

Algorithmics SAT - Friendship Network Part 3

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Abstract

‘How can a tourist best spend their day out?’. The first part of this SAT project aimed to model the Victorian public transport network and its proximity to friends’ houses in order to construct an algorithm and the second part considered the time complexity of said algorithms and analysed their impact on real life use-cases. In Part 3, we will finally design an improved algorithm for the original problem using more advanced algorithm design techniques

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This section of the Algorithmics SAT focuses improving the original data model and algorithm to solve the original problem more efficiently and effectively.

Throughout the analysis, note the following variables are used as shorthand:

Let F = number of friends

Let L = number of landmarks

Let R = number of routes

Suggested Improvements

From Part 2, there were various possible optimisations that became evident from the time complexity analysis. These read as follows:

1. The [current implementation of Dijkstra's](#) is far from optimal: the current algorithm has a cubic time complexity but with a min priority queue this can supposedly be reduced to $O(L + R \log L)$.
2. The abstraction of [soonest_time_at_node](#) can be implemented as a dictionary that is accessed in constant time but is currently implemented as two for loops that makes the [dist](#) function more complex than necessary.
3. The biggest optimisation needed is the caching of the Held-Karp outputs, meaning that subpaths are calculated once only, and all subsequent subpaths will be read in $O(1)$ time (basically dynamic programming by definition). This should probably help the factorial time complexity, though it might be hindered by the fact that a different starting time means that the whole subpath is different which decreases how effective this optimisation is.
4. Finally, it may be worth considering approximate solutions. This being said, the scope of the problem to solve does *just* fit into the practical input sizes that the algorithm allows, but definitely limits its usefulness and real world use cases. In many times, the *best* solution is not needed, just a relatively good one.

The first three can be implemented and compared relatively easily, so they will be the focus of this section.

Improving Dijkstra's Implementation

As stated above, the current implementation of Dijkstra's is naïve because each iteration of the while loop requires a scan over all edges to find the one with the minimum distance, but the relatively small change of using a [heap](#) as a min priority queue allows us to find the edge with minimum distance faster. In terms of the [pseudocode](#), this just means turning `unexplored_list` into a min priority queue, where the priority is based on the distance to the node.

Note that even though the `unexplored_list` simply appears as a priority queue in the pseudocode, for this change to be beneficial the priority queue data structure must itself be implemented efficiently, using something like a [heap](#).

See the [modified version of Dijkstra's](#) for the pseudocode.

Heaps In most implementations (such as the Python implementation we will be testing with), the inner workings of how a min priority queue works will be abstracted and hence doesn't *need* to be worried about. Nonetheless, it is worth exploring how they are actually implemented, a popular method being min heaps!

A heap is a special tree-based data structure in which the tree is a complete binary tree. In other words, each node has exactly two children and every level will be completely filled, except possibly the deepest level. In a min heap, the parent nodes are always smaller than their children, meaning that the root node is the very smallest element.

Interestingly, since there are no gaps in the tree, the heap can actually be stored simply as an array with additional logic for adding and removing from the priority queue.

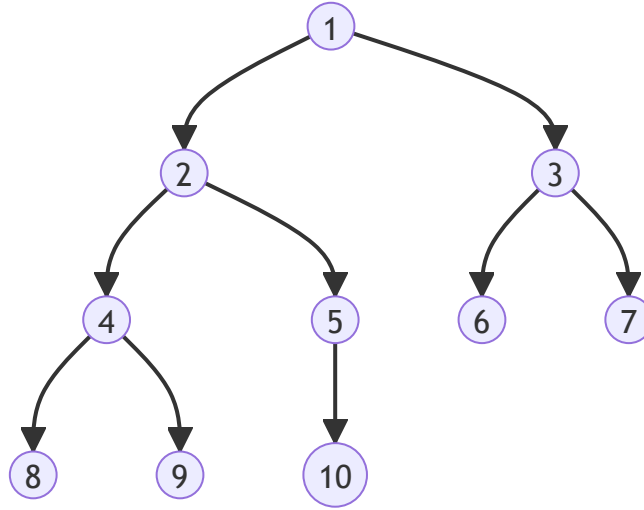


Figure 1: Complete Binary Tree

Insertion When inserting an element, it goes in the next empty spot looking top to bottom, left to right. If that’s not where the element should actually go, we can “bubble it up” until it is, meaning that we can swap that element with its parent node repeatedly until it has gone up the tree enough to be in the correct position. Since it is a binary tree, we can do this in $O(\log n)$ time.

