Maths Club - Term 2 Week 6 - Senior
Problem from the 2020 Oxford Uni Maths Admission Test

2. For **ALL APPLICANTS.**

The functions $f(n)$ and $g(n)$ are defined for positive integers n as follows:

$$f(n) = 2n + 1, \qquad\qquad g(n) = 4n.$$

This question is about the set S of positive integers that can be achieved by applying, in some order, a combination of fs and gs to the number 1. For example as

$$gfg(1) = gf(4) = g(9) = 36,$$

and

$$ffgg(1) = ffg(4) = ff(16) = f(33) = 67,$$

then both 36 and 67 are in S.

(i) Write out the binary expansion of 100 (one hundred).

[*Recall that binary is base* 2. *Every positive integer* n *can be uniquely written as a sum of powers of* 2, *where a given power of* 2 *can appear no more than once. So, for example,* $13 = 2^3 + 2^2 + 2^0$ *and the binary expansion of* 13 *is* 1101.]

(ii) Show that 100 is in S by describing explicitly a combination of fs and gs that achieves 100.

(iii) Show that 200 is not in S.

(iv) Show that, if n is in S, then there is only one combination of applying fs and gs in order to achieve n. (So, for example, 67 can only be achieved by applying g then g then f then f in that order.)

(v) Let u_k be the number of elements n of S that lie in the range $2^k \leqslant n < 2^{k+1}$. Show that

$$u_{k+2} = u_{k+1} + u_k$$

for $k \geqslant 0$.

(vi) Let s_k be the number of elements n of S that lie in the range $1 \leqslant n < 2^{k+1}$. Show that

$$s_{k+2} = s_{k+1} + s_k + 1$$

for $k \geqslant 0$.