BASIC MATHEMATICAL CONCEPTS

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SECTION 1:
NUMBERS

Numbers

- The concept of Number is the most basic concept in quantitative finance
- In finance, numbers are used for:
  - Counting (e.g., the number of trades in a given day in a given exchanges)
  - Ordering (e.g., the fifth largest stock by capitalization in a given day in a given exchange)
  - Representing measurable quantities (e.g., stock prices)

Numbers

- Five types of numbers are important in finance and economics:
  - Natural numbers
  - Integers
  - Rational numbers
  - Real numbers
  - Complex numbers

Natural numbers

- Natural numbers are the ordinary counting numbers: 1,2,3,….
- The set of all natural numbers is denoted with the bold capital **N** or with the capital N in the typeface Blackboard bold
- Often (but there is no universal agreement on this notation) 0 is included in the set **N** of natural numbers
- Natural numbers plus zero are sometimes called “whole numbers”

Integers

- Integers include the set of natural numbers including zero plus their negatives:
- ...-3,-2,-1,0,1,2,3,...
- The set of the integers is denoted with the capital **Z** or with the capital Z in Blackboard typeface

Infinite sets

- The sets **N** (natural numbers) and **Z** (integers) are both infinite
- An infinite set is characterized by the defining property that a proper part of the set can be put in a one-to-one correspondence with the entire set
- For example, the set of even natural numbers can be put in a one-to-one correspondence with the entire set of natural numbers…
- Because there is a one-to-one correspondence between n and 2xn

Countably infinite sets

- Any set that can be put in a one-to-one correspondence with the set of natural numbers 1,2,3,… is said to be countable, denumerable, or countably infinite
- For example, it is often convenient to consider time series of prices at discrete times $t_1, t_2, t_3, \ldots.$
- The set $t_1, t_2, t_3, \ldots.$ is countably infinite because there is a one-to-one correspondence between the discrete times and the set of numbers $t_1, t_2, t_3, \ldots.$

Rational numbers

- Intuitively, a rational number can be thought of as the quotient of a fraction with integer numerator and denominator, with the denominator non-zero
- Note: This definition is circular as the quotient is a rational number and we cannot define rational numbers as quotients
- In logically rigorous terms, a rational number is defined as a pair of integers
- A rational number can be represented as a decimal number with either a finite number of decimals or a pattern of a finite number of decimals which repeats indefinitely

The set of rational numbers

- The set of rational numbers, also called the “rationals” is denoted with the capital letter Q in bold typeface **Q** or in Blackboard bold typeface

- Exercise 1: Show that the set Q is a denumerable set (Hint: Create a table of natural numbers *i*,*j* and construct rational numbers following the path 11, 12,21,31,32,….)
- Exercise 2: Show that given any two rationals *p*,*q* it is always possible to find a rational *r* such that $p<r<q$

Real numbers

- Real numbers can be thought of as numbers whose decimal representation includes any possible finite or infinite sequence of integers
- Note: In rigorous logical terms, real numbers are defined as contiguous infinite classes of rational numbers
- The set of real numbers is denoted with the capital letter **R** in bold typeface or in Blackboard typeface

The set of real numbers

- The set **R** is infinite but not countably infinite
- The set **R** is much larger than any countably infinite set
- For example, if we exclude all rational numbers, the set of the remaining real numbers have the same size (called cardinality) as the entire set of real numbers
- We say that the set **R** forms a continuum
- Continuous-time finance assumes that time is continuous but sampled at discrete intervals
- Discrete-time finance assumes time moves in discrete increments

Complex numbers

- The square root of a non-negative (positive or zero) real number is a real number
- However, no real number can be the square root of a negative real number
- Complex numbers allow to take the square root of negative numbers
- First define the "imaginary unit" i through the property that $i^2 = i \times i = -1$
- Complex numbers are defined as $a + bi$ where a,b are real numbers and i is the imaginary unit

Complex numbers, ctd…

- And the usual rules of addition multiplication hold:

$$\alpha = a + bi,\ \beta = c + di$$

$$\alpha + \beta = (a + c) + (b + d)i$$

$$\alpha \times \beta = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

- The set of complex numbers is denoted with the capital letter **C** in bold or Blackboard bold typeface

Note: We cannot simply impose $i^2 = i \times i = -1$

Complex numbers are defined logically as pairs of real numbers

Algebraic equations

- Consider an algebraic equation of order *n*:

  $$a_0 + a_1 x + \cdots a_n x^n = 0$$

  where the coefficients are complex numbers
- The fundamental theorem of algebra states that any algebraic equation of order *n* with complex coefficients has *n* roots if each root is counted with its multiplicity
- Algebraic equations play a fundamental role in many problems in quantitative finance

