Quantitative Methods: ARCH and GARCH methods

Master

Financial Economics

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Time Series Concepts

Intuition

- A time series is a sequence of random variables
- The value of the series at each moment is uncertain
- But then we cannot think of a time series as a sequence of numbers
- But we have to imagine a bundle of paths

Stochastic processes and time series

- A *stochastic process* is a time-dependent random variable

- Consider a probability space (Ω, P), where Ω, is the set of possible states of the world and *P* is a probability measure

- A stochastic process is a bivariate function of time and states.

Paths

- A path of a stochastic process is a univariate function of time $X(t,\omega)$ formed by the set of all values $X(t,\omega)$ for a given $\omega \in \Omega$
- A stochastic process is the collection of all of its paths
- Two stochastic processes might have the same paths but different probability distributions.
- For example, consider a stock market.
- All stock price processes share the same paths but the probability distributions are different for different stocks.

Illustrations

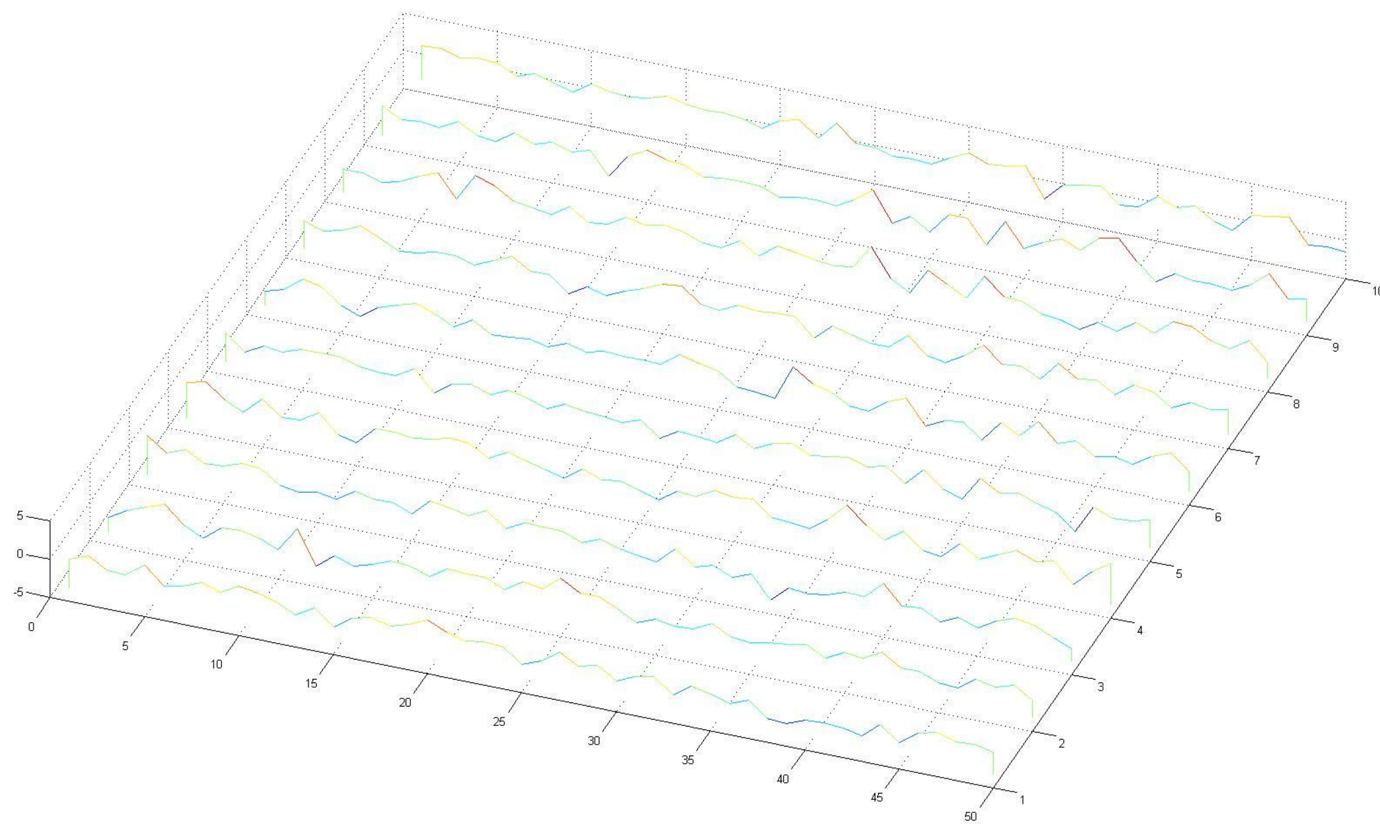
- We can illustrate the concept of a stochastic process starting from its simulation
- If we want to simulate a distribution, we generate a set of random numbers
- If we want to simulate a stochastic process or a time series we have to generate a set of random paths

