, let T be the tour in H which visits the nodes of H in the same order as these nodes are visited in an optimal tour of the original graph. Let LENGTH be the length of T. We have

$$\text{OPTIMAL} \geq \text{LENGTH} \tag{3}$$

This is because the tour with cost OPTIMAL passes through all the nodes that the tour with cost LENGTH passes through, and more. Thus if H has an edge (b, c), then the OPTIMAL tour will either have the edge (b, c) or take a less direct route through some of its extra nodes. So the triangle inequality implies (3).

Let (a_i, a_j) be an edge of T. If the nearest neighbour method adds point a_i before a_j, we have $d(a_i, a_j) \geq l_i$, where $d(a_i, a_j)$ is the distance between nodes a_i and a_j. We also see that if a_j is added first we have $d(a_i, a_j) \geq l_j$. This is because, say we added a_i first, we know there is a point l_i away from a_i that the nearest neighbour method makes the path to. This can be a_j, because we know a_j has not been added yet or another node. If it is another

node $d(a_i, a_j) \geq l_i$ because the nearest neighbour finds the closest node that has not yet been visited, or $d(a_i, a_j) = l_i$ if a_j is added next.

Since one has to be added before the other, we have

$$d(a_i, a_j) \geq \min(l_i, l_j) \tag{4}$$

Summing (4) over the edges of T, we get

$$\text{LENGTH} \geq \sum_{(a_i,a_j) \text{ in } T} \min(l_i, l_j) \tag{5}$$

If we let α_i be the number of edges (a_i, a_j) in T for which l_i is selected as $\min(l_i, l_j)$ we obtain

$$\sum_{(a_i,a_j) \text{ in } T} \min(l_i, l_j) = \sum_{a_i \text{ in } H} a_i l_i \tag{6}$$

Because a_i is the endpoint of two edges in T, $\alpha \leq 2$.

Because T has $\min(2k, n)$ edges (one for each node),

$$\sum_{a_i \text{ in } H} \alpha_i = \min(2k, n) \tag{7}$$

To get a lower bound on (6) we assume that $\alpha_i = 2$ for $k + 1 \leq i \leq \min(2k, n)$ and is zero of $i \leq k$. Thus,

$$\sum_{a_i \text{ in } H} a_i l_i \geq 2 \sum_{i=k+1}^{\min(2k,n)} l_i \tag{8}$$

Combining (3), (5), (6), and (8), we get

$$\text{OPTIMAL} \geq \text{LENGTH} \geq \sum_{(a_i,a_j) \text{ in } T} \min(l_i, l_j) = \sum_{a_i \text{ in } H} a_i l_i \geq 2 \sum_{i=k+1}^{\min(2k,n)} l_i$$

thus proving (2).

We now sum (2) for all values of k for all values of k equal to powers of two less than or equal to n i.e. $k = 2^j \leq n$ for $j = 0, 1, \ldots \lceil \log(n) \rceil - 1$. We then get

$$\sum_{j=0}^{\lceil \log(n) \rceil - 1} \text{OPTIMAL} \geq \sum_{j=0}^{\lceil \log(n) \rceil - 1} (2 \cdot \sum_{i=2^j+1}^{\min(2^{j+1},n)} l_i)$$

We have

$$\sum_{j=0}^{\lceil \log(n) \rceil - 1} \text{OPTIMAL} \geq 2 \cdot \sum_{i=2}^{2} l_i + 2 \cdot \sum_{i=3}^{4} l_i + 2 \cdot \sum_{i=5}^{8} + \sum_{j=3}^{\lceil log(n) \rceil - 1} (2 \cdot \sum_{i=2^j+1}^{\min(2^{j+1},n)} l_i)$$

$$\geq 2l_2 + 2l_3 + 2l_4 \cdots + 2l_8 + \sum_{j=3}^{\lceil log(n) \rceil - 1} (2 \cdot \sum_{i=2^j+1}^{\min(2^{j+1},n)} l_i)$$