Deletion Since we would want to remove the smallest node, this would of course be the root node. Removing the root node would create an empty spot, so when we remove the root, we instead fill that with the last element added. Similar to above, since this element might not be in the right spot, we take that element and “bubble it down” until it is, this time swapping with the smaller of the two children repeatedly. Similar to above, we can do this in $O(\log n)$ time.

Improvement

Visit Set Size	Initial Algorithm (s)	Improved Dijkstra’s (s)
8	1.4038	1.2842
9	3.9718	3.9315

All times are the average of 10 trials. Evidently, the improvement is slight, if any improvement at all.

Improving Distance Function

To find the `soonest_time_at_node`, the original Pythonic implementation was using a nested for loop to find when the next train/bus would arrive. This is thoroughly inefficient, namely due to the amount of times that the `dist` function is called, meaning that there would be a lot of overlap. This *could* be improved using dynamic programming, but since there is a fixed amount of time in a day (24 hours), it doesn’t actually take that long to precompute this waiting time and store it along with the rest of our data. The pseudocode for this function is below:

```

1 time_data = dictionary of dictionaries
2

```

```

3 for line in line_data:
4     for start_node in line_data[line]['timetable']:
5         for current_time in every minute of a day:
6             // calculate next time at node
7             for arrival_time at start_node:
8                 if arrival_time >= current_time and is first:
9                     next_time = arrival_time
10
11             wait_time = next_time - current_time
12             add wait_time to time_date

```

This produces a rather large dictionary of wait times, but the change to $O(1)$ time complexity pays off, even if space complexity is sacrificed.

Improvement

Visit Set Size	Initial Algorithm (s)	Improved Dijkstra's (s)	Improved Dist (s)
8	1.4038	1.2842	0.2746
9	3.9718	3.9315	2.2123
10	27.8881		24.4954

All times are the average of 10 trials and improvements are cumulative. The improvement seems quite large for smaller visit set sizes, but evidently this does not influence the Big O much as $\lim n \rightarrow \infty$.

Improving Held-Karp Implementation

Maybe the biggest flaw in the initial algorithm is that Held-Karp did not use dynamic programming. Due to the way Held-Karp works (explained previously), there are many overlapping problems and without the caching of these outputs, they will be calculated repeatedly unnecessarily. Since this main function is what contributes to the majority of the time complexity, improving it should make the algorithm scale better.

As we did with Dijkstra's in Part 1, caching can be done with an intermediary function, `fetch_hk`, which only runs `held_karp` if the value hasn't already been stored.

The pseudocode for this process is relatively simple and [can be found below](#).

After being implemented in Python, `fetch_hk` resembles the following:

```

1 def fetch_hk(start, end, visit):
2     name = f"{start}-{end}-{visit}@{current_time}"
3
4     global cached_hk
5     if name not in cached_hk:
6         cached_hk[name] = held_karp(start, end, visit, current_time)
7
8     return cached_hk[name]

```

Improvement

Visit Set Size	Initial Algorithm (s)	Improved Dijkstra's (s)	Improved Dist (s)	Improved Held-Karp (s)
8	1.4038	1.2842	0.2746	0.0264
9	3.9718	3.9315	2.2123	0.0579
10	27.8881		24.4954	0.1460
11				0.2339
12				0.5172
13				1.2122
14				2.8075

All times are the average of 10 trials and improvements are cumulative. The improvement from this change is much better than the previous changes, likely changing our Big O time from factorial to exponential, as seen by the roughly doubling running times. This can be verified by creating a line of best fit from the data above, which works out to be $t(n) \approx a^{n-b}$ where $a = 2.29792$ and $b = 12.7609$. This has an R^2 value of 0.9996, which provides us with a relatively high confidence that the new algorithm has $\Theta(2^n)$. According to this line of best fit, $n = 20$ would take about 7 minutes and 53 seconds, while $n = 30$ would take almost 3 weeks.