How do we generate paths?

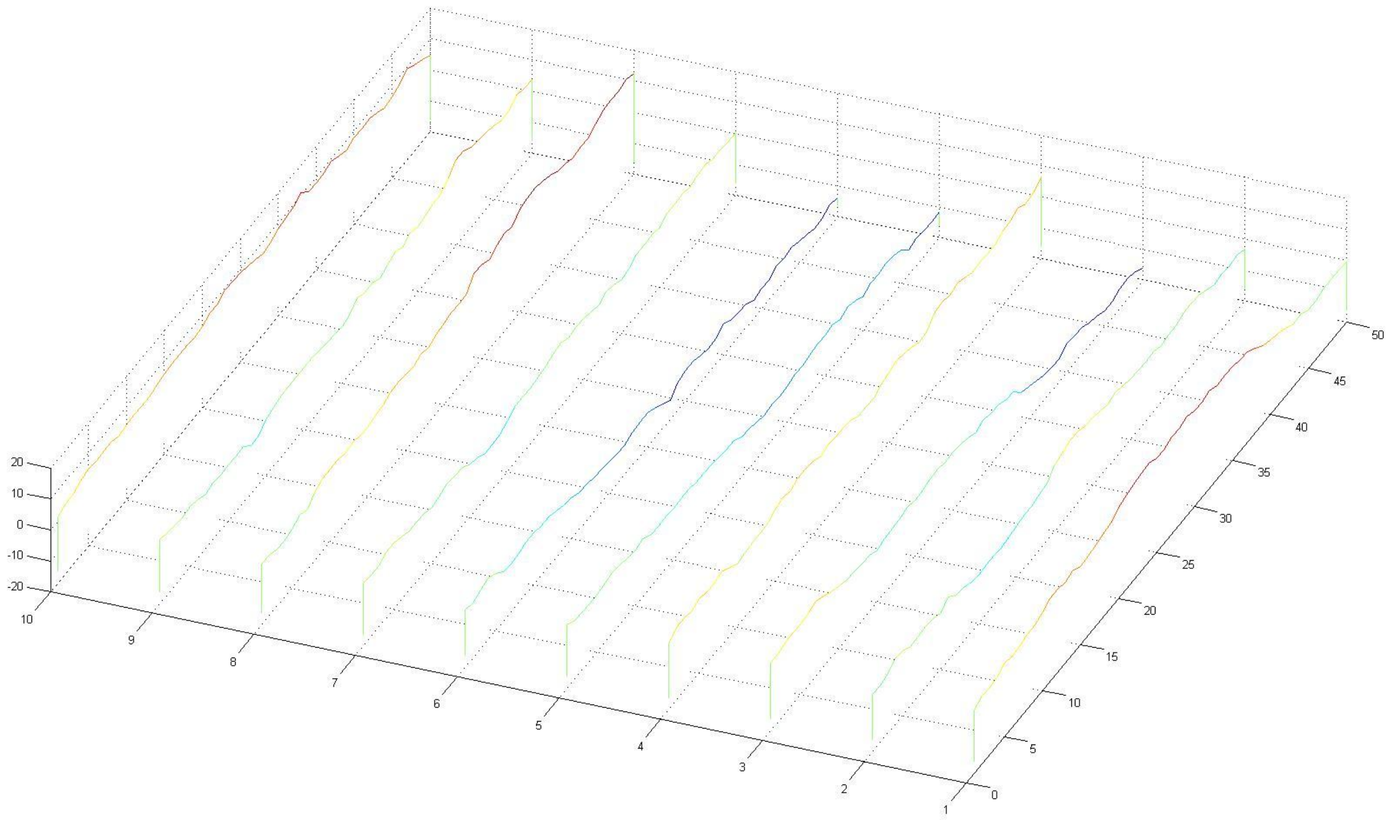
- We generate paths with a double loop, first on the path second on time:
- Suppose we want to generate 10 series of 50 iid returns

```
clear all
close all
clc
for i=1:10
   for t=1:50
      R(t,i)=randn
   end
End
P=cumsum(R)
figure
waterfall(R')
figure
waterfall(P')
```

