Fixed Income Analysis and Risk Management

Session 10 – Revision Session

Exercise 2.2.

The account balance in 7.14 years will be $\$2,500 \times e^{8\% \times 7.14} = \$4,425.98$

Exercise 2.7.

1. The price P of the bond is such as

$$P = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{N}{(1+r)^T} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{N}{(1+r)^T}$$

Here we obtain

$$P = \frac{10}{7\%}\left(1 - \frac{1}{(1+7\%)^5}\right) + \frac{100}{(1+7\%)^5} = 112.301$$

2. Similarly,

YTM	Price
8%	\$107,99
9%	\$103,89
10%	\$100,00

Bond prices decrease as rates increase.

Exercise 2.8

1. The price P is such as $P = \sum_{t=1}^{2T} \frac{C/2}{(1+r/2)^t} + \frac{N}{(1+r/2)^{2T}} = \frac{C}{r}\left(1 - \frac{1}{(1+r/2)^{2T}}\right) + \frac{N}{(1+r/2)^{2T}}$

$$P = \frac{10}{7\%}\left(1 - \frac{1}{(1+7\%/2)^{10}}\right) + \frac{100}{(1+7\%/2)^{10}} = 112.475$$

2.

YTM	Price
8%	$108,11
9%	$103,96
10%	$100,00

Exercise 2.13

1. Prices before and after the change in YTM are

YTM	Price
10%	$81,046
11%	$77,825

Hence the price has changed by (77.825 – 81.046) / 81.046 = -3.97%

2. Prices before and after the change in YTM are

YTM	Price
5%	$100
6%	$95,788

Hence the price has changed by (95.788 – 100) / 100 = -4.21%

3. In low IR environments the relative price volatility of a bond is higher than in high IR environments for the same yield change (in our example, 1%). This is due to the convex relationship between the price of a bond and its yield-to-maturity.

Exercise 2.18

1. The 1-year 0-coupon rate is such as $\frac{110}{1+R(0,1)} = 106.56$ or $R(0,1)$ = 3.228%.

   The 2-year 0-coupon rate is such as $\frac{8}{[1+R(0,1)]} + \frac{108}{[1+R(0,2)]^2} = 106.20$ or $R(0,2)$ = 4.738%.

   The 3-year 0-coupon rate is such as $\frac{8}{[1+R(0,1)]} + \frac{8}{[1+R(0,2)]^2} + \frac{108}{[1+R(0,3)]^3} = 106.45$ or $R(0,3)$ = 5.718%.

2. The price of bond 4 using the 0-coupon YC is

$$\frac{9}{[1+3.228\%]}+\frac{9}{[1+4.738\%]^2}+\frac{109}{[1+5.718\%]^3}=109.177\ .$$

3. The bond is underpriced by the market compared to its theoretical value. There is an arbitrage if the market price of this bond reverts to the theoretical value. We have to simply buy the bond at the $109.01 price and hope that it is mispriced by the market and will soon revert to around 109.177.

Exercise 3.5

1. The implied forward rates for year 1 to year *t* are such that
   - $(1+R(0,2))^2=(1+R(0,1))(1+f(1,2))$ or $f(1,2)=\dfrac{(1+R(0,2))^2}{(1+R(0,1))}-1=4.283\%$
   - $(1+R(0,3))^3=(1+R(0,1))(1+f(1,3))^2$ or $f(1,3)=\left[\dfrac{(1+R(0,3))^3}{(1+R(0,1))}\right]^{1/2}-1=4.698\%$
   - $(1+R(0,4))^4=(1+R(0,1))(1+f(1,4))^3$ or $f(1,4)=\left[\dfrac{(1+R(0,4))^4}{(1+R(0,1))}\right]^{1/3}-1=4.858\%$
   - $(1+R(0,5))^5=(1+R(0,1))(1+f(1,5))^4$ or $f(1,5)=\left[\dfrac{(1+R(0,5))^5}{(1+R(0,1))}\right]^{1/4}-1=4.972\%$

