## Senior Problems: Specialist Proofs & Matrices

CGS Maths Club

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**Problem 1.** Suppose n is a positive integer. How many values of n is  $9n^2 - 4$  prime for?

**Problem 2.** Let  $A = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , and  $C = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ .

- a. Describe the transformations done by each of the above matrices.
- b. Let D = CA. Find the value of D.
- c. Let E = DB. Show that  $E = \begin{bmatrix} -3 & 3 \\ 3 & 3 \end{bmatrix}$
- d. Let the triangle POQ have vertices (0,0), (-15,15) and (51,75) respectively. Find the vertices of the resulting triangle when POQ is transformed by E: P'O'Q.
- e. Finally, find the area of P'O'Q and hence find the area of POQ using matrix transformations.

**Problem 3.** Prove that  $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^k = \begin{bmatrix} k+1 & k \\ -k & -k+1 \end{bmatrix}$  given  $k \in \mathbb{Z}$ 

Problem 4.

- a. Prove that  $\frac{n-1}{n} \frac{n-2}{n-1} = \frac{1}{n(n-1)}$
- b. Using part **a**, evaluate  $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{100\times 101}$
- c. Using mathematical induction, prove that  $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- d. Using parts **b** and **c** evaluate  $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+100}$

**Problem 5.** Let  $P(n) = 4^n + 6n - 1$ . Prove that P(n) is divisible by 3 for any  $n \in \mathbb{Z}^+$ 

Email Garv if you have any question or want the solutions:)