7.1 SINGULAR VALUE AND SINGULAR VECTOR

Given a symmetric matrix S, there exist an orthogonal matrix Q and a diagonal matrix Λ such that $S = Q \Lambda Q^T$

Q: Orthogonal matrix
Λ: Diagonal matrix
Q^T: Orthogonal matrix

We want to extend this factorization to all matrices $A_{p \times q}$. To make this possible for every $A_{p \times q}$, we need one set of q orthonormal vectors $v_1, v_2, \ldots, v_q$ in $\mathbb{R}^q$ and a second orthonormal set $u_1, \ldots, u_p$ in $\mathbb{R}^p$. Instead of $Sx = \lambda x$, we want $Av = \sigma u$.

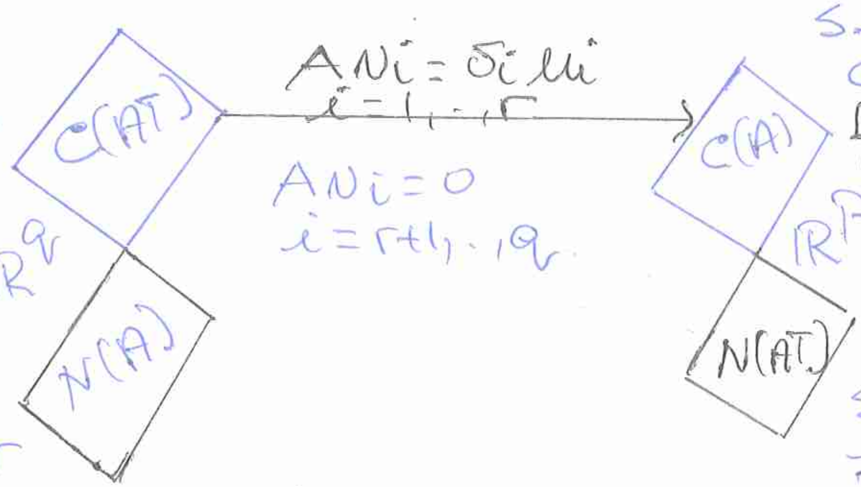
The u^s and v^s are called the singular vectors of $A_{p \times q}$ and they give bases for the four fundamental subspaces for $A_{p \times q}$.
σ is called singular value of $A_{p \times q}$.

Let $r = \text{rank}(A)$

Four Fundamental subspaces

Singular vector basis for $R(A)$: $\underbrace{\{v_1, \ldots, v_r\}}_{\beta_1}$

S.v. basis for $C(A)$: $\underbrace{\{u_1, \ldots, u_r\}}_{\beta_3}$



Singular vector basis for $N(A)$ $\underbrace{\{v_{r+1}, \ldots, v_q\}}_{\beta_2}$

S.v. basis for $N(A^T)$ $\underbrace{\{u_{r+1}, \ldots, u_p\}}_{\beta_4}$

$\beta_1 \cup \beta_2$ = basis for $\mathbb{R}^q$

$(v_i)_{i=1}^{q}$ is an orthonormal set

$\beta_3 \cup \beta_4$ = basis for $\mathbb{R}^p$

$(u_i)_{i=1}^{p}$ is an orthonormal set.

From $A v_i = \sigma_i u_i \quad i = 1, \ldots, r$

Let $U_r = [u_1\ u_2 \ldots u_r]_{p \times r}$ and $V_r = [v_1\ v_2 \ldots v_r]_{q \times r}$

* since the u's and v's are orthonormal vectors:

(a) $U_r^T U_r = I$ but $U_r U_r^T \neq I$ unless U_r is square

(b) $V_r^T V_r = I$ but $V_r V_r^T \neq I$ unless V_r is square

The equations $A v_i = \sigma_i u_i$, $i = 1, \dots, r$ tell us by column that:

$$\boxed{A V_r = U_r \Sigma_r}$$

Reduced form of the SVD (Nullspaces excluded)

Now include all v's in V and all u's in U to have: (*) $AV = U\Sigma$ where:

V: orthogonal matrix ($V^{-1} = V^T$)

U: orthogonal matrix ($U^{-1} = U^T$)