Paths iid



Paths prices



Representation of a stochastic process

- A possible way to represent a stochastic process is through all the finite joint probability distributions:

$$F(x_1,\ldots,x_n) = P(X(t_1) \le x_1,\ldots,X(t_n) \le x_n), t_1 \le \cdots \le t_i \cdots \le t_n$$

for any *n* and for any selection of *n* time points

- The finite distributions do not determine all the properties of a stochastic process and therefore they do not uniquely identify the process.

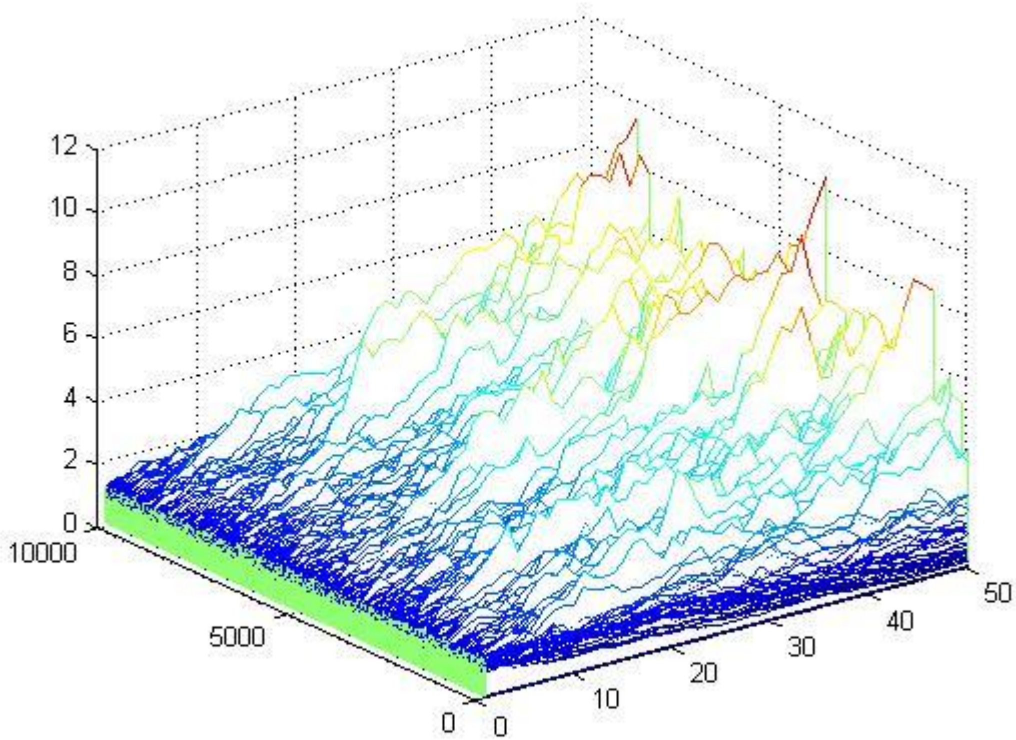
Stationary processes

- If the finite distributions do not depend on their absolute time location but only on the differences $\tau_i = t_i - t_{i-1}$