2. The implied forward rates for year 2 to year *t* are such that
   - $(1+R(0,3))^3=(1+R(0,2))^2(1+f(2,3))$ or $f(2,3)=\left[\dfrac{(1+R(0,3))^3}{(1+R(0,2))^2}\right]-1=5.115\%$
   - $(1+R(0,4))^4=(1+R(0,2))^2(1+f(2,4))^2$; $f(2,4)=\left[\dfrac{(1+R(0,4))^4}{(1+R(0,2))^2}\right]^{1/2}-1=5.147\%$
   - $(1+R(0,5))^5=(1+R(0,2))^2(1+f(2,5))^3$; $f(2,5)=\left[\dfrac{(1+R(0,5))^5}{(1+R(0,2))^2}\right]^{1/3}-1=5.203\%$

3. The implied forward rates for year 3 to year *t* are such that
   - $(1+R(0,4))^4=(1+R(0,3))^3(1+f(3,4))$; $f(3,4)=\left[\dfrac{(1+R(0,4))^4}{(1+R(0,3))^3}\right]-1=5.179\%$

- $(1+R(0,5))^5 = (1+R(0,3))^3(1+f(3,5))^2$; $f(3,5) = \left[\frac{(1+R(0,5))^5}{(1+R(0,3))^3}\right]^{1/2} - 1 = 5.247\%$

2. According to the pure expectations theory of the term structure of interest rates, the forward rate is an unbiased estimator of the future spot rates; i.e., $f(1,2)$ = 4.283% means that the market expects the 1-year rate to be equal to 4.283% in a year. Similarly the short (1-year) rate is expected to be equal to
   - 5.115% in 2 years,
   - 5.179% in 3 years.

   So the market expects the short (1-year) rates to rise in the future (relative to today's level of 3.702%). This is associated with scenarios of economic recovery.

   Similarly, the market expects
   - the 2-year rates in years 1, 2 and 3 (resp., 4.698%, 5.147% and 5.247%) to be more than the 2-year rate today (3.992%),
   - the 3-year rates in years 1 and 2 (resp., 4.858% and 5.203%) to be more than the 3-year rate today (3.992%),
   - the 4-year rate in year 1 (4.972%) to be more than the 4-year rate today (4.568%).

Exercise 4.2

Using the no-arbitrage relation we obtain the following equations for the 3 bond prices

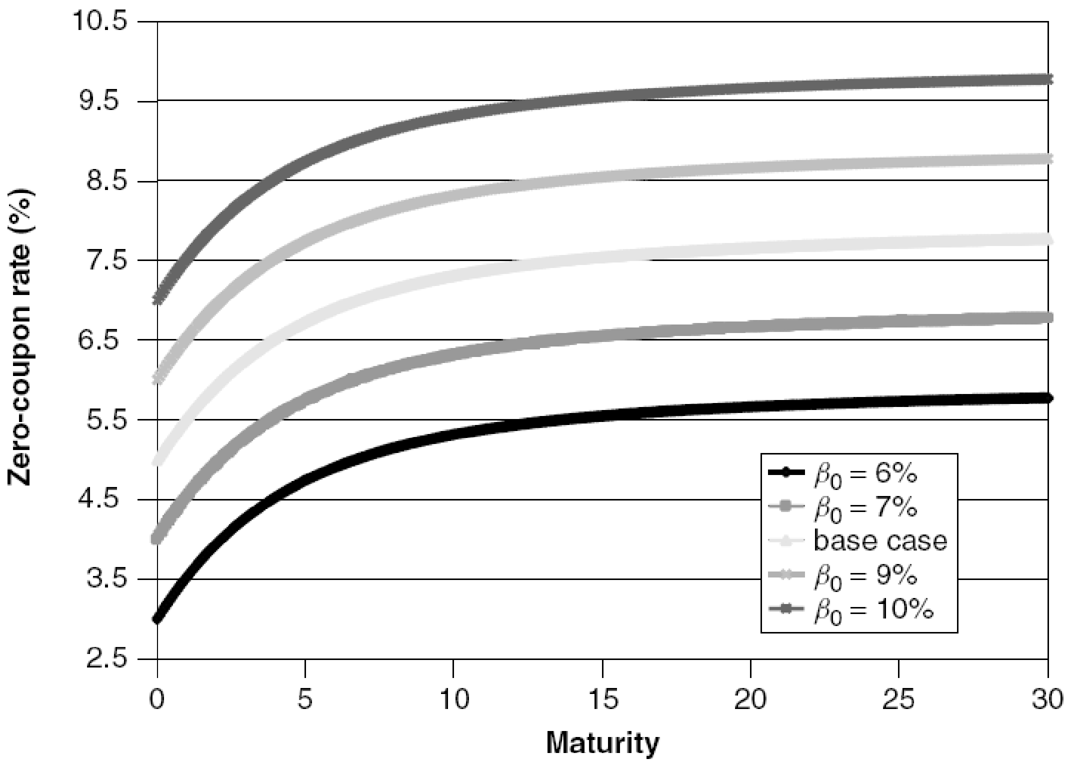
$$\begin{cases} 108 = 10 \times B(0,1) + 110 \times B(0,2) \\ 100.85 = 7.5 \times B(0,1) + 7.5 \times B(0,2) + 107.5 \times B(0,3) \\ 103.5 = 8.5 \times B(0,1) + 8.5 \times B(0,2) + 108.5 \times B(0,3) \end{cases}$$