(*) becomes $\boxed{A = U \Sigma V^T}$ Full SVD with nullspaces included

U: Left singular vector matrix

V: Right singular vector matrix

<u>SVD of $A_{p \times q}$ with rank r</u>

$$A_{p \times q} = U_{p \times p} \Sigma_{p \times q} V^T_{q \times q} = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$A_{p \times q}$ = sum of rank one matrices (sum of column vectors times row vectors)

Ideas of Proof of the SVD

$A = U\Sigma V^T$ $\quad A_{p\times q}$, $U_{p\times p}$, $\Sigma_{p\times q}$, $V_{q\times q}$

* the v's are the orthonormal eigenvectors of A^TA

$$A^TA = V\Sigma^T U^T U \Sigma V^T = \underset{q\times q}{V}\underbrace{(\Sigma^T\Sigma)}_{q\times q}\underset{q\times q}{V^T}$$

V: eigenvector matrix for A^TA

$\Sigma^T\Sigma$: eigenvalue matrix for A^TA

each σ^2 is an eigenvalue of A^TA

$Av_i = \sigma_i u_i$, $i = 1,\dots,r$ tell us the unit vectors u_1 to u_r

Ex.: From $Av_i = \sigma_i u_i$ $i = 1,\dots,r$ and $v_i^Tv_j = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases}$, show that: $u_i^T u_j = \begin{cases} 0 & i\neq j \\ 1 & i = j \end{cases}$

ANS

$$u_i^T u_j = \left(\frac{Av_i}{\sigma_i}\right)^T\left(\frac{Av_j}{\sigma_j}\right) = \frac{v_i^T(A^TAv_j)}{\sigma_i\sigma_j} = \sigma_j^2\frac{v_i^Tv_j}{\sigma_i\sigma_j} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$$

b/c the v_j's are the eigenvectors of A^TA with corresponding eigenvalues σ_j^2: $A^TAv_j = \sigma_j^2 v_j$

Ex. Show that the $(u_i$'s$)$ will be eigenvectors of AA^T

ANS

$$AA^Tu_i = AA^T\left(\frac{Av_i}{\sigma_i}\right) = A\left(\frac{A^TAv_i}{\sigma_i}\right) = \frac{A\sigma_i^2 v_i}{\sigma_i} = \sigma_i^2\underbrace{\frac{Av_i}{\sigma_i}}_{u_i}$$

Note: A^TA and AA^T have the same nonzero eigenvalues.

why?

Ans Let $\lambda \neq 0$ be an eigenvalue of A^TA with eigenvector x.

$$A^TAx = \lambda x \Rightarrow$$

Multiply both sides by A

$$AA^TAx = \lambda Ax$$

$$(AA^T)(Ax) = \lambda(Ax)$$

$Ax \neq \vec{0}$ b/c $Ax = \vec{0}$ would imply $A^TAx = \lambda x = 0$ ie $\lambda = 0$ which woud be a contradict°

Since $Ax \neq \vec{0}$, then Ax is an eigenvector of A^TA with eigenvalue $\lambda \neq 0$

Conclus°: A^TA and AA^T have same nonzero eigenvalues.

Ideas of Proof of the SVD

$$A = U\Sigma V^T \Rightarrow A^T = V\Sigma^T U^T \Leftrightarrow A^T U = V\Sigma^T$$

$$A^T u_i = V\begin{bmatrix} 0 \\ \vdots \\ \sigma_i \\ \vdots \\ 0 \end{bmatrix} = \sigma_i v_i$$

$i = 1, \dots, r$

$$AA^T = U\Sigma V^T V \Sigma^T U^T$$
$$= U(\Sigma\Sigma^T)U^T$$

U: matrix of eigenvectors for AA^T.

$\Sigma\Sigma^T$: eigenvalue matrix for AA^T

each σ^2 is an eigenvalue of AA^T

$A^T u_k = \sigma_k v_k$ tell the unit vectors v_1 to v_r.