Where do you find these concepts

- Integers, rational, and real numbers are obviously ubiquitous in finance
- You will find complex numbers in some eigenvalue problem and in the analysis of time series in the frequency domain

SECTION 2:

SETS

Sets

- A set is a collection of individuals called elements
- Sets are typically denoted with capital letters A,B,C,…
- Elements are typically denoted with lower-case letters $a, b, c, \ldots$
- An element a is said to "belong to" or "to be a member of" a set A
- If an element a belongs to a set A we write $a \in A$
- If an element a does not belong to a set B we write $a \notin A$

Properties of sets

- Consider two sets A and B; if each element of A belongs to B we say that A is contained in B and we write: $A \subset B$
- If every element of A belongs to B and vice-versa each element of B belongs to A then A and B are the same set and we write: $A = B$
- We generally work with a universe U, that is a set which includes all elements we want to consider, and every set $A \subset U$
- The complement A^C is the set of all elements of U that do not belong to A
- It is convenient to introduce the empty set $\varnothing$ which does not include any element

Set operations: Union

- Given two sets A and B, the set C which includes all the elements of A plus all elements of B is called the union of A and B denoted:

  $C = A \cup B$

- Alternatively, we can state that the union of A and B includes all and only the elements that belong either to A or to B or to both

Properties of set union

- Union is commutative $B \cup A = A \cup B$
- Union is associative $\left(A \cup B\right) \cup C = A \cup \left(A \cup B\right)$
- Hence we can write the union of *n* sets, which is independent of their order: $A_1 \cup A_2 \cup \cdots \cup A_n$
- The union of any set A with the empty set is the set A:

$$A \cup \varnothing = \varnothing \cup A = A$$

Set operations: Intersection

- Given two sets A and B, their intersection C is the set which includes all and only the elements that belong to A and to B
- Set intersection is also called set product and is denoted as:

$$C = A \cap B$$

- The following properties of set product hold:

$$A \cap B = B \cap A$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$A \cap A^C = \varnothing$$

- Hence we can define the product of *n* sets

Distributive property and disjoint sets

- Distributive property:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Two sets are said to be disjoint if they do not have elements in common
- Hence, the intersection of disjoint sets is the empty set
- The intersection of any set with the empty set is the empty set

Examples

- Consider the gains or losses made by stocks in a given day in a given exchange; if A is the set of stocks with positive gains, B is the set of stocks with gains less than 5%, C is the set of stocks with losses:
- The union $C = A \cup B$ is the set of all stocks
- The intersection $A \cap B$ is the set of all stocks with positive gains less than 5%
- The intersection $A \cap B \cap C$ is the empty set

Where do you find these concepts

- The notion of set and the properties of sets are fundamental in probability and statistics
- You will need to understand concept and the properties of sets in every discipline with a quantitative basis, finance, econometrics, corporate finance, economics, etc…

SECTION 3:

LIMITS, SEQUENCES, AND SERIES

Sequences of real numbers

- Consider an infinite sequence of real numbers:

$$a_1, a_2, a_3, \ldots$$

- An infinite sequence is an idealization, a mathematical construct as empirical sequences are all finite

Limits of sequences

- Consider a sequence $a_1, a_2, a_3, \ldots$
- We say that the real number L is the limit of the sequence $a_1, a_2, a_3, \ldots$
- Or that the sequence $a_1, a_2, a_3, \ldots$ tends to the limit L or that it converges to L if the following holds:
- For any arbitrarily small real number ε we can determine a natural number n_0 such that, for any $n > n_0$ then $|a_n - L| < \varepsilon$

Limits of sequences, ctd…

- The intuition behind the concept of limit of a sequence is that….
- A sequence tends to *L* if it gets closer and closer to *L* so that the distance (absolute value of the difference) of its terms from *L*….
- Can be made less than any arbitrarily small number as *n* grows

Diverging sequences

- A sequence is said to diverge or to tend to infinity if the following holds:
- For any arbitrarily large real number D we can determine a natural number n_0 such that, for any $n > n_0$ then $|a_n| > D$
- That is, the absolute value of the terms of the sequence gets larger and larger with growing n

Diverging sequences, ctd…

- We can additionally distinguish if the sequence tends to plus or minus infinite if
- For any arbitrarily large real number D we can determine a natural number n_0 such that, for any $n > n_0$ then $a_n > D$ or, respectively, $a_n < -D$

Series

- The sum of the terms of a sequence is called a series:

$$S = \sum_{i=0}^{\infty} a_i$$

- A series is said to be convergent if the sequence of the partial sums

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ &\vdots \\ S_n &= a_1 + \cdots + a_n \end{aligned}$$

converges to a limit S

Note: convergence of a sequence and convergence of a series, that is, convergence of the partial sums, are different concepts not to be confused. A sequence might converge even if its partial sums do not