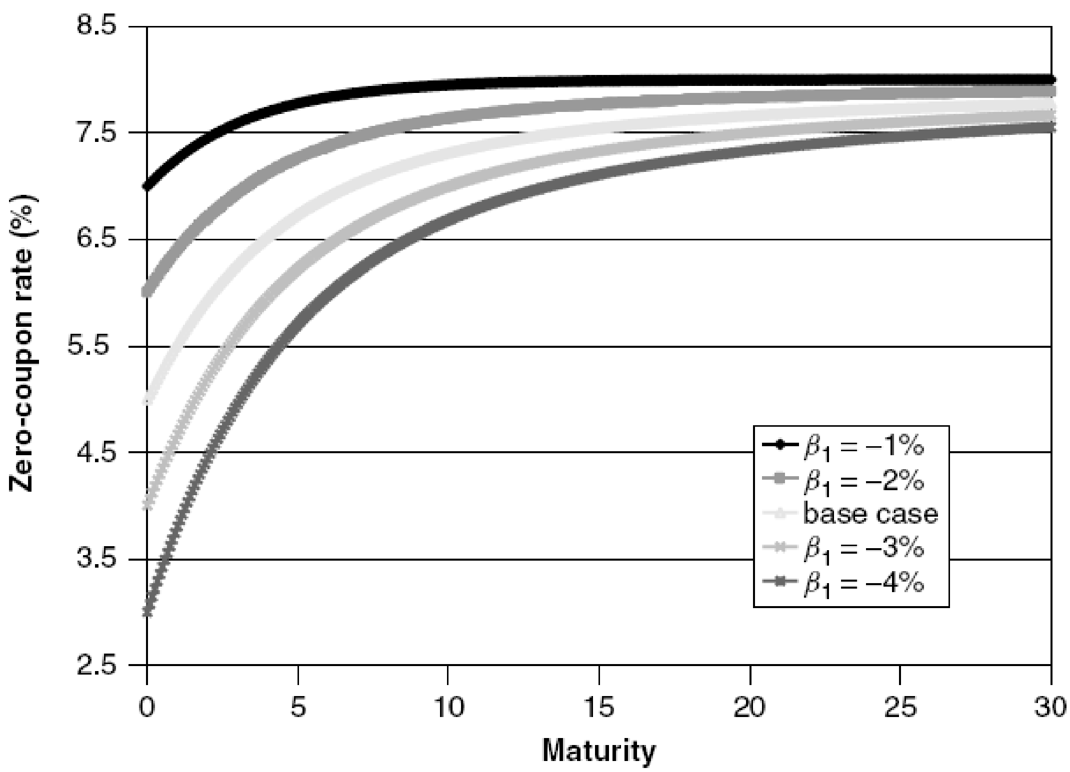
Or in matrix form $\begin{pmatrix} 108 \\ 100.85 \\ 103.5 \end{pmatrix} = \begin{pmatrix} 10 & 110 & \\ 7.5 & 7.5 & 107.5 \\ 8.5 & 8.5 & 108.5 \end{pmatrix} \times \begin{pmatrix} B(0,1) \\ B(0,2) \\ B(0,3) \end{pmatrix}$