$k = 1, \dots, r$

$\boxed{\text{Ex}}$: show that $v_i^T v_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$\boxed{\text{Ans}}$

$$v_i^T v_j = \left(\frac{A^T u_i}{\sigma_i}\right)^T \left(\frac{A^T u_j}{\sigma_j}\right) = \frac{u_i^T AA^T u_j}{\sigma_i \sigma_j} = \sigma_j^2 \frac{u_i^T u_j}{\sigma_i \sigma_j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$\boxed{\text{Recall}}$: $AA^T u_j = \sigma_j^2 u_j$

$\boxed{\text{Ex}}$ The v_i's are eigenvectors of A^TA

$\boxed{\text{Ans}}$ $$A^TAv_i = A^TA\left(\frac{A^T u_i}{\sigma_i}\right) = A^T\left(\frac{AA^T u_i}{\sigma_i}\right) = \frac{A^T \sigma_i^2 u_i}{\sigma_i} = \sigma_i^2 \underbrace{\frac{A^T u_i}{\sigma_i}}_{v_i}$$

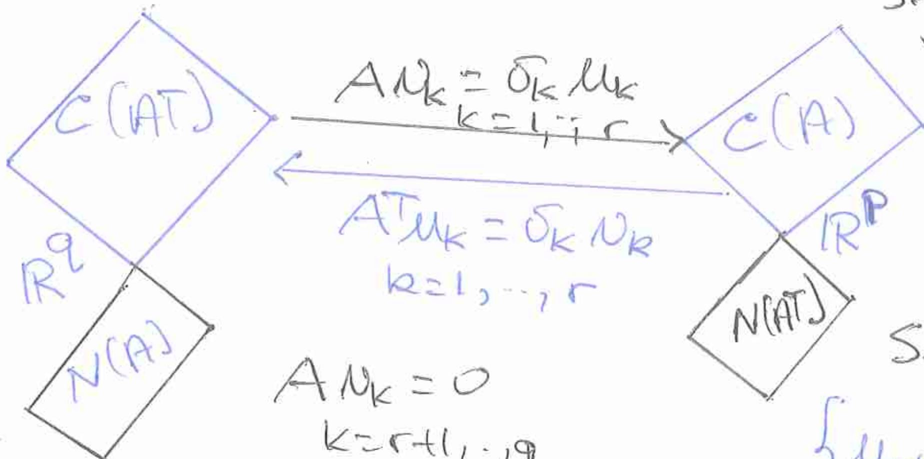
Given $A_{p\times q}$

Four Fundamental subspaces of A

$r = \text{rank}(A)$

S.v. basis for R(A)

$\{v_1, \dots, v_r\}$



S.v. basis for C(A)

$\{u_1, \dots, u_r\}$

S.v. basis

$\{u_{r+1}, \dots, u_p\}$

$Av_k = 0$

$k = r+1, \dots, q$

$A^T u_k = 0$

$k = r+1, \dots, p$

S.v. basis for N(A)

$\{v_{r+1}, \dots, v_q\}$

* From R(A) to C(A)

$$Av_k = \sigma_k u_k \quad k=1,\dots,r$$

$$u_k = \frac{Av_k}{\sigma_k}$$

* From C(A) to R(A)

$$A^T u_k = \sigma_k v_k \quad k=1,\dots,r$$

$$v_k = \frac{A^T u_k}{\sigma_k}$$

The v's and u's diagonalize the matrix $A_{p\times q}$

$$V = [v_1 \dots v_q]_{q\times q} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_r & 0 \\ \underbrace{0}_{p-r} & & \underbrace{}_{q-r} \end{bmatrix}_{p\times q} \qquad U = [u_1 \dots u_p]$$

SVD: $A = U\Sigma V^T$

$$A = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

SVD of A $\quad A_{p\times q}$

Suppose $r = \text{rank}(A)$

A has r singular values $\sigma_1, \sigma_2, \dots, \sigma_r$ such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

The singular values are the square root of the nonzero eigenvalues of the matrix A^TA or AA^T. $\quad \sigma'^s = \sqrt{\lambda(A^TA)}$ or $\sigma'^s = \sqrt{\lambda(AA^T)}$

Note: since A^TA and AA^T have same nonzero eigenvalues, given a matrix $A_{p\times q}$, between A^TA and AA^T, choose the matrix with smaller size to compute the r singular values and their corresponding singular vectors.