Concepts of convergence

- A sequence $a_1, a_2, a_3, \ldots$ is said to be absolutely convergent or absolutely summable if the sequence of the partial sums of the absolute values of its terms $S_n = |a_1| + \cdots + |a_n|$ converges to a limit S

- A sequence $a_1, a_2, a_3, \ldots.$ is said to be square summable if the sequence of the partial sums of the squares of its terms
- $S_n = a_1^2 + \cdots + a_n^2$

  converges to a limit S

- A sequence $a_1, a_2, a_3, \ldots$ is said to be 1-summable if the sequence of the partial sums $S_n = 1 \times |a_1| + \cdots + n \times |a_n|$ converges to a limit S

Where do you find these concepts

- The concepts of sequences, series, and limits are fundamental in time series analysis
- You will find these concepts in finance, corporate finance, econometrics, economics, etc…

SECTION 4: FUNCTIONS

Real valued functions

- A real valued function of real numbers is a correspondence between the real numbers x in a set D called the Domain to the real numbers $y=f(x)$ in a set R called the Range
- Example: $y=+sqrt(x)$ associates to any non-negative real number x a non-negative real number y equal to the square root of x
- The range and the domain of $y=+sqrt(x)$ are the set of non-negative real numbers

Real valued functions, ctd…

- Note: the same value y can correspond to many (more than 1) values x
- Example: $y=\sin(x)$
- If to each x corresponds one and only one value y then the function $y=f(x)$ is invertible
- In the sense that we can define $x=g(y)$ such that $x=g(f(x))$

Increasing and decreasing functions

- A function y=f(x) is said to be (strictly) increasing if
  $y_2 = f(x_2) > y_1 = f(x_1)$ for all $x_2 > x_1$
- A function y=f(x) is said to be non-decreasing if
- $y_2 = f(x_2) \geq y_1 = f(x_1)$ for all $x_2 > x_1$
- A function y=f(x) is said to be (strictly) decreasing if
  $y_2 = f(x_2) < y_1 = f(x_1)$ for all $x_2 > x_1$
- A function y=f(x) is said to be non-increasing if
  $y_2 = f(x_2) \leq y_1 = f(x_1)$ for all $x_2 > x_1$

Limits of functions

- A function $y=f(x)$ is said to tend to the limit l for x that tends to a and we write $\lim_{x\to a} f(x)=l$ if the following holds:
- For any arbitrarily small real number ε we can find a real number δ such that $|f(x)-L|<\varepsilon$ for $|x-a|<\delta$
- If $f(a)$ exists and $f(a)=l$ then the function f is said to be continuous in a.

Examples

- Example: the function $y = x^2$ is continuous for any real number x
- The function $y = \begin{cases} x^2 \text{ for } x < 0 \\ x^2 + 1 \text{ for } x > 0 \end{cases}$ is not continuous for x=0 because it makes a jump
- The function $y = x$ for $-1 < x < 1$ is not continuous in x=1 because the limit $\lim_{x \to 1} y = 1$ exists but y=x(1) is not defined

Derivatives

- Given a continuous function $y = f(x)$ its derivative
  $$\frac{dy}{dx} = \frac{df(x)}{dx} \equiv f'(x)$$
  is defined as:
  $$g(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)}{\Delta x}$$
- The derivative can be interpreted as the angular coefficient of the tangent to the curve f(x)
- We can define derivatives of higher order as derivatives of derivatives; for example a second order derivative is:
  $$\frac{d^2 f(x)}{dx^2} \equiv f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

Properties of derivatives

The following rules apply:

$$(af + bg)' = af' + bg', \text{a, b, constants}$$

$$(fg)' = f'g + fg'$$

$$(f(h(x)))' = f'(h(x))h'(x)$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

Examples

- Show using the definition that $y = x \to y' = 1$
- The following holds (properties of derivatives):

$$(x^2) = (x \times x)' = 1 \times x + x \times 1 = 2x$$

$$(2x + 3x^2)' = 2 + 6x$$

$$((x^2)^2)' = 2(x^2)2x = 4x^3$$

Frequently used derivatives:

$$f(x) = x, \quad f'(x) = 1$$

$$f(x) = x^2, \quad f'(x) = 2x$$

$$f(x) = x^3, \quad f'(x) = 3x^2$$

$$f(x) = x^4, \quad f'(x) = 4x^3$$

$$f(x) = x^n, \quad f'(x) = nx^{n-1}, \text{n integer}$$

$$f(x) = x^\alpha, \quad f'(x) = \alpha x^{\alpha-1}, \alpha \text{ real}$$

$$f(x) = \log(x), \quad f'(x) = \frac{1}{x}$$

Frequently used derivatives:

$$f(x) = x^{-1} = \frac{1}{x}, \quad f'(x) = -x^{-2} = -\frac{1}{x^2}$$

$$f(x) = x^{-2}, \quad f'(x) = -2x^{-3}$$

$$f(x) = x^{-3}, \quad f'(x) = -3x^{-4}$$

$$f(x) = x^{-4}, \quad f'(x) = -4x^{-5}$$

$$f(x) = x^{-\alpha}, \quad f'(x) = -\alpha x^{-\alpha-1}$$

$$f(x) = e^{x}, \quad f'(x) = e^{x}$$

Definite Integral (Riemann)

