Garry Huang

Professor Stacey Suver

CSE 300

24 March 2017

The Satisfiability Problem in showing NP-Completeness of Modern Problems

The satisfiability problem (SAT), also known as the boolean satisfiability problem, determines if a given boolean expression can be satisfied. This problem is the first proven to be NP-complete as this was the first problem to be proven that it could not be computed in polynomial time. From this point on, NP-complete refers to the set of problems in which no algorithm exists that is able to solve the problem in polynomial time. In order to show that a problem is NP-complete, one needs to reduce SAT into the specified problem that is already known to be NP-complete. In all cases, if a problem is considered to be NP-complete, then it can be reduced from the SAT problem. As a result, if it were shown that SAT was solvable in polynomial time, it is possible that all other algorithms reduced from SAT could also be solved in polynomial time. This leads to the million dollar question of $p = np$. The reduction of SAT to a given problem has been proven in many areas which include graph algorithms, brain games, as well as designing firewalls.

"The NP-Completeness of Edge-Coloring" written by Ian Holyer describes the classic problem of edge coloring, as well as the process of reducing a given clause in the 3SAT problem, a more specific variation of the SAT problem, to the 3-edge coloring problem given that the clause was satisfiable in order to show NP-completeness. Holyer describes the edge-coloring problem in which a graph is presented and the goal is to find out how many colors would be needed to color all the edges if no two adjacent edges can have the same color. Holyer describes this reduction starting with a graph describing a given 3SAT problem clause that is satisfiable. This graph contains information carried between edges, such as whether or not two edges have the same color or distinct colors. Later in the article, Holyer provides a more in-depth algorithm for converting a clause from the 3SAT problem into the edge-coloring problem. Holyer's proof of NP-completion using reduction would help layout some of the foundation, such as the use of gadgets, for future proofs including the minimum Manhattan network and the ferry cover problem.

"The Ferry Cover Problem" by Michael Lampis and Valia Mitsou describe a problem in which given a number of *n* items, the minimum size of the ferry required to safely transport all *n* items. In Lampis and Mitsou's research, they used the reduction of the MAX-NAE-{3}-SAT problem, a special type of SAT problem, into a graphing coloring problem, into the ferry cover problem. In their reduction to the ferry cover problem, Lampis and Mitsou state their first step is that given a boolean formula of *m* clauses, to reconstruct the same boolean formula but with a complement of each clause, extending the new boolean formula to have *2m* clauses. Next, the researchers describe that for every variable in these formulas, construct a point in a bipartite graph. These points are then connected to a respective clause inside the boolean formula. Lampis and Mitsou continue and say in order to properly color the graph in a given 3 colors, would be to assign the first color to true literals, the third color for false literals, and the second color to the remaining literal. By completing this process, Lampis and Mitsou are able to successfully reduce their MAX-NAE-{3}-SAT problem which can then be used to show a graph coloring problem into the ferry cover problem. The result of this reduction shows that the ferry cover problem is NP-Complete unless $p = np$. The research article written by Lampis and Mitsou show how the reduction from SAT problem can lead to an already known and solvable problem to solve the given problem. The research article also shows that reducing a problem can take more than one step in order to achieve the given problem. The previous problem of edge-coloring also laid out foundations of using gadgets in order to help prove reductions which the minimum Manhattan network relies heavily on.

"Minimum Manhattan Network is NP-Complete" by researchers Francis Chin, Zeyu Guo, He Sun, explain the reduction of 3SAT to the minimum Manhattan network(MMN) in order to show that MMN is strongly NP-Complete. The researchers describe the problem as a Manhattan path being a rectilinear path between two points and the Manhattan network as there existing a Manhattan path between all points in a graph. As stated by the researchers, the reduction from 3SAT to MMN rely on various gadgets and three principles. These three principles are that a horizontal/vertical line always exists between two points which have the same x/y-coordinate, there are many ways to connect two points with different x/y-coordinates, and there are several ways of connecting line segments to staircase points. The researchers then state the 3SAT clause generated from these three principles is $(E_F \cup E_S \cup E_C)$, where E_F represents the line segments in a graph, E_S represents sets of strip paths, and E_C represents consists of line segments within the gadgets. Similar to the Ferry Cover problem, the MMN follows the same principle that MMN is NP-Complete unless $p = np$. Although it may seem evident that the usage of reductions is commonplace in proving that certain algorithms are NP-complete, this reduction can also be done on other processes.