Note $\quad N(A^TA) = N(A)$

0-eigenspace of $A^TA = N(A)$

(I) Suppose A^TA has the smaller size.

From $A = U\Sigma V^T$, $A^TA = V(\Sigma^T\Sigma)V^T$

The right singular vectors v's of A are the eigenvectors of A^TA.

Find the eigenpairs $(\lambda_k \neq 0, v_k)$ of A^TA $k = 1$ to r.

The $\sqrt{\lambda_k}$'s are the singular values of A.

The v_k's are the right singular vectors of A.

Sort the singular values $\sqrt{\lambda_k}$'s in descending order and derive the singular values σ_i's $i = 1, r$ such that $\sigma_1 \geq \dots \geq \sigma_r \geq 0$.

* At this point, the pairs (σ_i, v_i) $i = 1$ to r are known.

Derive the pairs (σ_i, u_i) $i = 1$ to r

$$A = U\Sigma V^T \iff AV = U\Sigma$$

$$\begin{cases} \text{(1) Column } i \text{ of } AV = Av_i \\ \text{(2) Column } i \text{ of } U\Sigma = U\begin{bmatrix} 0 \\ \vdots \\ \sigma_i \\ \vdots \\ 0 \end{bmatrix} = \sigma_i u_i \end{cases}$$

$(1) = (2) \iff Av_i = \sigma_i u_i$ $i = 1$ to r ie $u_i = \dfrac{Av_i}{\sigma_i}$ $i = 1$ to r

* Now we know (σ_i, v_i) and (σ_i, u_i) $i = 1$ to r
* Use Gram Schmidt to ~~sele~~ construct an orthonormal basis for $N(A)$ to find the remaining v's $\{v_{r+1}, \dots, v_q\}$
* Repeat the above to construct basis $\{u_{r+1}, \dots, u_p\}$ for $N(A^T)$

II Suppose AA^T has the smaller size.

From $A = U\Sigma V^T$, $AA^T = U(\Sigma\Sigma^T)U^T$

The left singular vectors of A are the eigenvectors of AA^T.

Find the eigenpairs $(d_k \neq 0, u_k)$ of AA^T $k=1$ to r.

the $\sqrt{d_k}$'s are the singular values of A.

the u_k's are the left singular vectors of A.

At this point, the pairs (σ_i, u_i) $i=1$ to r are known.

<u>Derive the pairs (σ_i, v_i) $i=1$ to r</u>

$A = U\Sigma V^T \Leftrightarrow A^T = V\Sigma^T U^T$

$A^T U = V\Sigma^T$

$$\begin{cases} ① \text{ column } k \text{ of } A^T U = A^T u_k \\ ② \text{ column } k \text{ of } V\Sigma^T = V\begin{bmatrix} 0 \\ \vdots \\ \sigma_k \\ \vdots \\ 0 \end{bmatrix} = \sigma_k v_k \end{cases}$$

$① = ② \Leftrightarrow A^T u_k = \sigma_k v_k \Rightarrow \boxed{v_k = \frac{A^T u_k}{\sigma_k} \quad k=1 \text{ to } r}$
$k=1$ to r

For the remaining v's and u's, repeat the process described in (I)

Question: $A_{p\times q}$

Find a nonzero vector x that maximizes the ratio $\frac{\|Ax\|}{\|x\|}$. What is the maximum value?

$\boxed{\text{ANS}}$ The x that maximizes the ratio $\frac{\|Ax\|}{\|x\|}$ is the same x that maximizes the ratio $\frac{\|Ax\|^2}{\|x\|^2}$.