Therefore we can write

$$\lceil \log(n) \rceil \cdot \text{OPTIMAL} \geq 2 \sum_{i=2}^{n} l_i \tag{9}$$

Now OPTIMAL must be longer than twice any edge in the graph because it contains two paths between any given pair of points and these paths are , by the triangle inequality, longer than the distance of the edge connecting the points directly, i.e. OPTIMAL $\geq 2l_i$ for $i = 1, 2, \ldots, n$. Specifically,

$$\text{OPTIMAL} \geq 2ł_1 \tag{10}$$

Summing (9) and (10) we get

$$(\log(n) + 1) \cdot \text{OPTIMAL} \geq 2 \sum_{i=1}^{n} l_i$$

By definition, $\sum_{i=1}^{n} l_i = \text{NEARNEIBR}$, thus we have

$$\text{NEARNEIBR} \leq (\frac{1}{2} \lceil \log(n) \rceil + \frac{1}{2}) \text{OPTIMAL}$$

□

We have thus shown that when formulating and solving our problem as a travelling salesman problem with a dummy node, we get the same solution as the nearest neighbour search algorithm. This search algorithm then has a bound on the ratio of the resulting path to the optimal path i.e.

$$\frac{\text{NEARNEIBR}}{\text{OPTIMAL}} \leq (\frac{1}{2} \lceil \log(n) \rceil + \frac{1}{2})$$

We now must remove the dummy node and provide a bound for the true cost that we will get from our search.

NEARNEIBR and OPTIMAL as above are costs of Hamaltonian circuits. By definition every node in a Hamaltonian circuit is passed through exactly once. Therefore the dummy node will be passed through exactly once, and we have shown that it will be the last node passed through in the NEARNEIBR. In the NEARNEIBR path, because the dummy node is length $\max_{i,j} c_{i,j}$ it will never be the closest next node, unless we are given the choice to go from π_i to π_j for i and j being the maximum edge cost in the complete subgraph. In this case we can break the tie arbitrarily and choose to go to π_j instead of the dummy node. Thus the path found by the nearest neighbour search will be the path found by our our algorithm, and then going to the dummy node for a cost of $\max_{i,j} c_{i,j}$, then from there going to the initial node to for a cost of $\max_{i,j} c_{i,j}$. Therefore the cost of our algorithm, denote ALGOR is

$$\text{ALGOR} = \text{NEARNEIBR - } 2 \max_{i,j} c_{i,j}$$

The path OPTIMAL, however is not guaranteed to have the dummy node be the last node visited. The cost of the path which is optimal and requires that the dummy node is the last node visited, is then greater than or equal to OPTIMAL. This is because of the freedom taken away by requiring the dummy node to be visited last, and less freedom in a minimization problem results in a larger value. Let ACCEPT be the cost of the accepted algorithm for path planning. ACCEPT $+2 \max_{i,j} c_{i,j}$ is then equal to the cost of the optimal travelling salesman solution which requires that the dummy node is the last node visited. This is because we have already established that the accepted algorithm will find the optimal path. Therefore we have

$$\text{ACCEPT} + 2 \max_{i,j} c_{i,j} \geq \text{OPTIMAL}$$

Plugging into the travelling salesman bound, we get

$$\begin{aligned}
\text{NEARNEIBR} &\leq (\frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2})\text{OPTIMAL} \\
\text{ALG} + 2 \max_{i,j} c_{i,j} &\leq (\frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2})\text{OPTIMAL} \\
\text{ALG} + 2 \max_{i,j} c_{i,j} &\leq (\frac{1}{2}\lceil \log(n) \rceil + \frac{1}{2})(\text{ACCEPT} + 2 \max_{i,j} c_{i,j})
\end{aligned}$$

We can check with our previously calculated values for ALG and AC-

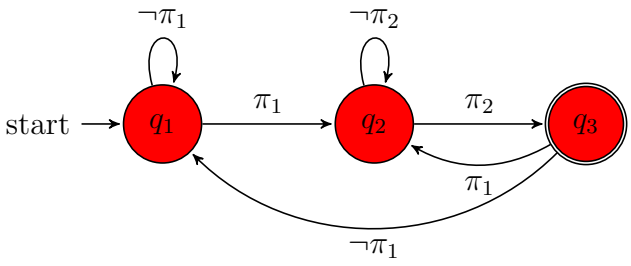


Figure 13: Büchi Automaton for $\square(\diamond\pi_1 \wedge \diamond\pi_2 \wedge \diamond\pi_3)$ 1

CEPT

$$\begin{aligned} \text{ALG} + 2\max_{i,j} c_{i,j} &\leq (\frac{1}{2}\lceil\log(n)\rceil + \frac{1}{2})(\text{ACCEPT} + 2\max_{i,j} c_{i,j} \\ 62 + 2(35) &\leq (\frac{1}{2}3 + \frac{1}{2})(59 + 2(35)) \\ 132 &\leq 258 \end{aligned}$$

We can see that this is still a conservative bound, and emphasize that it is the worst case. Usually the algorithm will preform much better.