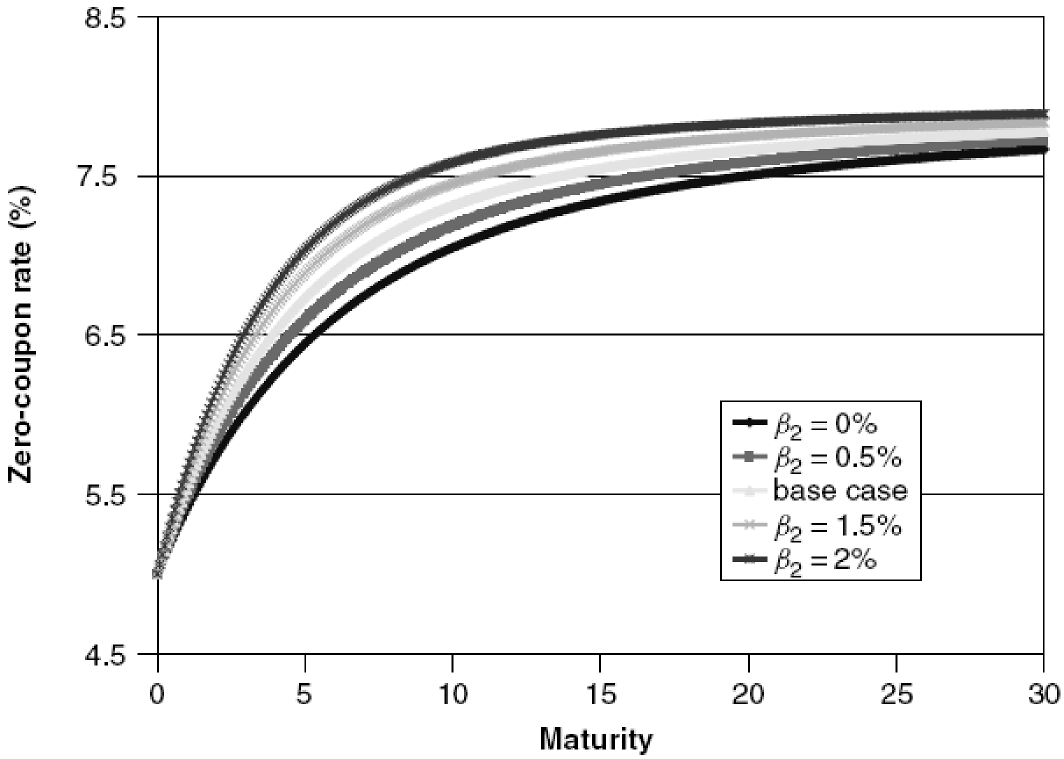
We get the following discount factors and discount rates

$$\begin{pmatrix} B(0,1) \\ B(0,2) \\ B(0,3) \end{pmatrix} = \begin{pmatrix} 0.944275 = 1 \div (1+R(0,1)) \\ 0.895975 = 1 \div (1+R(0,2))^2 \\ 0.80975 = 1 \div (1+R(0,3))^3 \end{pmatrix} \text{ and } \begin{pmatrix} R(0,1) \\ R(0,2) \\ R(0,3) \end{pmatrix} = \begin{pmatrix} 5.901\% \\ 5.646\% \\ 7.288\% \end{pmatrix}$$

Exercise 4.14







Exercise 5.8

1. Last coupon date: 15 May 2000
   Settlement date: 13 Sept 2000
   Next coupon date: 15 Nov 2000

   Number of days since last coupon payment = 16 (May) + 30 (June) + 31 (July) + 31 (Aug) + 13 (Sept) = 121
   Number of days in between 2 coupon payments = 16 (May) + 30 (June) + 31 (July) + 31 (Aug) + 30 (Sept) + 31 (Oct) + 15 (Nov) = 184

   Accrued interests = 121 / 184 × 7% / 2 = 2.30163%

   To compute the dirty price, we calculate the PV(CF) using the YTM of 6.75%.