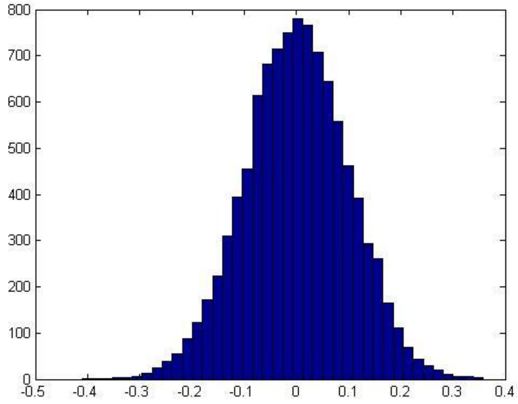
- then the process is called *stationary* or *strictly stationary*

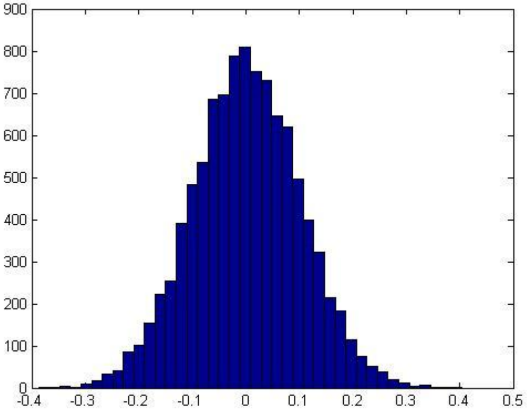
- In this case, the finite distributions can be written as $F(\tau_1, \ldots, \tau_{n-1})$