0.3.4 Recurrence (Liveness)

Recurrence is coverage over and over again, and can be expressed as $\square(\diamond\pi_1 \wedge \diamond\pi_2 \wedge \cdots \wedge \diamond\pi_n)$. This example is interesting for two reasons: it is prone to Büchi automata that are not tight, and it an accepting path for it does cannot stay in one state (in contrast to the other formulas, in which all accepting states have self loops). We first look at the tightness.

To illustrate our point, we consider the formula $\square(\diamond\pi_1 \wedge \diamond\pi_2 \wedge \diamond\pi_3)$. The Büchi automaton corresponding to this formula, as calculated by [8] is

Note: The actual automaton generated has much more edges. For example, there is an edge from q_4 to q_2 which is labelled $\pi_1\&\&\pi_2$. It is impossible for us to make this transition because π_i for all i is a region in our partition. This is because the requirements of our partition are chosen specially to guarantee that we are never in two regions at once. Thus they are excluded in the interest of the reader. In this automation, $d(q_1) = 2$, $d(q_2) = 1$, and $d(q_3) = 0$. So, to get from $q'_{init} = \langle\pi_2, q_1\rangle \in Q'_0$, we have to first get down to level 2. Given the Büchi automaton 13 the only way to do this is to go to region π_1. Our algorithm does this, and then starts a new Dijkstra search. In this case the same statement holds for π_2. Therefore the optimal prefix is to concatenate the optimal paths down from each each level (first to π_1,

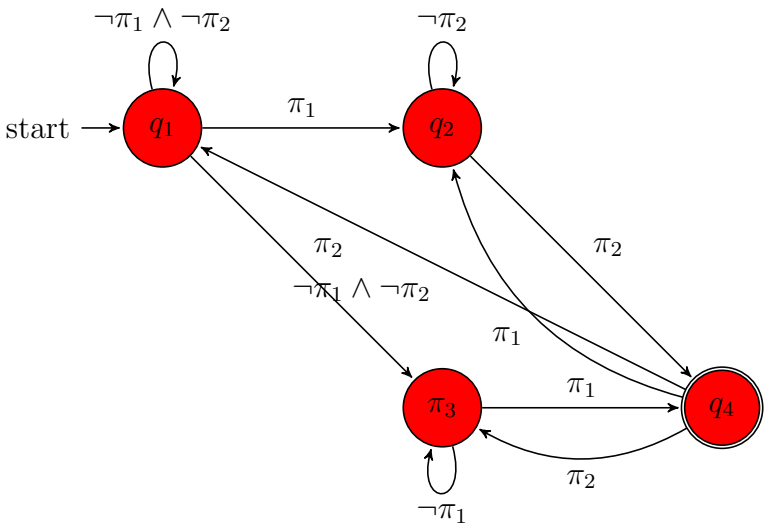


Figure 14: Büchi Automaton for $\square(\diamond\pi_1 \wedge \diamond\pi_2 \wedge \diamond\pi_3)$ 2

etc). Our algorithm does a Dijkstra search at each level so it will return this path as the prefix. The accepted algorithm will also return this prefix.

This path however is in general not truly optimal. It is because the Büchi automaton given in figure 13 not a tight Büchi automaton [16]. A Büchi automaton is tight if it accepts the shortest lasso (prefix and suffix). The loss of this optimality property is due to the fact that the algorithm in [8] simplifies the Büchi automaton which is usually a good thing because it leads to a lower computational complexity in most applications. We take a look at a different automaton corresponding to the formula $\square(\diamond\pi_1 \wedge \diamond\pi_2)$, shown in figure 14.