It is worth noting that although this does improve the time complexity by a large factor, the cache also takes up a lot of space, making the space complexity worse. This tradeoff is quite good in most cases since modern devices have plenty of memory and storage, but in the case that space complexity is a constraint, this may be an unideal optimisation.

Practicalities of an Exact Algorithm

Though the algorithm has seen a dramatic improvement from factorial time to likely exponential time, it still maintains a lot of the issues that the previous version possessed. Namely, because exponential time still does not scale very well, the practical input size for n is still very limited, changing from about $n \leq 9$ to $n \leq 14$.

As stated in Part 2, this is mostly sufficient for the specific use case of the problem outlined in most cases since the amount of friends people would hang out with in this fashion is intrinsically small, as it only applies itself to close friends. Because of this, even if someone does have a large amount of close friends, it is unlikely that the visit set that gets computed is larger than 14 (the current input data has 18 friends but a visit set of size 7). As such, for the practical cases of this specific problem, the exact algorithm is sufficient, and also works for adjacent scenarios such as mapping applications (Google Maps, etc) wanting to have certain pickup points along the way.

The algorithm begins to become impractical once the problem is scaled up more as a general solution for the TSP. For example, if a truck driver for a logistics company wanted an optimal route given a list of pickup points, this would very quickly surpass the practical limit of $n \leq 14$, and the graph would be much larger as well. In wider applications like this, using an exact algorithm is simply not useful and we would rather want paths that have a “small enough” cost but have a feasible time complexity. This is where we get into the realm of [approximate algorithms](#).

Tractability

It is important to note that the problem that was initially described can simply be generalised as the Travelling Salesman Problem, which is famously NP-Hard meaning that there is no known polynomial time solution for the problem.

Due to the fact that our final exact algorithm implementation had its execution time double every time n was increased by 1, it is safe to assume that the algorithm runs in exponential time at best, meaning that it is still considerably intractable for large inputs due to the exponential growth.

From this, it is clear that the problem does not become tractable based on the above implementation and it will be hard to make an exact algorithm that is much faster. This is why [approximate algorithms](#) are worth considering, namely those that have performance guarantees of worst cases that are within a certain factor of the minimal cost solution. They provide a trade-off between speed and optimality, and while they make the problem more tractable than exact algorithms, they do not make it completely tractable due to their approximate nature and how they do not always produce the optimal solution.

Approximate/Heuristic Algorithms

The general idea of most approximation algorithms is we can start with an initial candidate solution and then keep making changes to see if we can get better. The initial candidate solution need not be good, but it would certainly help produce results closer to the global optimum after a certain amount of iterations.

One of the most intuitive ideas to generate an initial candidate solution would be to visit the closest node in the visit set from any given node, and this can more formally be described as the [Nearest Neighbour Heuristic](#).

Nearest Neighbour Heuristic

The Nearest Neighbour (NN) algorithm is a greedy (and somewhat naïve) approach where the closest unvisited city is selected as the next destination. This method produces a reasonably short route, but usually not the optimal one. The informal steps of this approximate algorithm are listed below:

1. Mark every vertex as unvisited.
2. Set the starting vertex as the current vertex **u**, marking it as visited.
3. Find the shortest outgoing edge from **u** to an unvisited vertex **v**.
4. Set **v** as the current vertex **u** and mark it as visited.
5. If all vertices have been visited, terminate, if not, go to step 3.

This is a very simple algorithm, but as is the case with most greedy approaches, it can quite easily miss shorter routes. For this specific use case step 3 may cause a few issues in terms of time complexity, as unlike the normal TSP, our graph is not complete. This means that at this step, we would need to run Dijkstra's at every single node in the graph and then sort them to find the shortest path, which is inefficient.