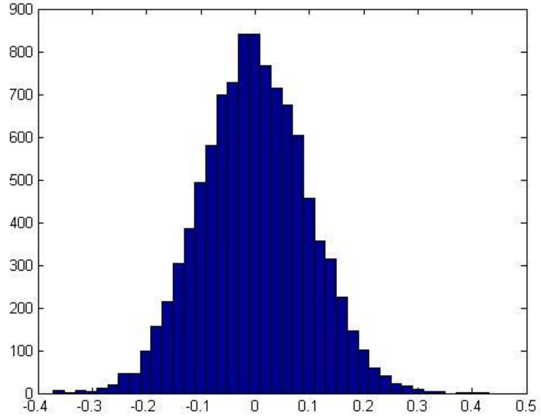
Paths geometric random walk



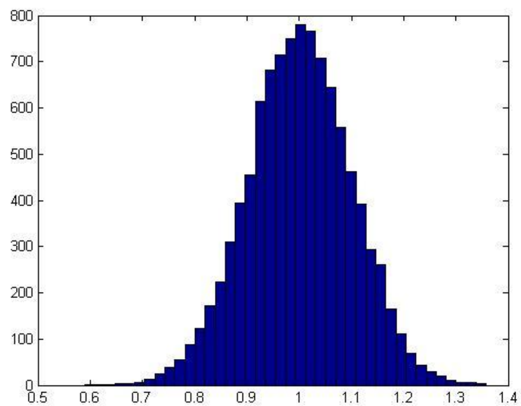
Distribution of returns

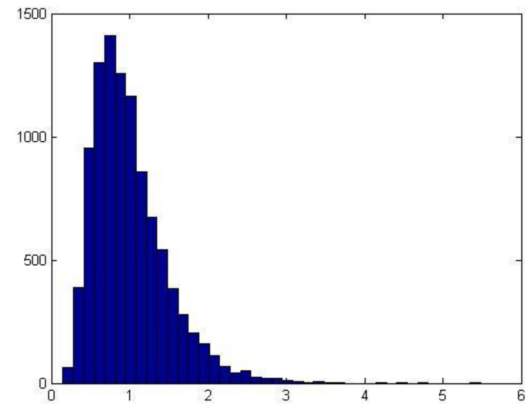


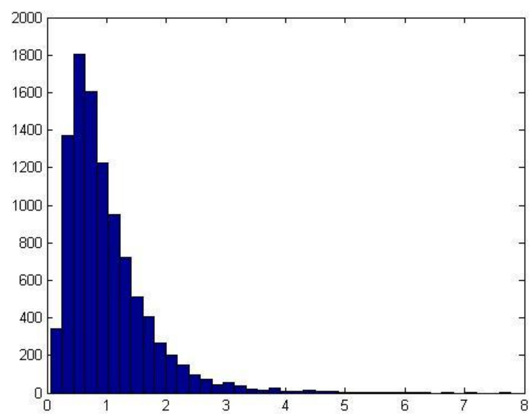




Distribution of prices (non stationary)







Ergodic processes

- A stationary stochastic process is called *ergodic* if all the (time independent) moments are equal to the limit of the corresponding time average when time goes to infinity.

- For example, if a process is ergodic, the mean equals the time average.

Multivariate stochastic processes

- The definition of a stochastic process extends naturally to multivariate processes

- A *multivariate stochastic process* is a time-dependent random vector: $X_t = \left(X_{1,t}, \ldots, X_{p,t}\right)$

- Therefore, a multivariate stochastic process is a set of *p* bivariate functions: $X(t,\omega) = \left[X_1(t,\omega), \ldots, X_p(t,\omega)\right]$

- The finite distributions are joint distributions of the *p* variables at *n* different instants of time

- Stationarity is defined as in the univariate case.

Time series

- If the time parameter moves in discrete increments,

- a stochastic process is called a *time series*.

- A univariate time series is a sequence of random variables: $X(t_1), \ldots, X(t_s), \ldots$

Spacing of time points

- Time points can be equally spaced, that is, the difference between adjacent time points is a constant: $t_s - t_{s-1} = \Delta t$

- Time points can also be spaced randomly or follow a deterministic law

- In the latter case, the time series is more properly called a *point process*

Multivariate series

- A *multivariate time series* is a sequence of random vectors:

$$X(t_1), \ldots, X(t_s), \ldots$$

$$X(t_i) = \begin{bmatrix} X_1(t_i) \\ \vdots \\ X_n(t_i) \end{bmatrix}$$

Length of a time series

- The number of time points of a time series is generally considered to be infinite

- Time series can be infinite in both directions, from $-\infty$ to $+\infty$ or they can have a starting point

- Any empirical time series can be considered a sample extracted from an infinite time series

- Strictly speaking, a time series can be stationary only if it is infinite in both directions

Covariance stationary

- A multivariate series is called *covariance stationary* or *weakly stationary* or *wide-sense stationary* if:

$$E\left(X_1(t_i),\ldots,X_N(t_i)\right)=E\left(X_1(t_j),\ldots,X_N(t_j)\right),\ \forall i$$

- and if all the covariances and autocovariances, correlations ,and autocorrelations depend only on the time lags:

$$\mathrm{corr}\left(X_i(t_r),X_j(t_s)\right)=\mathrm{corr}\left(X_i(t_{r+q}),X_j(t_{s+q})\right),\ \forall i,j=1,\ldots,N,\ \forall q,r,s=\ldots-2,-1,0,1,2,\ldots$$
$$\mathrm{cov}\left(X_i(t_r),X_j(t_s)\right)=\mathrm{cov}\left(X_i(t_{r+q}),X_j(t_{s+q})\right),\ \forall i,j=1,\ldots,N,\ \forall q,r,s=\ldots-2,-1,0,1,2,\ldots$$

ARCH and GARCH models

Empirical Regularities (Stylized Facts)

- Thick tails
- Volatility clustering
- Thick tails and volatility clustering intimately related: link between dynamic (conditional) volatility behavior and (unconditional) heavy tails.
- Leverage effects: changes in stock prices negatively correlated with changes in stock volatility
- Long-memory and persistence: volatility is highly persistent
- Co-movements in volatilities
- Volatility and information arrival: links between volatility and trading volume, dividend announcements or macroeconomic data releases.