- The definite (Riemann) integral of a function

$$\int_a^b f(x)dx$$

- is the area below the curve in the interval [a,b]; It is defined as the limit of the approximating sums:

$$\int_a^b f(x)dx = \lim_{\|\Delta\to 0\|} \sum f(c_i)\Delta_i$$

Indefinite integral

- Consider the function $F(x)=\int_a^x f(u)du$ called the Indefinite Integral of *f*.
- The Fundamental Theorem of Calculus states that the derivative of the integral of a function $f(x)$ is the function itself: $F' = f$
- Given a function $f(x)$ there are infinite indefinite integrals
- Any two indefinite integrals of *f* differ by a constant

Properties of indefinite integrals

- The following properties of indefinite integrals hold:

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\int fg'\,dx = fg - \int f'g\,dx$$

$$\int f'(g)g'\,dx = \int f'(g)\,dg = f(g)$$

- Given any indefinite integral $F = \int f(u)\,du$ the definite integral

$$\int_a^b f(x)\,dx$$

can be computed as: $\displaystyle\int_a^b f(x)\,dx = [F]_a^b = F(b) - F(a)$

Frequently used indefinite integrals:

$$\int 1dx = x + C$$

$$\int xdx = \frac{1}{2}x^2 + C$$

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \text{n integer}$$

$$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C, \alpha \text{ real}$$

Frequently used indefinite integrals:

$$\int \frac{1}{x} dx = \log(x) + C$$

$$\int x^{-2} dx = -\frac{1}{x} + C$$

$$\int x^{-3} dx = -\frac{1}{2} x^{-2} + C$$

$$\int x^{-4} dx = -\frac{1}{3} x^{-3} + C$$

$$\int x^{-\alpha} dx = -\frac{1}{\alpha + 1} x^{-\alpha + 1} + C$$

$$\int e^{x} dx = e^{x} + C$$

Examples

- Example: Compute the area below the segment of parabola

  $y = x^2$ for $-1 \le x \le 1$

$$\int_{-1}^{+1} x^2 dx = \left[\frac{1}{3}x^3\right]_{-1}^{+1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

- Exercise: Compute the area below the exponential function in the interval (-1,+1)

Functions of more than one variable

- The concept of function carries over to functions of more than one variable
- For simplicity let's consider only two variables as all concepts easily extend to more than two variables
- A real-valued function of two real-valued variables $z = f(x, y)$ is a correspondence between pairs of real numbers x,y in a set D called the Domain to the real numbers $z = f(x, y)$ in a set R called the Range
- The domain D is a set of the two-dimensional plane x,y which might coincide with the entire plane

Limits and continuous functions in two Variables

- There are different notions of limit as a function of two variables might tend to some limit along different directions
- A broad definition of limit without specifying any limit is the following
- A real-valued function $z = f(x, y)$ tends to a limit L when (x, y) tend to (x_0, y_0) and we write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if for any real number ε there is a positive real number δ such that in the disk $\left[(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \delta\right]$ the inequality $|f(x, y) - L| < \varepsilon$ holds
- A real-valued function $z = f(x, y)$ is said to be continuous in (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

Partial derivatives

- Given a real-valued function of two variables $z = f(x, y)$ we define the partial derivatives $f_x \equiv \frac{\partial f}{\partial x}$ and $f_y \equiv \frac{\partial f}{\partial y}$ are respectively the derivative with respect to *x* of the function *f* keeping constant the variable *y* and the derivative with respect to *x* of the function *f* keeping constant the variable *x:*

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}$$

- We can define partial derivatives of higher order as partial derivatives of partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right); \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right); \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Examples

- Consider the function: $z = x^2 y^2$
- Its partial derivatives are: $\frac{\partial z}{\partial x} = 2xy^2; \ \frac{\partial z}{\partial y} = 2x^2 y$
- And the second order partial derivatives are:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(2xy^2\right) = 2y^2; \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(2x^2 y\right) = 2x^2;$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2x^2 y\right) = 2xy; \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2xy^2\right) = 2xy$$

Maxima and minima

- Consider a real-valued function of one variable x defined in a domain D: $y = f(x)$
- The function $y = f(x)$ is said to have a relative maximum or minimum in $x_0 \in D$ if there is a real number ε such that the neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon) \subset D$ is included in D and for any real number $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ the following holds: $f(x) \leq f(x_0)$ for a maximum and $f(x) \geq f(x_0)$ for a minimum
- If the relationships $f(x) \leq f(x_0)$ or $f(x) \geq f(x_0)$ hold for any $x \in D$ then the function $f(x)$ is said to have a global or an absolute maximum or minimum respectively

Maxima and minima, ctd...