To make this slightly faster we *could* simply choose the first unvisited node in the visit set to go to, but that would still require Dijkstra's to run at every node to find a path, meaning that only the time spent sorting would be saved (which is minimal since Dijkstra's will already have them sorted from the min heap). The problem with this approach is that it will produce a less optimal solution, causing the algorithm to have to spend a longer amount of time improving the solution in the simulated annealing phase. This means that it is a bit of a tradeoff, and for now the shortest node will be chosen.

To avoid using Dijkstra's at all, it is worth considering candidate solutions based on the MST, such as those created by Christofides' Algorithm, which may turn out to be faster. This can be further considered to optimise the algorithm, but for simplicity's sake, the NN Heuristic will be continued with.

Below is the pseudocode to generate an initial candidate solution. Note that in this pseudocode, `fetch_djk` only has the input of the starting node and visit set and returns the path to the closest node in the visit set, so it is a slightly modified version of the `fetch_djk` outlined above.

```

1 // creates a candidate solution using the NN Heuristic
2 function candidate_solution (
3     start: node,
4     end: node,
5     visit: set of nodes,
6     current_time: datetime,
7 ):
8     path = [start]
9     current_vertex = start
10    cost = 0
11
12    while len(visit) != 0:
13        closest_node = fetch_djk(current_vertex, visit, current_time)
14        path.add(closest_node)
15        cost += closest_node.cost
16        visit.remove(closest_node)
17        current_vertex = closest_node
18
19    // go back to the end node
20    closest_node = fetch_djk(current_vertex, end, current_time)
21    path.add(closest_node)
22    cost += closest_node.cost
23
24    return {'path': path, 'cost': cost}
25 end function

```

Pairwise Exchange

Once we have an initial candidate solution that has a reasonable cost for the traversal, a natural question to ask is “how can we make it better?” More specifically, it is worth considering how we could make modifications to generate a better solution.

One way to do this is random swapping, where we randomly pick two cities in the current tour order, and swap them. The goal is to see if these random swaps will ever create a lower cost tour, and if they do, we can accept the new solution. This is a form of the Hill Climbing heuristic, where we keep moving around the sample space to see if we can improve our solution at all.

A slightly more sophisticated technique than randomly swapping the nodes is a method called Pairwise Exchange or 2-opt. The main idea is that we can select any two edges and reconfigure them in the only other way possible with the hopes that this may result in a lower cost tour.

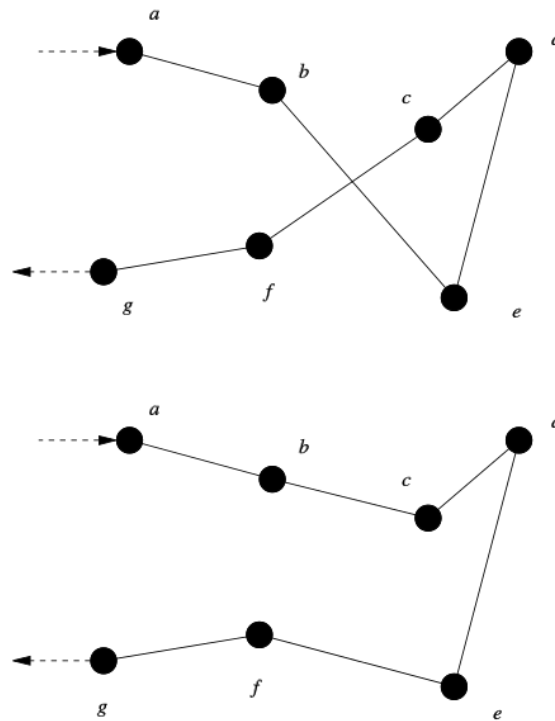


Figure 2: Demonstration of the 2-opt Technique

For example, in the diagram above, it can be seen that the pairs $b - e$ and $c - f$ cross over each other, so the edges can be swapped so that they do not.