In this automaton, $d(q_1) = 2$, $d(q_2) = d(q_3) = 1$, and $d(q_4) = 0$. So, we are starting at the same level i.e. 2, however this time we have two choices about what to do to get down to level 2; we can go to π_1 or π_2. Being able to choose is good in the sense that we can now find the optimal path, and bad in the sense that the extra state in the *Büchi* automaton increased the size of the product automaton by 33% (hence increasing the time it takes to search the automaton). This very well illustrates the trade off between the search time and optimally/cost of the resulting run. We propose that this is a good way to think about our algorithm. There is a trade off that sometimes it will not find the optimal run, even if this is possible, though it will be faster.

The second aspect of this problem that we wish to look at is fact that it does not have a trivial suffix. In the other examples we have looked at, the

suffix of the calculated path (with our algorithm and the accepted algorithm) was a single state; that is, the formula could be satisfied by staying in one state indefinitely. In this example, π_1, π_2, and π_3 must all be visited infinitely often, and thus these states must be in the suffix.

The applicability of our algorithm to find the suffix has to be considered. For the total run, R, to be accepting, $\texttt{Inf}(R) \cap \mathcal{F}$ must not be empty. We are specifically looking for runs of the form

$$R = \langle R_{pre}, R_{suf} \rangle = q_0 q_1 \dots q_f [q_f q_{f+1} \dots q_n]^\omega$$

where $q_f \in \mathcal{F}$. Thus when calculating to the suffix we must find the path back to the *same* accepting state. We cannot not just look for any accepting state as we do in the prefix calculation. Our algorithm in general only looks for an accepting state, not a specific accepting state; however in certain circumstances it can find a specific accepting state. We illustrate this using the same examples above.

$\square(\diamond\pi_3 \wedge \pi_5)$ We notice how in figure 13 there is only one arrow to the accepting state, labelled π_5. This implies that the only way to get down to level 0 is to go to π_5, and thus go to the accepting state $\langle \pi_5, q_3 \rangle$. There is no self loop on q_3, so we leave q_3 immediately. This implies that the only reachable accepting state is $\langle \pi_5, q_3 \rangle$. So because there is only one accepting state, our algorithm will find this state again, and thus is appropriate for finding the suffix.

In 14 on the other hand, there are two arrows going to the accepting state and there is no self loop. This implies that there are two reachable accepting states i.e. $\langle \pi_3, q_4 \rangle$ and $\langle \pi_5, q_4 \rangle$. This poses a problem to our algorithm that is only guaranteed to reach an accepting state. We thus propose using Dijkstra's search algorithm to find the path from the accepting node back to itself.

0.4 More Complex Formulas

The formulas in the previous section are common formulas, but are fairly simple. The benefit of using temporal logics is that a wide variety of behaviours can be expressed, including propositions about the robot *and* about the workspace. Up to now, we have not looked at any formulas that include atomic propositions about potential tasks. We will show through examples that the same ideas presented in the previous chapter still hold true for this complex tasks, and show the speed up we get by using our algorithm compared to the accepted algorithm.

0.4.1 Example 1

We look at the example from [9] which says "eventually pick up the red ball. Once it is done, move to one basket and drop it. At last come back to room one and stay there". This task can be written as the LTL formula $\varphi = \diamond(\text{rball} \wedge \diamond\text{basket}) \wedge \diamond\Box r1$. The Büchi automaton corresponding to this formula as translated by [8] is shown in figure 0.4.1