- In other words, the global or absolute maximun or minimum of a function is the maximun or minimum value of its range
- The extreme value theorem (first proved by Bernard Bolzano in 1830, not to be confused with statistical extreme value theory) states that any continuous function in a given closed interval (an interval is closed if it contains its endpoints, open if it does not) attains its maximum and minimum at least once in the given interval
- Fermat's theorem states that any local maximum or minimum must occur at a point where the function is not differentiable or, if differentiable, where its derivatives are zero

Finding maxima and minima

- Given a real valued function of one variable $y = f(x)$
- A necessary condition for a point x_0 to be a local maximum or a local minimum of a function f differentiable in x_0 is that its derivative is zero: $f'(x_0) = 0$
- This is a necessary but not sufficient condition as derivatives are zero also on inflection points and maxima and minima might occur where functions are not differentiable
- If the function f admits also a second derivative (derivative of the derivative) then a point is a local maximum if $f'(x_0) = 0,\ f''(x_0) < 0$ and a local minimum if $f'(x_0) = 0,\ f''(x_0) > 0$
- Global maxima or minima are found by inspecting all local maxima or minima plus the boundary points

Finding maxima and minima, ctd…

- A necessary condition for local maxima and minima of a real valued function of two variables is that the partial derivatives exist and are zero: $\frac{\partial f(x_0, y_0)}{\partial x} = 0,\ \frac{\partial f(x_0, y_0)}{\partial y} = 0$
- This condition is necessary but not sufficient
- To determine local minima and maxima we need to compute the second partial derivatives test
- This test is performed by computing the Hessian matrix which is the matrix formed with the second partial derivatives
- There is a maximum if the Hessian is positive and the second partial derivative with respect to *x* is positive, a minimum if the Hessian is positive and the second partial derivative with respect to *x* is negative
- If the Hessian is zero the test is inconclusive

Lagrange multipliers

- It is often necessary to find the local maxima or minima of a function of two variables *f* subject to constraints $g(x, y) = 0$
- Constraints of this type might define a line in the plane, for example a circle $x^2 + y^2 = 1$
- To solve the constrained problem we can form a new function *L* adding a new variable λ in this way:

  $$L = f(x, y) + \lambda g(x, y)$$
- And solve the unconstrained problem max *L* or min *L* by setting to zero the three partial derivatives:

  $$\frac{\partial L}{\partial x} = 0, \ \frac{\partial L}{\partial y} = 0, \ \frac{\partial L}{\partial \lambda} = 0$$

Optimization

- More general maxima and minima problems cannot be generally solved analytically but require iterative numerical procedure
- This is the field of optimization
- Recent progress in optimization makes it suitable for optimizing large portfolios
- Reference: *Robust Portfolio Optimization and Management*, F.J. Fabozzi, P.N. Kolm, D.A. Pachamanova, and S.M. Focardi (Wiley, 2007)

Examples

- Find local maxima of the function $y = -3x^2 + 2x + 3$
- Let's first determine the points where the derivative is equal to zero:

$$\frac{dy}{dx} = -6x + 2$$

$$\frac{dy}{dx} = 0 \rightarrow -6x + 2 = 0 \rightarrow x = \frac{1}{3}$$

- There is only one point where derivatives go to zero; let's check if it is a local maximum computing the second derivative:

$$\frac{d^2 y}{dx^2} = -6 < 0$$

- The second derivative is negative and therefore $x = \frac{1}{3}$ is a maximum

Examples

- Find local minima of the function: $z = x^2 + y^2 - 3x + 4y - 6$
- Compute the first partial derivatives and see where they are equal to zero:

$$\frac{\partial z}{\partial x} = 2x - 3; \ \frac{\partial z}{\partial y} = 2y + 4; \begin{cases} \dfrac{\partial z}{\partial x} = 0 \rightarrow x = \dfrac{3}{2} \\ \dfrac{\partial z}{\partial y} = 0 \rightarrow y = -\dfrac{1}{2} \end{cases}$$

- Compute the Hessian matrix and determinant:

$$\frac{\partial^2 z}{\partial x^2} = 2; \ \frac{\partial^2 z}{\partial y^2} = 2; \ \frac{\partial^2 z}{\partial x \partial y} = 0; \ H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}; \ M = \det(H) = 4 > 0$$