More simply, when imagined as a one dimensional array, this could be viewed as the following transformation where we simply reversed the order of the path $e \leftrightarrow d \leftrightarrow c$:

1. $a \leftrightarrow b \leftrightarrow e \leftrightarrow d \leftrightarrow c \leftrightarrow f \leftrightarrow g$
2. $a \leftrightarrow b \leftrightarrow c \leftrightarrow d \leftrightarrow e \leftrightarrow f \leftrightarrow g$

In essence, this “untangles” our candidate solution and can go through all the possible edge combinations much faster than simply randomly switching nodes (which has a much lower chance of being any better).

It is worth noting that the 2-opt technique (where 2 edges are selected and reconfigured) can actually

be extended to any number of edges, known as k -opt for k edges. It might be worth working with a larger amount of edges (3-opt for example), but for simplicity's sake, 2-opt will be the one continued with.

The above notion of reversing the order of a certain path can be expanded upon to develop our pseudocode. The informal steps for this process are listed below:

Let u and v be the first vertices of the edges that are to be swapped. Let tour be an array of vertices that defines our candidate path. 1. Add all vertices up to and including u in order. 2. Add all vertices after u up to and including v in reverse order. 3. Add all vertices after v in order.

In the example above, u would have been b and v would have been c .

This basic logic can be combined with the Hill Climbing Heuristic to provide a simple way to improve the initial candidate solution. Here, the `calculate_cost` function would simply add up the cost of traversing the graph in the input order, using Dijkstra's at every vertex.

```
1 function pairwise_swap (
2     u: integer,
3     v: integer,
4     path: path of nodes
5 ):
6     new_tour = []
7
8     for i in [0, u]:
9         new_tour.add(path[i])
10    for i in [v, u]:
11        new_tour.add(path[i])
12    for i in (v, len(path)):
13        new_tour.add(path[i])
14
15    return new_tour
16 end function
17
18 function calculate_cost (
19     path: path of nodes,
20     current_time: datetime
21 ):
22     cost = 0
23     time = current_time
24
25     for i from 0 to len(path) - 1:
26         djik = fetch_djik(path[i], path[i + 1], current_time)
27         cost += djik['cost']
28         time += djik['cost'] number of minutes
29
30     return cost
31 end function
32
33 function hill_climbing (
34     candidate: path of nodes,
35     current_time: datetime,
36     fail_count: int = 0
37 ):
38     if fail_count < 200:
```

```

39     cost = calculate_cost(candidate, current_time)
40     u = random number from 1 to len(candidate) - 1 inclusive
41     v = random number from u to len(candidate) - 1 inclusive
42
43     new_tour = pairwise_swap(u, v, candidate)
44     new_cost = calculate_cost(new_tour, current_time)
45
46     if new_cost <= cost:
47         // new cost is better/equal -> accept
48         return hill_climbing(new_tour, current_time, 0)
49     else:
50         // new cost is worse -> go again
51         return hill_climbing(candidate, current_time, fail_count + 1)
52 else:
53     return candidate
54 end function

```

Note that the above range of u and v values has been chosen to prevent them from referring to the start or end of the tour, since in our particular use case we would like to force the tour to start and end at particular locations

Simulated Annealing

One of the problems with the above solution is that it will quite easily get stuck on a local minimum. Demonstrated by the graph below, the Hill Climbing Heuristic is blind to anything besides its local vicinity. As such, there may be an overall better solution, but not one that can be achieved by constantly improving the current candidate solution. In other words, sometimes things have to get worse before they get better, especially for the TSP.