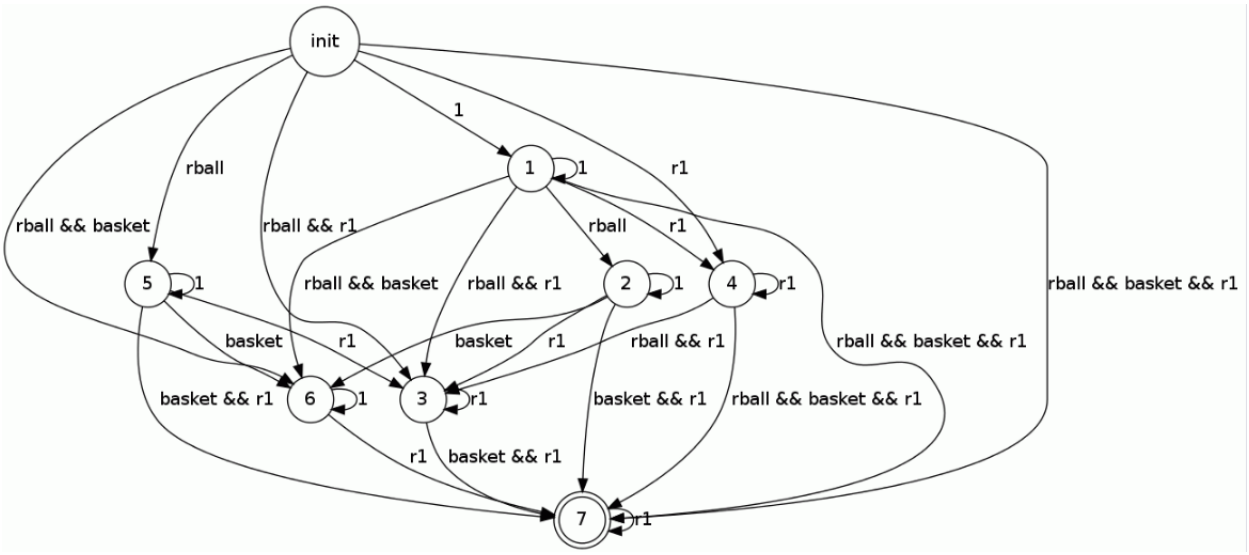


Figure 15: Büchi Automaton Corresponding to $\varphi = \diamond(\text{rball} \wedge \diamond\text{basket}) \wedge \diamond\Box r1$

As we can see, there are many edges in this automaton and edges that have && in the label. These paths can only be taken if we satisfy both of the propositions at the same time. However, because in our example the propositions do not overlap (the ball is not in the same room as the

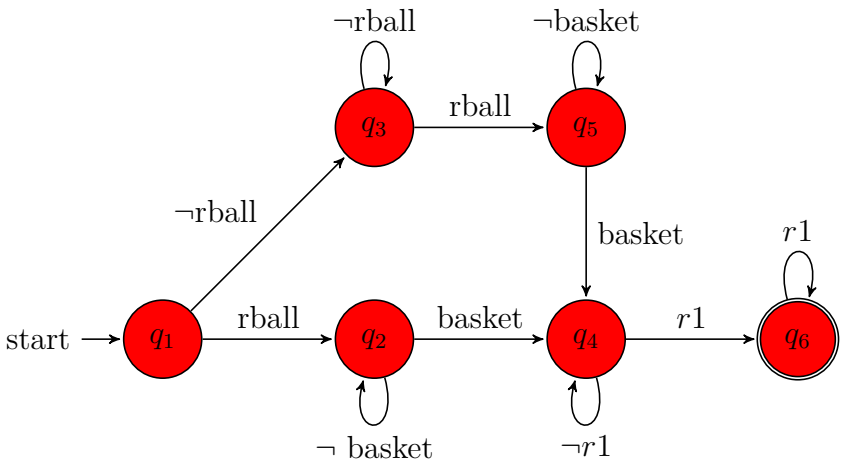


Figure 16: Simplified Büchi Automaton for $\varphi = \diamond(\text{rball} \wedge \diamond\text{basket}) \wedge \diamond\Box r1$ 1