Volatility

- Intuition:
- Basic concept: volatility is the magnitude of returns' fluctuations
- This notions ignores models of expected returns, hence...
- Volatility is the magnitude of unexpected return fluctuations
- Volatility is residual uncertainty after modeling

Volatility modeling

- Modeling volatility a major theoretical step in modern finance
- ARCH/GARCH and other similar models primary modelling tools
- But also stochastic volatility through coupled models
- And now realized volatility (UHFD)

Volatility modelling

- Define the mathematical models and their properties
- Estimate the models
- Test models and understand their applicability

Define the models

- Model of conditional mean
- Models of conditional variance

Models of conditional mean and conditional variance

- Consider a stationary time series $(X_1, \ldots, X_t, \ldots)$
- The unconditional mean and variance are constant $\mu = E(X_t)$

$$\sigma^2 = E\left[(X_t - \mu)^2\right]$$

- If a stationary time series is generated by an AR process the conditional mean depends on past values of the X_t

$$X_t = \alpha_0 + \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p}$$

Conditional mean and conditional variance

- We can also define stationary processes with a non constant conditional variance or conditional volatility (std)
- The ARCH/GARCH is a family of stationary models with time varying conditional variance
- Stationary models have constant unconditional mean and variance but can have time varying conditional mean and variance

Conditional Variance

- The conditional variance σ_t^2 of an innovation process e_t is by definition:

$$E_{t-1}(\varepsilon_t) = 0,\ \ \mathrm{t} = 1,2,\ldots$$

$$\sigma_t^2 = \mathrm{var}_{t-1}(\varepsilon_t) = E_{t-1}(\varepsilon_t^2),\ \ \mathrm{t} = 1,2,\ldots$$

- In autoregressive conditional heteroscedastic models, this conditional
- variance depends on the past of the innovations $[\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots]$

- Standardized process $z_t = \varepsilon_t (\sigma_t^2)^{-\frac{1}{2}}$
  so that $\varepsilon_t = (\sigma_t^2)^{\frac{1}{2}} z_t$

- Mean-zero, time invariant and variance of unity
- If conditional distribution ε_t of z_t is assumed to be time invariant and Gaussian, the unconditional distribution for ε_t is leptokurtic.
- The key insight of GARCH lies in the distinction between conditional and unconditional variances of the innovations process

Conditional Variance
Linear ARCH (q) model: Engle (1982)

$$\sigma_t^2 = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

- Parameters must satisfy $w > 0,\ \alpha_i \geq 0,\ i = 1,\ldots,q$
- Defining: $\nu_t = \varepsilon_t^2 - \sigma_t^2$ the model can be rewritten as

$$\varepsilon_t^2 = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \nu_t$$

- Since $E_{t-1}(\nu_t) = 0$, the model corresponds directly to an AR(q) for the ε_t^2 innovations squared,
- The process is covariance stationary if and only if $\sum_{i=1}^{q} \alpha_i < 1$ then:

$$\operatorname{var}(\varepsilon_t) = \sigma^2 = \frac{w}{1 - \alpha_1 - \cdots - \alpha_q}$$

- In empirical applications of ARCH(q) models, a long lag length and a large number of parameters are often called for.

GARCH (p,q) model

- To circumvent this problem, Bollerslev (1986) proposed a generalized ARCH, GARCH (p,q)

$$\sigma_t^2 = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

- For the conditional variance to be well-defined, all coefficients in infinite ARCH representation must be positive.
- For GARCH (1,1), positivity of σ_t^2 requires $w \geq 0,\ \alpha_1 \geq 0,\ \beta_1 \geq 0,$

$$\sigma_t^2 = w + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Estimate ARCH/GARCH models

- Maximum likelihood estimation
- Given the density of the model
- the likelihood is the probability distribution computed on the data