Senior Problems: Specialist Proofs & Matrices

CGS Maths Club

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Problem 1. Suppose n is a positive integer. How many values of n is $9n^2 - 4$ prime for?

Problem 2. Let $A = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

a. Describe the transformations done by each of the above matrices.

b. Let $D = CA$. Find the value of D.

c. Let $E = DB$. Show that $E = \begin{bmatrix} -3 & 3 \\ 3 & 3 \end{bmatrix}$.

d. Let the triangle POQ have vertices $(0, 0)$, $(-15, 15)$ and $(51, 75)$ respectively. Find the vertices of the resulting triangle when POQ is transformed by E: $P'O'Q$.

e. Finally, find the area of $P'O'Q$ and hence find the area of POQ using matrix transformations.

Problem 3. Prove that $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^k = \begin{bmatrix} k+1 & k \\ -k & -k+1 \end{bmatrix}$ given $k \in \mathbb{Z}$

Problem 4.

a. Prove that $\frac{n-1}{n} - \frac{n-2}{n-1} = \frac{1}{n(n-1)}$

b. Using part **a**, evaluate $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{100\times101}$

c. Using mathematical induction, prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

d. Using parts **b** and **c** evaluate $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+100}$

Problem 5. Let $P(n) = 4^n + 6n - 1$. Prove that $P(n)$ is divisible by 3 for any $n \in \mathbb{Z}^+$

Email Garv if you have any question or want the solutions :)