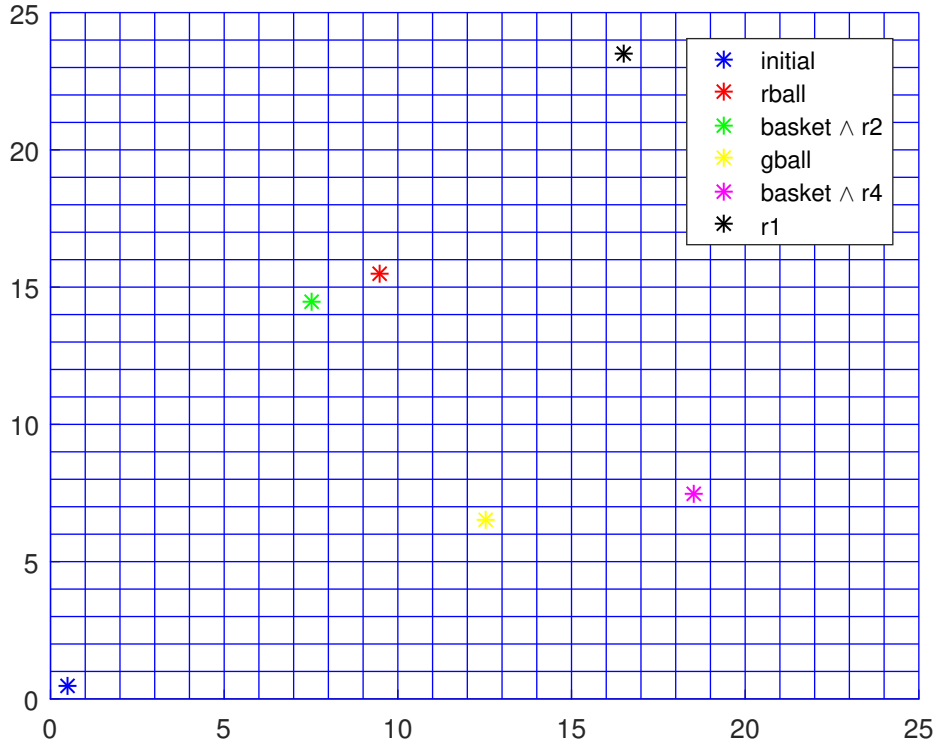
basket, and the neither the ball or basket is located in room 1) these edge are impossible to take. Therefore we remove these edges from the automaton (show the code for this). We then have a much simpler automaton that is shown in figure 16

In this automaton, we can see that $d(q_1) = 3$, $d(q_2) = 2$, $d(q_3) = 3$, $d(q_4) = 1$, $d(q_5) = 2$, $d(q_6) = 0$. For the first time, we have a node that connects to the initial node which is on the same level as the initial node. Examining our algorithm, we see that we will not start a new Dijkstra search until we find a node which is a level bellow our current level. Therefore we will not start a new search until we find a node in the product automaton with projection onto q_2 or q_5.

We can also see that from the illustration of the workspace, that the ball (rball) is not located next to the initial node, so the first proposition must be ¬rball. Examining the automaton in figure 16 that then we are guaranteed to take a path through nodes with projection q_3 and that we will never go to a node with the projection of q_2. Therefore we are in the same situation as for sequencing i.e. there is only one sequence of actions that will satisfy the formula, implying that our algorithm will find the same path as the accepted algorithm, just much faster.

0.4.2 Example 1 Overlapping Regions

We now look at the same example, except now we have a different workspace. We choose this example to show what happens if the regions of interest are overlapping. We show multiple scenarios. First, if rball is with the basket.



If this is the case, then rball and basket can be satisfied simultaneously. Therefore we have to admit paths with rball && basket into the automaton. The automaton is now

In this automaton, we now have $d(q_1) = 2$, $d(q_2) = 2$, $d(q_3) = 2$, $d(q_4) = 1$, $d(q_5) = 2$ and $d(q_6) = 0$. We see again that rball and basket are not one step away from the initial node, implying that we cannot take the first step rball or rball && basket. This means that we cannot ever go to q_2.

0.4.3 Example 2

We now look at the example taken from [9] in which the robot has to pick up and deliver two different balls (rball and gball) to two different baskets, and the robot cannot carry two balls at once. After this is done the robot is to go to r1 and stay there. This task is formalized as $\varphi = \diamond(\text{rball} \wedge \diamond(\text{basket} \wedge \text{r2})) \wedge \diamond(\text{gball} \wedge \diamond(\text{basket} \wedge \text{r4})) \wedge \Box(\text{rball} \rightarrow \mathbf{X}(\neg\text{gball}\boldsymbol{\mathcal{U}}\text{basket})) \wedge \Box(\text{gball} \rightarrow \mathbf{X}(\neg\text{rball}\boldsymbol{\mathcal{U}}\text{basket}))\&\&\diamond\Box r1$. This formula formalizes the basket