$$\frac{\|Ax\|^2}{\|x\|^2} = \frac{(Ax)^T(Ax)}{x^Tx} = \frac{x^TA^TAx}{x^Tx}$$

Since x is in $\mathbb{R}^q$, let $(v_i)_{i=1}^{q}$ be the singular vector basis for $\mathbb{R}^q$

$$x = c_1v_1 + c_2v_2 + \cdots + c_qv_q$$

$$\frac{x^TA^TAx}{x^Tx} = \frac{(c_1v_1^T + \cdots + c_qv_q^T)A^TA(c_1v_1 + \cdots + c_qv_q)}{(c_1v_1^T + \cdots + c_qv_q^T)(c_1v_1 + \cdots + c_qv_q)}$$

$$= \frac{(c_1v_1^T + \cdots + c_qv_q^T)(c_1A^TAv_1 + \cdots + c_qA^TAv_q)}{(c_1v_1^T + \cdots + c_qv_q^T)(c_1v_1 + \cdots + c_qv_q)}$$

Recall: $A^TA = V(\Sigma^T\Sigma)V^T$ ie $A^TAv_j = \sigma_j^2 v_j$

$$v_i^Tv_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\frac{\|Ax\|^2}{\|x\|^2} = \frac{c_1^2\sigma_1^2 + \cdots + c_q^2\sigma_q^2}{c_1^2 + \cdots + c_q^2} \leq \sigma_1^2\frac{(c_1^2 + \cdots + c_q^2)}{c_1^2 + \cdots + c_q^2} = \sigma_1^2$$

Q: what x makes $\frac{\|Ax\|^2}{\|x\|^2} = \sigma_1^2$?

ANS: For $x = v_1$, $\frac{\|Av_1\|^2}{\|v_1\|^2} = \frac{\|\sigma_1 u_1\|^2}{1} = \sigma_1^2\|u_1\|^2 = \sigma_1^2$

$$\boxed{\text{For } x = v_1 \quad \frac{\|Ax\|}{\|x\|} = \sigma_1}$$

Conclusion:

The x that maximizes the ratio $\frac{\|Ax\|}{\|x\|}$ is $x = v_1 \neq 0$.

The maximum ratio is: $\frac{\|Av_1\|}{\|v_1\|} = \frac{\|\sigma_1 u_1\|}{1} = \sigma_1 \|u_1\| = \sigma_1$

Second method:

$A = U \Sigma V^T$

$\|Ax\| = \|U \underbrace{\Sigma V^T x}_{y}\| = \|y\|$ b/c U is orthogonal

$\|Uy\|^2 = (Uy)^T Uy = y^T \underbrace{U^T U}_{I} y = \|y\|^2$

$\|Uy\| = \|y\|$

$\|Ax\| = \|y\| = \|\Sigma V^T x\|$

$\leq \|\Sigma\| \, \|V^T x\|$ but $\|V^T x\| = \|x\|$ b/c V^T is orthogonal

and $\|\Sigma\|_1 = \max_{1 \leq j \leq q} \sum_{i=1}^{p} |\Sigma_{ij}|$

or $\|\Sigma\|_\infty = \max_{1 \leq i \leq p} \sum_{j=1}^{q} |\Sigma_{ij}|$

$\|Ax\| \leq \sigma_1 \|x\|$ ie

$\frac{\|Ax\|}{\|x\|} \leq \sigma_1$. The maximum value of $\frac{\|Ax\|}{\|x\|}$ is σ_1

σ_1 is reached @ $x = v_1$

b/c $\frac{\|Av_1\|}{\|v_1\|} = \frac{\|\sigma_1 u_1\|}{1} = \sigma_1 \|u_1\| = \sigma_1$ $\square$

$\boxed{Ex.}$ Find the SVD for $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

$\boxed{ANS}$ $\operatorname{rank}(A) = 0$ $R(A) = \{\vec{0}\}$, $C(A) = \{\vec{0}\}$

Orthonormal basis for $N(A)$:

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $s_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $s_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Right singular vectors: $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$V = [v_1 \; v_2 \; v_3]$

Orthonormal basis for $N(A^T)$

$A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $s_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $s_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Left singular vectors: $u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$U = [u_1 \; u_2]$

$$A = U \Sigma V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The decomposition is not unique, any orthogonal matrices U and V work since $\Sigma = 0$ matrix.