   Number of days till next coupon payment = 17 (Sept) + 31 (Oct) + 15 (Nov) = 63

Date	Nber of semi-annual periods t	F_t	$\frac{F_t}{(1+y/2)^t}$	$\frac{tF_t}{(1+y/2)^{t+1}}$
15/11/2000	0.34239	3.5	3.4604	1.1461
15/05/2001	1.34239	3.5	3.3475	4.3469
15/11/2001	2.34239	3.5	3.2382	7.3375
15/05/2002	3.34239	3.5	3.1325	10.1281
15/11/2002	4.34239	103.5	89.6071	376.4055
		Price	102.7857	399.3641
		MD		1.9427

with

$$\$Duration = \frac{1}{2} \times \sum_{t=1}^{2N} \left[(-t) F_t \left(1 + \frac{y}{2} \right)^{-t-1} \right]$$

$$MD = -\frac{\$Duration}{P} = \frac{\sum_{t=1}^{2N} \left[tF_t \left(1 + \frac{y}{2} \right)^{-t-1} \right]}{2P}$$

The dirty price is 102.7857.
The clean price is 102.7857 – 2.30163 = 100.48405
The MD is 1.9427

2. Approximation of $dP = \$Duration \times dy$ = -MD $\times P \times dy$

dy	dP	New P
0.25%	-0.49921	102.2865
-0.25%	0.49921	103.2849

3. Let us discount the CF at YTM = 7%

Date	Nber of semi-annual periods t	F_t	$\frac{F_t}{(1+y/2)^t}$
15/11/2000	0.34239	3.5	3.4590
15/05/2001	1.34239	3.5	3.3420
15/11/2001	2.34239	3.5	3.2290
15/05/2002	3.34239	3.5	3.1198
15/11/2002	4.34239	103.5	89.1381
		Price	102.2881

Even though YTM = c, the price differs from par since we are pricing the bond in between 2 coupon periods and not at the coupon payment date. So the discounting takes place over a fraction of a semi-annual period (as opposed to an integer number of semi-annual periods), while the payment of the coupons takes place every 6 months.

Actual prices and changes in prices:

YTM	*P*	*dP*
7%	102.2881	-0.4976
6.50%	103.2865	0.5008

$Duration gives a good approximation of the actual change in prices for small *dy*. At 7%, the price went down by slightly less than what was predicted. At 6.5%, the price went up by slightly more than was predicted. This is because duration only gives a linear approximation to the price change. For large changes in yield, there will be errors due to the convex relationship between price and yield.

Exercise 6.4

1. The dirty price and *MD* of each bond are given by
   - $P = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{Par}{(1+y)^T}$,
   - $\$Duration = -\sum_{t=1}^{T} \frac{tF_t}{(1+y)^{t+1}}$; i.e., 1st derivative of the price function with respect to the YTM,
   - $MD = -\frac{\$duration}{P}$.

   For example for bond 2,

Timing of *F* (*t*)	F_t	$\frac{F_t}{(1+y)^{t+1}}$	$\frac{tF_t}{(1+y)^{t+1}}$
1	6	5.3907	5.3907
2	6	5.1097	10.2194
3	6	4.8433	14.5299
4	6	4.5908	18.3632
5	6	4.3515	21.7574
6	6	4.1246	24.7477
7	6	3.9096	27.3672
8	6	3.7058	29.6462
9	6	3.5126	31.6133
10	106	58.8205	588.2051
		$Duration	-771.8400
		MD	7.4381

Bond	Price	MD
1	100	1.859
2	103.769	7.438
3	113.765	13.394

2. a) The YTM of each bond decreases instantaneously by 0.2%.

The new price is calculated using $P = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{Par}{(1+y)^T}$.

The 1st order Taylor expansion is used to approximate dP as:
$dP = \$Duration \times dy = -MD \times P \times dy$.
The estimated price is then calculated as $P - MD \times P \times dy$.