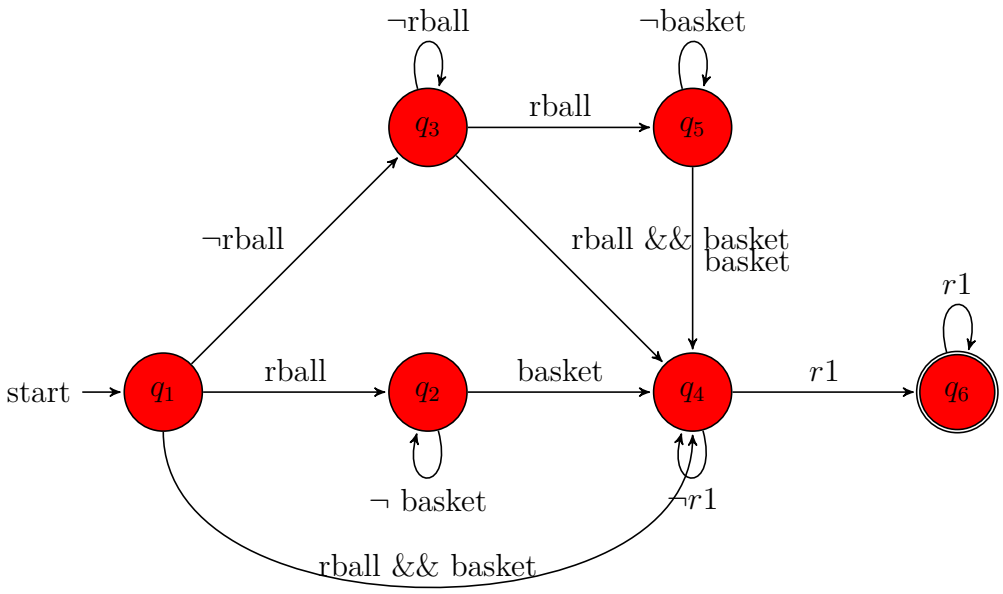


Figure 17: Simplified Büchi Automaton Corresponding to $\varphi = \diamond(\text{rball} \wedge \diamond\text{basket}) \wedge \diamond\Box r1$ 2

corresponding to rball is in region r2 and the basket corresponding to gball is in r4. The Büchi automaton corresponding to this formula is much to large to show. It has 56 states and 673 edges. If the reader is interested, the automaton can be found using the online tool [1] with the input F(rball && F(basket && r2)) && F(gball && F(basket && r4)) && G(rball -> X(! gball U basket)) && G(gball -> X(! rball U basket)) && F(G(r1)).

To analyse the performance of our algorithm on this problem, we are going to break up this problem into the choices that the robot has. The robot has to pick up one of the balls, return it to the corresponding basket, then pick up the second ball and return it to its corresponding basket. Assuming that everything else is done in the optimal way, the only choice that must be made is which ball to pick up first. Our algorithm chooses the ball that is closest to us.

0.4.4 OR Operator

As we have seen in the LTL semantics, LTL formulas can contain an OR Boolean connective i.e. $\varphi = \varphi_1 \vee \varphi_2$. In all the other examples that we have seen, the formulas specify tasks and the algorithm has to *at most* choose the order of the tasks. The OR connective introduces the idea that the