Ex. Find the reduced SVD of $A = xy^T$, $x \neq 0, y \neq 0$. Check that $|\lambda_{max}(A)| \leq \sigma_1$

$\boxed{\text{ANS}}$ rank$(A) = 1$, A has one singular value $\sigma_1 > 0$

Find (σ_1, v_1)

$$A^TA = yx^Txy^T = x^Tx\, yy^T = \|x\|^2 yy^T$$

Eigenvectors of A^TA

$z \neq 0$

$$A^TAz = \|x\|^2 yy^Tz = \|x\|^2 y^Tz\, y$$

For $z = y$ $\quad A^TAy = \|x\|^2\|y\|^2 y$

$$A^TA\underbrace{\frac{y}{\|y\|}}_{v_1} = \underbrace{\|x\|^2\|y\|^2}_{\sigma_1^2}\underbrace{\frac{y}{\|y\|}}_{v_1} \qquad A^TAv_1 = \sigma_1^2 v_1$$

From $v_1 = \frac{y}{\|y\|}$ find u_1 : $Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{Av_1}{\sigma_1}$

$$u_1 = \frac{xy^T}{\|x\|\|y\|}\frac{y}{\|y\|} = \frac{y^Ty\, x}{\|x\|\|y\|\,\|y\|}$$

$$u_1 = \frac{\|y\|^2 x}{\|x\|\,\|y\|^2} = \frac{x}{\|x\|}$$

Reduced SVD : $AV_r = U_r\Sigma \quad V_r = v_1 = \frac{y}{\|y\|}$, $U_r = u_1 = \frac{x}{\|x\|}$

$$\Sigma_{r\times r} = \sigma_1 = \|x\|\|y\|$$

$$A = \sigma_1 u_1 v_1^T = \|x\|\|y\|\frac{x}{\|x\|}\frac{y^T}{\|y\|}$$

Eigenvalues of $A = xy^T$: $Az = xy^Tz = (y^Tz)x \quad z = x$

$$\text{e.v}(A) = y^Tx$$

By schwarz inequality : $|\lambda_{max}(A)| = |y^Tx| \leq \|x\|\|y\| = \sigma_1$

$\boxed{Ex}$ $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$ Find SVD of B

$\boxed{ANS}$ rank(B) = 2 $\Rightarrow \sigma_1 \geq \sigma_2 > 0$.

$\times \underbrace{B^TB}_{4\times 4}$, $\underbrace{BB^T}_{2\times 2}$ (smaller size)

Find eigenvalues of BB^T

$BB^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Eigenvalues: 2, 2

singular values of B: $\sigma_1 = \sigma_2 = \sqrt{2}$

Left singular vectors u_1, u_2 (basis for C(A))

$BB^T - 2I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hookrightarrow u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hookrightarrow u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ u_is in $\mathbb{R}^2$

$N(B^T) = \{\vec{0}\}$

Right singular vectors v_1, v_2 (basis for R(A))

$v_i = \frac{A^T u_i}{\sigma_i}$ $v_1 = \frac{A^T u_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$v_2 = \frac{A^T u_2}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

v_is in $\mathbb{R}^4$

Remaining right singular vectors v_3, v_4

$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ P.C F.C $s_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ x_3 \\ x_4 \end{matrix}$ $s_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \\ \\ x_3 \\ x_4 \end{matrix}$ $\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_4 = 0 \end{cases} \Leftrightarrow Bx = 0$

$\hookrightarrow v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ b/c $s_1 \perp s_2$ otherwise use Gram Schmidt

$B = U \Sigma V^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$

Q. / maximum ratio $\frac{\|Bx\|}{\|x\|} = \sigma_1$ reached @ $x = v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Ex. $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$ Find SVD of C

$\boxed{\text{Ans}}$ $\text{rank}(C) = 2 \Rightarrow \sigma_1 \geq \sigma_2 > 0$

$\underbrace{CC^T}_{2\times 2}$ $\quad \underbrace{C^TC}_{3\times 3}$

Smaller size

Find eigenvalues of CC^T

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = (2-\lambda-1)(2-\lambda+1) = (1-\lambda)(3-\lambda)$$

Eigenvalues of CC^T: 3, 1 $\rightarrow \sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$