- The Hessian determinant and the second partial derivatives are positive and therefore the point $\left(\frac{3}{2}, -\frac{1}{2}\right)$ is a local minimum

Examples

- Consider the same function $z = x^2 + y^2 - 3x + 4y - 6$
- Solve the problem: find local minima s.t. (subject to) the constraint: $y = 3x + 2$
- Compute the Lagrangian L and its derivatives:

$$L(x, y, \lambda) = x^2 + y^2 - 3x + 4y - 6 + \lambda(y - 3x - 2)$$

$$\frac{\partial L}{\partial x} = 2x - 3 - 3\lambda; \ \frac{\partial L}{\partial y} = 2y + 4 + \lambda; \ \frac{\partial L}{\partial \lambda} = y - 3x - 2$$

- Solve the system of linear equation equating derivatives to zero:

$$\begin{cases} 2x - 3\lambda = 3 \\ 2y + 4 + \lambda = -4 \\ y - 3x = 2 \end{cases} \rightarrow A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = A^{-1}B = \begin{bmatrix} -1.05 \\ -1.15 \\ -1.70 \end{bmatrix}$$

Where do you find these concepts

- Functions, derivatives, and integrals appear in many problems in finance, corporate finance, economics
- In particular, functions, derivatives, and integrals are essential prerequisite for understanding probability concepts when variable are continuous, such as prices and returns
- You will find problems of maxima and minima in most methods of portfolio management (modern portfolio theory)

SECTION 5:

MATRIX ALGEBRA

What is a matrix?

- A *nxm* matrix is a bidimensional array of numbers:

$$A = \{a_{ij}\} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

- The first index identifies the rows, the second one the columns
- The numbers a_{ij} can be real or complex numbers
- If $n=m$ the matrix A is called a square matrix

Vectors

- A column n-vector V is a *nx1 matrix:*

$$V = \begin{bmatrix} v_{11} \\ \vdots \\ v_{n1} \end{bmatrix}$$

- A row vector is a *1xn* matrix:

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \end{bmatrix}$$

Diagonal matrices

- The terms a_{ii} form the main diagonal
- A matrix whose entries are all zero except those on the main diagonal is called a diagonal matrix
- A diagonal matrix with the diagonal terms equal to 1 is called the Identity matrix I

$$I_N = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

Matrix operations

- Addition: the sum of two matrices is a matrix whose entries are the sum of the corresponding elements

$$C = \{c_{ij}\} = A + B = \{a_{ij} + b_{ij}\}$$

Properties of addition

- Addition is commutative:

$$A+B=B+A$$

- Additionis is associative:

$$\left(A+B\right)+C=A+\left(B+C\right)$$

- Hence we can recursively define the sum of n matrices:

$$A+B+C+D=\left(\left(\left(A+B\right)+C\right)+D\right)$$

Transpose and symmetric

- The transpose of a matrix is obtained switching colums with rows:

$$A = \{a_{ij}\}, \; A' \equiv A^T = \{a_{ji}\}$$

- All elements of row i become the column j

- If A is a square matrix and $A' = A$ the matrix A is called symmetric

Triangular matrices

- A matrix whose elements below (above) the main diagonal are all zero i called upper (lower) triangular
- Triangular matrices whose diagonal elements are all zero are called strictly triangular
- If $A' = A$ the upper triangular part of A is equal to the lower triangular part:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{12} & a_{22} & \ddots & a_{n2} \\ \vdots & \ddots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Multiplication

- Several different multiplication operations can be defined
- Scalar-matrix multiplication:

$$cA = Ac = \{ca_{ij}\}$$

- Matrix-matrix multiplication:

$$A = \{a_{ij}\}_{nm},\ B = \{b_{ij}\}_{mp},\ C = \{c_{ij}\}_{np}$$

$$C = AB,\ \left\{c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}\right\}$$

Multiplication

- The matrices A and B can be multiplied only if the number of columns of A equals the number of rows of B
- Matrix multiplication is not commutative:

$$\underset{n\times m}{A}\,\underset{m\times p}{B} \neq \underset{m\times p}{B}\,\underset{n\times m}{A},\; n=m=p$$

- It is distributive with addition:

$$\underset{n\times m}{A}\left(\underset{m\times p}{B}+\underset{m\times p}{C}\right)=\underset{n\times p}{AB}+\underset{n\times p}{AC}$$

- And associative:

$$\underset{n\times m}{A},\; \underset{m\times p}{B},\; \underset{p\times s}{C},\; \underset{n\times p}{AB},\; \underset{m\times s}{BC},\; (AB)C = A(BC) = ABC$$