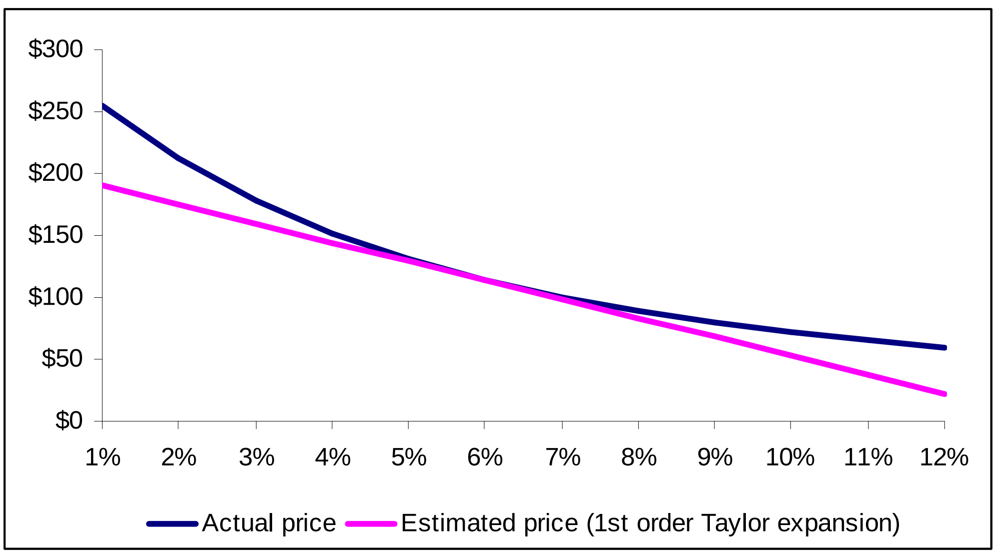
Bond	YTM	New price	Actual *dP*	1st order approximation of *dP*	Estimated price	Pricing error
1	4.80%	100.3729	0.3729	0.3719	100.3719	0.001
2	5.30%	105.3273	1.5585	1.5437	105.3125	0.015
3	5.80%	116.8774	3.1126	3.0475	116.8124	0.065

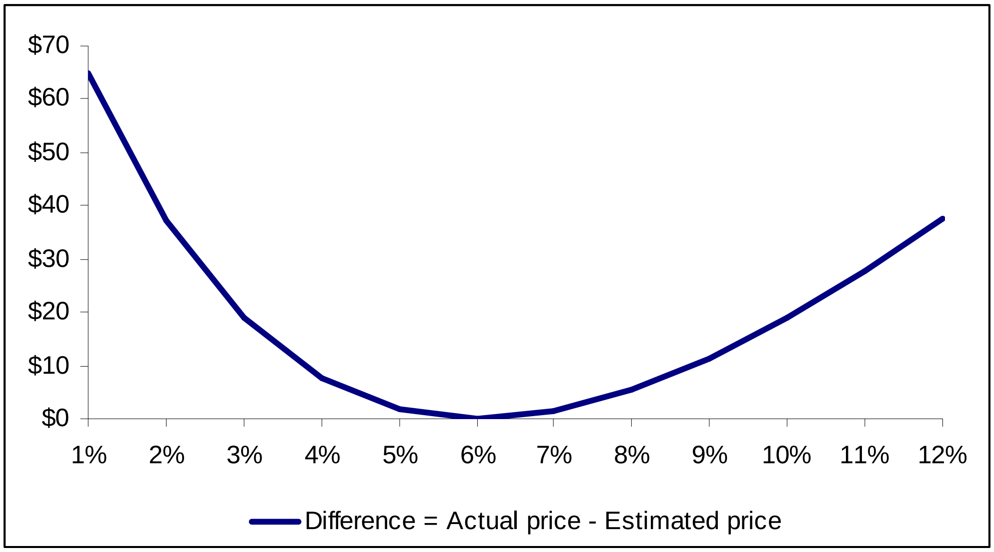
b) The YTM of each bond decreases instantaneously by 1%