Written by researchers Inês Lynce and Joël Ouaknine, "Sudoku as a SAT problem" show the reduction of the SAT to the popular puzzle, sudoku. First, the researchers describe there are two different encodings for solving sudoku, a minimal encoding which specifies the problem and an extended encoding which adds redundant constraints to the already existing rules in the minimal encoding. Lynce and Ouaknine then describe the constraints to both of these encodings. In the minimal encoding, the most basic rules of sudoku are encoded, such as one number in each entry, each number appears at most once in each row, each number appears at most once in each column, and each number appears at most once in each 3x3 sub-grid. On the other hand, the researchers describe that the extended encoding includes all of the previous rules, but add on the following rules as well: there is at most one number in each entry, each number appears at least once in each row, each number appears at least once in each column, and each number appears at least once in each 3x3 sub-grid. The researchers described that in order to encode all of these rules, many propositional variables which are notated as S_{xyz}, where *xy* represent the row and column of the number *z,* are required and many clauses are required to represent a solution to the sudoku puzzle. Lynce and Ouaknine also determined that a sudoku puzzle could be solved in polynomial time based on two properties. The first being that the sudoku puzzle had only solution and the other being that the puzzle could be solved using only reasoning. The research done shows the flexibility in the SAT problem and the usage of constraints, albeit requiring many propositional clauses and constraints. Unlike the previous two problems which used a form of 3SAT, this problem used SAT in order to show more constraints within each clause of the SAT formula.

In the research article, “Hardness of Firewall Analysis,” Ehab Elmallah and Mohamed Gouda explain various problems that firewalls encounter as well as the proof of hardness for those problems. Specifically, Elmallah and Gouda use the reduction of 3SAT into the slice probing problem. As specified by Elmallah and Gouda, this problem requires the design of an algorithm where the input is an accept/discard slice and determines whether or not it accepts/discards at least one packet. The researchers provide an easy reduction of 3SAT to the firewall slicing probe problem. This reduction as stated by the researchers starts with every variable in the 3SAT formula to be represented as its own slice. Then each clause in the 3SAT formula is translated to an accept rule and then at the end, adds a discard rule to the bottom of the slice. The reduction from 3SAT to the firewall slice probing problem shows that across many fields and industries, the usage of reductions is prominent and effective at proving NP-completeness.

By solving the reduction from SAT to a given problem, researchers are able to relate these now solved problems to even harder and complicated problems. The field in which reduction can occur is extensive, and in the future it is highly likely that more and more algorithms will be designed across various fields which are proven to be NP-complete given by previous reductions. It is also important to note that if SAT or another problem were proven to be completed in polynomial time, then all other problems that were reduced from SAT would also be polynomial. This change would result in massive changes in how society is currently structured.

Works Cited

Chin, Francis Y. L., Zeyu Guo, and He Sun. "Minimum Manhattan Network Is NP-Complete." *Discrete & Computational Geometry* 45.4 (2011): 701-22. *Google Scholar*. Web. 26 Mar. 2017.

Elmallah, Ehab S., and Mohamed G. Gouda. "Hardness of Firewall Analysis." *Networked Systems Lecture Notes in Computer Science* (2014): 153-68. *Google Scholar*. Web. 28 Mar. 2017.

Holyer, Ian. "SIAM J. Comput." *SIAM Journal on Computing*. Society for Industrial and Applied Mathetmatics, 13 July 2006. Web. 30 Mar. 2017.

Lampis, Michael, and Valia Mitsou. "The Ferry Cover Problem." *Theory of Computing Systems* 44.2 (2008): 215-29. *Google Sholar*. Web. 26 Mar. 2017.

Lynce, Inês Lynce, and Joël Ouaknine. "Sudoku as a SAT Problem." (2006): n. pag. *Google Scholar*. Web. 28 Mar. 2017.