Multiplication

- Suppose the two matrices A,B are two vectors:

$$A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}$$

- The product

$$A'B = \begin{bmatrix} a_{11} \cdots a_{n1} \end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix} = \sum_{i=1}^{n} a_{i1} b_{i1}$$

- is the usual scalar product (dot product) betwen vectors

Multiplication

- If we represent matrices with row and column vectors:

$$A = [a_i], B = [b_j]$$

- The i,j entry of the product matrix is the dot product of the respective row and column:

$$AB = \{a_i b_j\}$$

Trace

- The trace of a matrix is the sum of its diagonal entries:

$$trace(A) = trace\{a_{ij}\}_{nn} = \sum_{i=1}^{n} a_{ii}$$

- The following properties hold

$$trace(AB) = trace(BA)$$

$$trace(A+B) = trace(A) + trace(B)$$

Linear independence

- Consider a *mxn* matrix: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{12} & a_{22} & \ddots & a_{n2} \\ \vdots & \ddots & \ddots & \\ a_{m1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}$
- The rows (columns) of A are said to be linearly independent if

$$\begin{cases} k_1 a_{11} + \cdots k_m a_{m1} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ k_1 a_{n1} + \cdots k_m a_{mn} = 0 \end{cases} \quad \text{iff} \quad k = 0 \qquad \begin{cases} h_1 a_{11} + \cdots h_n a_{1m} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ h_1 a_{n1} + \cdots h_n a_{mn} = 0 \end{cases} \quad \text{iff} \quad h = 0$$

- If $s \leq min(m,n)$ rows (columns) are linearly independent then *s* columns (rows) are linearly independent

Rank

- The rank of a matrix is the number of linearly independent columns or row
- If A is nxn and of full rank the number of linearly independent columns (rows) is n
- The following holds:

$$rank(AB) \leq \min(rank(A), rank(B))$$

Determinant

- The determinant of a square *nxn* matrix is defined as follows: $\det(A) \equiv |A| = \sum (-1)^{t(j_1,\ldots,j_n)} \prod_{i=1}^{n} a_{ij}$
- Where $(j_1,\ldots,j_n)$ is a permutation of the set of integers 1,…,*n* and $t(j_1,\ldots,j_n)$ is the number of transpositions to reach $(j_1,\ldots,j_n)$
- I.e. the det is the sum of all possible signed products
- Ex. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ |A| = a_{11}a_{22} - a_{12}a_{21}$

Computing the determinant

- Determinants can be computed, and defined, recursively as follows:

$$|A| = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} |A_{1i}|$$

- Where A_{1i} is the matrix obtained deleting the first row and the i-th column in the matrix

A

Properties

The following properties of determinants hold:

$$|AB| = |A||B|$$

$$|cA| = c^n |A|$$

$$|A'| = |A|$$

$$rank(A) = n \text{ iff } |A| \neq 0$$

Inverse

- The identity matrix is the *nxn* matrix:

$$I_n = \begin{bmatrix} 1 & \ddots & 0 \\ \ddots & \ddots & \ddots \\ 0 & \ddots & 1 \end{bmatrix}$$

- Multiplication by the identity matrix leaves any matrix unchanged: $AI = IA = A$
- Matrix A is said to be non singular if $rank(A)=n$, singular if $det(A)=0$ i.e. $rank(A)<n$
- If a matrix A is non singular we can define its inverse:

$$A^{-1},\ AA^{-1} = A^{-1}A = I$$

Properties of inverse and transpose

- The inverse of a product is the product of the inverses in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}$$

- The transpose of a product is the product of the transposes in reverse order:

$$(AB)' = B'A'$$

Minors and cofactors

- The minor M_{ij} of the entry *i,j* of a matrix A is the determinant of the matrix obtained removing the row *i* and the column *j*
- The cofactor is defined as: $$A_{ij} = (-1)^{i+j} M_{ij}$$
- The adjoint of a matrix is: $$adj(A) = \{A_{ij}\}^T$$
- The following holds: $$A^{-1} = \frac{adj(A)}{|A|}$$

Orthogonal matrices

- A square matrix is called orthogonal if:

  $$AA' = A'A = I$$

- A matrix is called orthogonal if any two rows or columns are orthogonal, i.e.:
- The dot product between any two different rows (columns) of an orthogonal matrix is zero, the norm of each row (column) is 1
- If a matrix is orthogonal $\quad A^{-1} = A'$

Bilinear forms and positive/negative definite/semi-definite matrices

- Given a *nxn* matrix $A = \{a_{ij}\}$ and two vectors *x*,*y* the expression: $x' Ay = \sum_{i,j=1}^{n} a_{ij} x_i y_j$ is called a bilinear form
- A symmetric matrix A is said to be positive definite if,