Bond	YTM	New price	Actual *dP*	1st order approximation of *dP*	Estimated price	Pricing error
1	4.00%	101.8861	1.8861	1.8594	101.8594	0.027
2	4.50%	111.8691	8.1003	7.7184	111.4872	0.382
3	5.00%	130.7449	16.9801	15.2377	129.0025	1.742

The 1st order approximation only works well for small *dy*.

c)





3. $Convexity is the second derivative of the price function with respect to the YTM

$$\$Convexity = \sum_{t=1}^{T} \frac{t(t+1)F_t}{(1+y)^{t+2}}$$

$$RC = \frac{\$Convexity}{P}$$

	Bond 1 (c = 5%, T = 2)		Bond 2 (c = 6%, T = 10)		Bond 3 (c = 7%, T = 30)	
t	F_t	$\frac{t\times(t+1)\times F_t}{(1+y)^{t+2}}$	F_t	$\frac{t\times(t+1)\times F_t}{(1+y)^{t+2}}$	F_t	$\frac{t\times(t+1)\times F_t}{(1+y)^{t+2}}$
1	5	8.64	6	10.22	7	11.75
2	105	518.30	6	29.06	7	33.27
3			6	55.09	7	62.77
4			6	87.03	7	98.69
5			6	123.74	7	139.66
6			6	164.20	7	184.46
7			6	207.52	7	232.02
8			6	252.91	7	281.43
9			6	299.65	7	331.88
10			106	6132.94	7	382.67
11					7	433.21
12					7	482.99
13					7	531.60
14					7	578.66
15					7	623.89
16					7	667.05
17					7	707.96
18					7	746.46
19					7	782.45
20					7	815.86
21					7	846.65
22					7	874.80
23					7	900.31
24					7	923.20
25					7	943.52
26					7	961.33
27					7	976.67
28					7	989.64
29					7	1000.31
30					107	15419.81
$Convexity		526.94		7362.37		31964.99
RC		5.27		70.95		280.97

4. The 1st and 2nd order Taylor expansion is used to approximate dP as

$$dP = \$Duration \times dy + \frac{1}{2}\times \$Convexity \times (dy)^2 = -MD \times P \times dy + \frac{1}{2}\times \$Convexity \times (dy)^2$$

The estimated price is then calculated as $P - MD \times P \times dy + \frac{1}{2}\times \$Convexity \times (dy)^2$

Bond	YTM	New price	Actual dP	1st and 2nd order approximation of dP	Estimated price	Pricing error
1	4.00%	101.8861	1.8861	1.8858	101.8858	0.000
2	4.50%	111.8691	8.1003	8.0865	111.8553	0.014
3	5.00%	130.7449	16.9801	16.8359	130.6008	0.144

Exercise 8.2

Compute

- the \$Duration of the bonds as $\$Duration = -\sum_{t=1}^{T} \frac{tF_t}{(1+y)^{t+1}}$,
- the *MD* of the bonds as $MD = -\frac{\$Duration}{P}$,
- the absolute change in prices as $dP = \$Duration \times dy$ where dy = -0.3%,
- the relative change in prices as $\frac{dP}{P} = \frac{\$Duration}{P} \times dy = -MD \times dy$.

Maturity T	Coupon rate	YTM	Price	\$Duration	Absolute gain (dP)	MD	Relative gain (dP/P)
5	7%	4%	113.355	-482.70	1.448	4.2583	0.013
7	6%	4.50%	108.839	-621.62	1.865	5.7113	0.017
15	8%	5%	131.139	-1248.44	3.745	9.5200	0.029
20	5%	5.25%	96.949	-1194.64	3.584	12.3223	**0.037**
22	7%	5.35%	121.042	-1463.99	**4.392**	12.0948	0.036

Exercise 8.3

1. The price of the 183-day T-bill is \$97.817, so that buying and holding it till maturity will provide a total return of

$$\frac{100 - 97.817}{97.817} = 2.231\%.$$

The alternative is to ride the yield curve by buying the 274-day T-bill and holding it for 183 days. If we assume, as the "riding the YC" strategy does, that the YC is going to remain unchanged over the next 183 days, we will be able to sell the 274-day T-bill in 183 days at 99.027 (the price today of the 91-day T-bill). The total return of the strategy is then

$$\frac{99.027 - 96.542}{96.542} = 2.574\%;$$

that is, 0.343% surplus of total return compared to the buy-and-hold strategy.