Singular values of C: $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$

Left singular vectors u_1, u_2 (basis for C(A))

$$CC^T - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{x_2} \rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$CC^T - 1.I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 1 \\ \underbrace{0}_{P.C} & \underbrace{0}_{F.C} \end{bmatrix} \quad S = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{x_2} \rightarrow u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

u_i's in $\mathbb{R}^2$

Right singular vectors v_1, v_2 (basis for R(A))

$$v_i = \frac{C^T u_i}{\sigma_i} \qquad v_1 = \frac{C^T u_1}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

v_i's in $\mathbb{R}^3$

$$v_2 = \frac{C^T u_2}{\sigma_2} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Remaining right singular vector v_3

$$C = \begin{bmatrix} 1 & 1 & 0 \\ \underbrace{0 \quad 1}_{P.C} & & \underbrace{1}_{F.C} \end{bmatrix} \quad S = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{x_3} \qquad Cx = 0 \Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$U = [u_1\ u_2]$

$V = [v_1\ v_2\ v_3]$ $\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}_{2\times 3}$

$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

. maximum ratio $\frac{\|Cx\|}{\|x\|} = \sigma_1 = \sqrt{3}$ reached @ $x = v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\boxed{Ex.}$ $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Find $A = Q \Lambda Q^T$ and $A = U \Sigma V^T$

$\boxed{1}$ $A = Q \Lambda Q^T$

C.E: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 - 4 = (-1-\lambda)(3-\lambda)$

Eigenvalues of A: $-1, 3$.

$A - (-1)I = A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $s = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$

$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$s = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = Q \Lambda Q^T = \begin{bmatrix} q_2 \,|\, q_1 \end{bmatrix} \Lambda \begin{bmatrix} q_2^T \\ q_1^T \end{bmatrix}$ $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$A = \lambda_{max} q_2 q_2^T + \lambda_{min} q_1 q_1^T$

$A = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}$

Q./ Is this a SVD of A?

$\boxed{\text{ANS}}$ NO, Find SVD

$A = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}$

SVD of A

$$A^TA = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix} = \begin{bmatrix}5&4\\4&5\end{bmatrix}$$

$$C.E = \begin{vmatrix}5-\lambda & 4\\ 4 & 5-\lambda\end{vmatrix} = (5-\lambda)^2-16 = (5-\lambda-4)(5-\lambda+4)$$

$$|A^TA-\lambda I| = (1-\lambda)(9-\lambda) \rightsquigarrow \lambda(A^TA): 9, 1$$

Find v's $\qquad \sigma_1 = 3,\ \sigma_2 = 1$

$$A^TA - 9I = \begin{bmatrix}-4&4\\4&-4\end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix}-4&4\\0&0\end{bmatrix} \xrightarrow{R_1/4} \begin{bmatrix}-1&1\\0&0\end{bmatrix} \quad S = \begin{bmatrix}1\\1\end{bmatrix}_{x_2=1}$$

P.C $\quad \hookrightarrow v_1 = \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$

$$A^TA - I = \begin{bmatrix}4&4\\4&4\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}4&4\\0&0\end{bmatrix} \xrightarrow{R_1/4} \begin{bmatrix}1&1\\0&0\end{bmatrix} \quad S = \begin{bmatrix}-1\\1\end{bmatrix}_{x_2=1}$$

v_i's in $\mathbb{R}^2$ $\quad$ P.C $\quad \hookrightarrow v_2 = \frac{1}{\sqrt2}\begin{bmatrix}-1\\1\end{bmatrix}$

Find u's: $\quad Av_k = \sigma_k u_k \Rightarrow u_k = \dfrac{Av_k}{\sigma_k}$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix}1&2\\2&1\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{3\sqrt2}\begin{bmatrix}3\\3\end{bmatrix} = \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{1}\begin{bmatrix}1&2\\2&1\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix}$$

u_i's in $\mathbb{R}^2$

$$A = U\Sigma V^T = \begin{bmatrix}1/\sqrt2 & 1/\sqrt2\\ 1/\sqrt2 & -1/\sqrt2\end{bmatrix}\begin{bmatrix}3&0\\0&1\end{bmatrix}\begin{bmatrix}1/\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 1/\sqrt2\end{bmatrix}$$

$$A = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}-1&1\end{bmatrix}$$