  $x' Ax > 0, \ \forall \mathrm{x} \neq 0$
- negative definite if $x' Ax < 0, \ \forall \mathrm{x} \neq 0$
- A symmetric matrix A is said to be positive semi-definite if $x' Ax \geq 0, \ \forall \mathrm{x} \neq 0$
- negative semi-definite if $x' Ax \leq 0, \ \forall \mathrm{x} \neq 0$

Eigenvectors and eigenvalues

- Consider a matrix A and a vector x
- If $Ax = \lambda x$ the vector x is called a (right) eigenvector and the scalar λ is called an eigenvalue of the matrix A
- If $x'A = \lambda x'$ the vector x is called a left eigenvector and the scalar λ is called an eigenvalue of the matrix A
- Without specifications, eigenvectors are right eigenvectors

Conditions for eigenvalues

- Write $Ax = \lambda x$ as $(A - \lambda I)x = 0$
- The condition $(A - \lambda I)x = 0$ has non trivial solutions if the characteristic equation holds

$$\det(A - \lambda I) = 0$$

- Eigenvalues are the solutions of the characteristic equation
- If A is positive definite all eigenvalues are real and positive
- If A nxn then:

$$trace(A) = \sum_{i=1}^{n} \lambda_i$$

Examples

Matrix-matrix multiplication

A =

1	2	3
4	5	6

B =

7	8	9
10	11	12

AB’ =

50	68
122	167

Multiplication

A(1,:) =

1 2 3

B(1,:) =

7 8 9

A(1,:)B(1,:)' =50

B(2,:) =

10 11 12

A(1,:)B(2,:)' =68

AB' =

50 68

122 167

Matrix multiplication is not commutative

H=[1 2;3 4]

H =

```
    1     2
    3     4
```

K=[5 6;7 8]

K =

```
    5     6
    7     8
```

HK =

```
   19    22
   43    50
```

KH =

```
   23    34
   31    46
```

Matrix multiplication is associative

M=[1 2;3 4;5 6]

M =

1	2
3	4
5	6

N=[7 8 9;10 11 12]

N =

7	8	9
10	11	12

Q=[13 14 15 16;18 19 20 21;22 23 24 25]

Q =

13	14	15	16
18	19	20	21
22	23	24	25

MNQ =

1617	1707	1797	1887
3667	3871	4075	4279
5717	6035	6353	6671

Trace

```
F=[1 2 3;4 5 6;7 8 9]
F =
    1     2     3
    4     5     6
    7     8     9

G=[10 11 12;13 14 15;16 17 18]

G =
   10    11    12
   13    14    15
   16    17    18

F+G =
   11    13    15
   17    19    21
   23    25    27

trace(F) = 15

trace(G) = 42

trace(F+G) = 57
```

Trace

FG =

$$\begin{matrix} 84 & 90 & 96 \\ 201 & 216 & 231 \\ 318 & 342 & 366 \end{matrix}$$

GF =

$$\begin{matrix} 138 & 171 & 204 \\ 174 & 216 & 258 \\ 210 & 261 & 312 \end{matrix}$$

trace(FG) = 666

trace(GF) = 666

Rank

V=[1 2 3;4 5 6;5 7 9]

V =

1	2	3
4	5	6
5	7	9

rank(V) = 2

Det(V)=0

Determinant

W=[1 2 3;4 5 6;5 8 9]

W =

1	2	3
4	5	6
5	8	9

$$|A| = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} |A_{1i}|$$

rank(W) = 3

det(W) = 6

Minors:

det([5 6;8 9]) = -3

det([4 6;5 9]) = 6

det([4 5 ;5 8]) = 7

1*det([5 6;8 9])-2*det([4 6;5 9])+3*det([4 5 ;5 8]) = 6

Determinant of the product

F=[1 2 3;4 5 6;5 8 10]

F =

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 8 & 10 \end{matrix}$$

det(F) = 3

det(W) = 6

det(WF) = 18

Eigenvalues and eigenvectors

C=

0.0267	-0.0333	-0.0067
-0.0333	0.0692	0.0433
-0.0067	0.0433	0.0467

[V D]=(eig(C))

V =

-0.6413	0.6889	-0.3379
-0.6053	-0.1836	0.7745
0.4715	0.7012	0.5348

D =

0.0001	0	0
0	0.0288	0
0	0	0.1136

Eigenvalues and eigenvectors

Trace(C) = 0.1425

sum(diag(D)) = 0.1425

Det(C)=3.3333e-007

prod(diag(D))=3.3333e-007

Where do you find these concepts

- You will find matrices in most problems related to asset management as covariance matrices are essential inputs for modern portfolio theory
- Matrices are also fundamental concepts for understanding the behaviour of time series