2. If however the 0-coupon YC rises by 2%, the price of the 91-day T-bill will fall to

$$\frac{100}{(1+6\%)^{\frac{91}{360}}} = 98.5379.$$

So riding the YC only brings a total return rate of

$$\frac{98.5379 - 96.542}{96.542} = 2.067\% .$$

In this case, riding the YC is worse than the buy-and-hold strategy.

Exercise 8.7

1.

Maturity	Coupon rate	YTM	Price	$Duration	Quantity
2	6%	6%	100	-183.34	q_s
10	6%	6%	100	-736.01	-10,000
30	6%	6%	100	-1,376.48	q_l

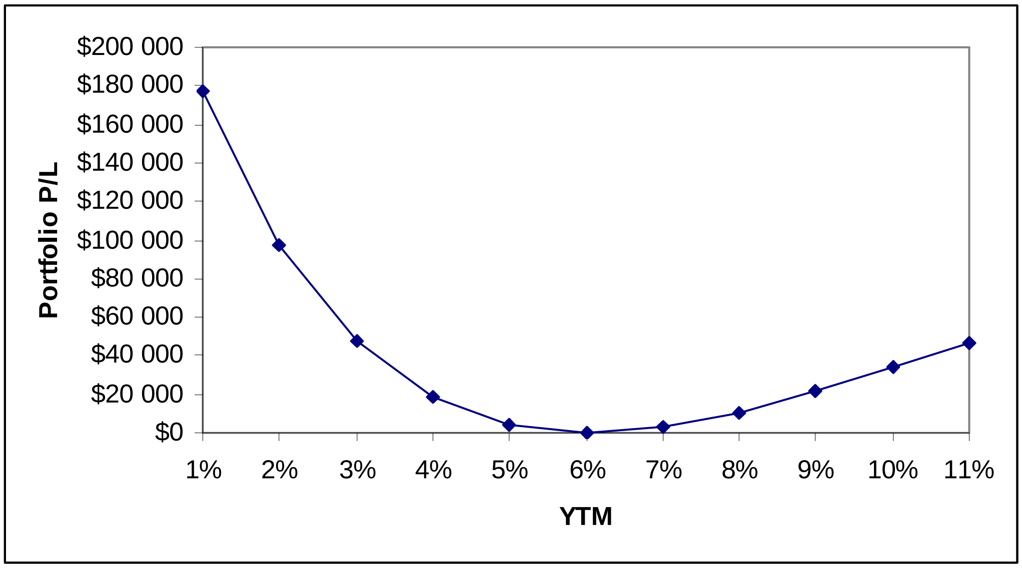
q_s and q_l are such as the portfolio is cash and $duration neutral; i.e.,

$$\begin{cases} q_s \times 183.34 + q_l \times 1{,}376.48 = 10{,}000 \times 736.01 \\ q_s \times 100 + q_l \times 100 = 10{,}000 \times 100 \end{cases} \quad \text{or} \quad \begin{cases} q_s = 5{,}368 \\ q_l = 4{,}632 \end{cases}$$

2. If the YTM rises to 7% or falls to 5%, the portfolio benefits due to its positive convexity

Maturity	Coupon rate	YTM	Price	$Duration	Quantity	Price if YTM = 7%	Price if YTM = 5%
2	6%	6%	100	-183.34	5368	98.19	101.86
10	6%	6%	100	-736.01	-10000	92.98	107.72
30	6%	6%	100	-1376.48	4632	87.59	115.37
					Portfolio	$3,051.70	$3,969.16

3. The butterfly has a positive convexity. Whatever the value of the YTM, as long as the YC moves in a parallel way, the strategy always generates a gain. This gain is all the more substantial as the YTM reaches a level further away from 6%.


$200 000
$180 000
$160 000
$140 000
$120 000
$100 000
$80 000
$60 000
$40 000
$20 000
$0
Portfolio P/L
1%
2%
3%
4%
5%
6%
7%
8%
9%
10%
11%
YTM