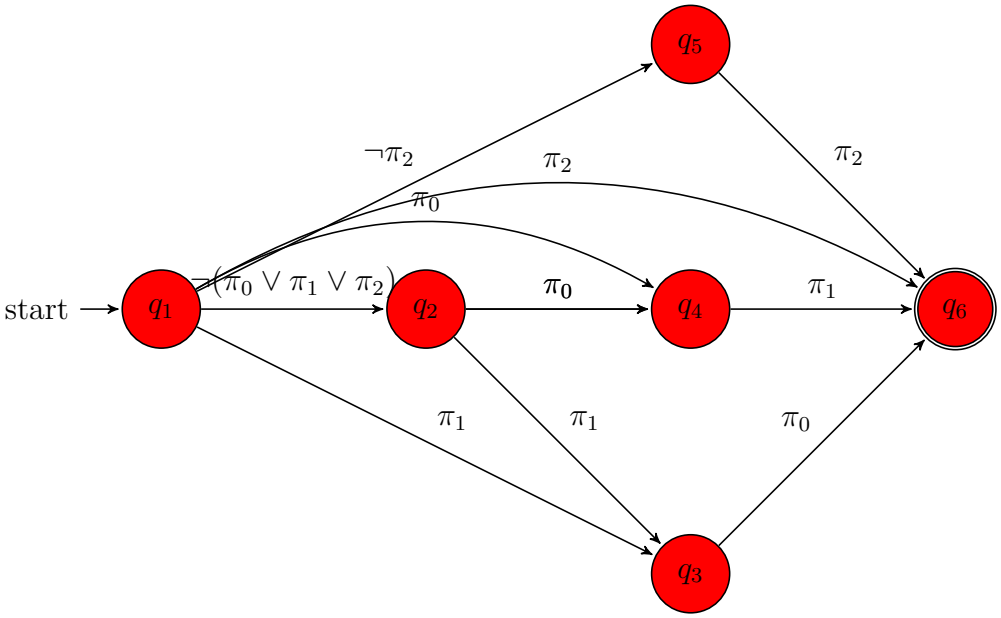


Figure 18: Simplified Büchi Automaton Corresponding to $(\diamond\pi_0 \wedge \diamond\pi_1) \vee \diamond\pi_2$

algorithm has to choose *which* tasks to do. Let us first look at the formula $(\diamond\pi_0 \wedge \diamond\pi_1) \vee \diamond\pi_2$. The Büchi automaton corresponding to this formula as calculated by [8] is shown in figure 18.

The distances corresponding to this automaton are $d(q_1) = 1$, $d(q_2) = 2$, $d(q_3) = 1$, $d(d_4) = 1$, $d(q_5) = 1$ and $d(q_6) = 0$. As one can see, starting with a distance of 1, the only lower level is 0, which is the accepting level. Therefore our algorithm will only do one Dijkstra search, which is the same as the accepted algorithm. Our algorithm therefore gives the optimal result.

0.5 Appendix

```
from collections import deque
from heapq import heappush, heappop
from itertools import count
import networkx as nx
from networkx.utils import generate_unique_node
import warnings as _warnings

def adapted_dijkstra_multisource(G, source, cutoff=None, target=None):
    """Uses Dijkstra's algorithm to find shortest weighted paths
    Parameters
    ----------
    G : NetworkX graph
    sources : non-empty iterable of nodes
        Starting nodes for paths. If this is just an iterable containi
        a single node, then all paths computed by this function will
        start from that node. If there are two or more nodes in this
        iterable, the computed paths may begin from any one of the sta
        nodes.
    target : node label, optional
        Ending node for path. Search is halted when target is found.
    cutoff : integer or float, optional
        Depth to stop the search. Only return paths with length <= cut
    Returns
    -------
    dist : dictionary
        A mapping from node to shortest distance to that node from one
        of the source nodes.
    next_node : tuple
        The first node, n, the search finds that is one level below th
        i.e. d_p(n) = lev - 1
    paths: dictionary
        dict to store the path list from source to each node, keyed by
    Notes
    -----
    The optional predecessor and path dictionaries can be accessed by
    the caller through the original pred and paths objects passed
    as arguments. No need to explicitly return pred or paths.
    """
```

```
paths = {source: [source]}

# define weight function
weight = lambda u, v, data: data.get('weight', 1)

# succ = successors
G_succ = G.succ if G.is_directed() else G.adj

# rename functions
push = heappush
pop = heappop

dist = {}  # dictionary of final distances
seen = {}
# fringe is heapq with 3-tuples (distance,c,node)
# use the count c to avoid comparing nodes (may not be able to)
c = count()
fringe = []

# current level of starting node
cur_level = G.node[source]['dist']

#for source in sources:
seen[source] = 0
push(fringe, (0, next(c), source))
while fringe:
    (d, _, v) = pop(fringe)
    if v in dist:
        continue  # already searched this node.
    dist[v] = d
    if G.node[v]['dist'] < cur_level:
        next_node = v
        break
    for u, e in G_succ[v].items():
        cost = weight(v, u, e)
        if cost is None:
            continue
        vu_dist = dist[v] + cost
        if cutoff is not None:
            if vu_dist > cutoff:
```

```
                continue
            if u in dist:
                if vu_dist < dist[u]:
                    raise ValueError('Contradictory␣paths␣found:',
                                     'negative␣weights?')
            elif u not in seen or vu_dist < seen[u]:
                seen[u] = vu_dist
                push(fringe, (vu_dist, next(c), u))
                if paths is not None:
                    paths[u] = paths[v] + [u]
    print next_node
    print type(next_node)
    return dist, next_node, paths
```

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