SF2521: Homework Assignment 3

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1 Well–posedness and von Neumann analysis

In this section, we consider the 2π-periodic Cauchy problems

$$\begin{aligned} u_t &= \alpha u_{xx} + \beta u_{xxxx} \\ u(x,0) &= sin(x) \end{aligned}$$

1.1 Well-posed for $\alpha > 0$ and $\beta = 0$

We now show that the problem is well-posed in the L_2 -norm for for $\alpha > 0$ and $\beta = 0$.

$$\begin{aligned} u_t &= \alpha u_{xx} \\ u(x,0) &= sin(x) \end{aligned}$$

It is seen immediately that we now have the one dimensional heat equation. The solution to this problem is well documented and, given our initial condition, we have $u(x,t) = sin(x)e^{-at}$

1.2 Ill-posed for $\beta > 0$

1.3 Stability for $\alpha > 0$ and $\beta = 0$

We wish to derive a condition on Δt which guarantees stability in the max norm for a scheme using central difference in space and forward Euler in time. Our scheme is thus

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \\ \implies u_j^{n+1} &= u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \\ &= \frac{\alpha \Delta t}{\Delta x^2}(u_{j+1}^n + u_{j-1}^n) + (1 - 2\frac{\alpha \Delta t}{\Delta x^2})u_j^n \end{aligned}$$

We see that for $\frac{\alpha \Delta t}{\Delta x^2} \in [0, \frac{1}{2}]$ that u_j^{n+1} is a weighted average of u_{j-1}^n, u_j^n and u_{j+1}^n ($\frac{\alpha \Delta t}{\Delta x^2}$ weight for u_{j+1}^n and u_{j-1}^n and $1 - 2\frac{\alpha \Delta t}{\Delta x^2}$ for u_j^n). This implies that

$$\begin{aligned} u_j^{n+1} &\leq \max(u_{j-1}^n, u_j^n, u_{j+1}^n) \qquad \forall j \qquad (1) \\ \implies \max_j(u_j^{n+1}) &\leq \max_j(u_j^{n+1}) \end{aligned}$$

We quickly note that if $\frac{\alpha \Delta t}{\Delta x^2} > \frac{1}{2}$, then it is possible to pick $u_{j-1}^n, u_j^n, u_{j+1}^n$ such that(1) is not fulfilled. Therefore are stability criterion is

$$\begin{aligned} \frac{\alpha \Delta t}{\Delta x^2} &\leq \frac{1}{2} \\ \Delta t &\leq \frac{\Delta x^2}{2\alpha} \end{aligned}$$