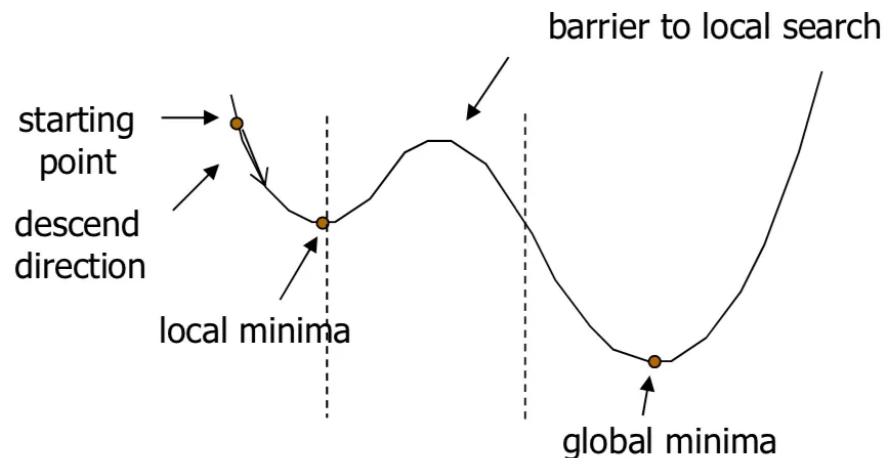


Figure 3: Example of the Limitations of Hill Climbing

Currently, once the Hill Climbing algorithm is implemented in Python, it produces a somewhat suboptimal result. It is hardcoded to terminate after it has had 200 consecutive iterations that have

seen no improvement. Sometimes, it can terminate on a relatively good result, but in other cases it gets stuck on much more sub-par candidates. This can be demonstrated by the two paths below, both of which the Hill Climbing algorithm terminated on.

```
1 The cost has been improved from 234.0 to 227.0
2 ['Brandon Park', 'Oakleigh', 'Wheelers Hill Library', 'CGS WH', 'Chadstone',
   'Caulfield', 'Flinders Street', 'Camberwell', 'Parliament', 'Melbourne
   Central', 'Brighton Beach', 'Richmond', 'Mount Waverley', 'Glen Waverley',
   'Brandon Park']

1 The cost has been improved from 277.0 to 270.0
2 ['Brandon Park', 'CGS WH', 'Glen Waverley', 'Mount Waverley', 'Camberwell',
   'Chadstone', 'Caulfield', 'Brighton Beach', 'Flinders Street', 'Melbourne
   Central', 'Parliament', 'Richmond', 'Oakleigh', 'Wheelers Hill Library',
   'Brandon Park']
```

Simulated Annealing is a concept that builds off of this idea of possibly selecting a worse solution to hopefully get to the global optimum. Namely, it tries to explore as much of the search space as possible at the start (by being more likely to select worse candidates) and then gradually reduces this chance so that it can converge on a better solution.

The logic behind this is quite similar to Hill Climbing:

1. Start with a candidate solution, from a previous algorithm or just a random tour.
2. Modify this candidate by trying to apply some tour improvements, in this case 2-opt.
3. Decide whether to accept the new solution or stay with the old one.

The key difference here is step 3. In both algorithms, if the new tour's cost is lower than the previous one, we will always accept it. If the cost is more than the current solution, with some probability, we will actually accept the higher cost solution but this probability will decrease over time.

How this probability is determined is mostly based on a parameter called the "Temperature" T . At the start we will initialise this to a high value, and a higher temperature means we are more likely to select a worse solution. Any $T \in [0, 1]$ will work, but we want to gradually reduce our temperature over time, so that it can influence some probability function.

There are usually three main types of temperature reduction functions, where α is the factor by which the temperature is scaled after n iterations:

1. Linear Reduction Rule: $T = T - \alpha$
2. Geometric Reduction Rule: $T = T \times \alpha$
3. Slow-Decrease Rule¹: $T = \frac{T}{1+\beta T}$

Each of these reduction rules decreases the temperature at a different rate, so they may be better for different use cases. For now, we will settle upon the Geometric Reduction Rule (as it is the most common).

Starting at the initial temperature, the algorithm will loop through n iterations and then decrease the temperature according to the selected temperature reduction function at the end of every iteration. This loop will stop once the terminating condition is reached, generally some low cutoff temperature where we have determined an acceptable amount of the search space has been explored.

¹This rule is not often used, but β is a different constant that we'll get to later.

Finally, within each iteration, we will use our temperature, the old cost and the new cost to determine whether we accept the new solution or not. This follows the formula below where $\Delta c = \text{new cost} - \text{old cost}$:

$$P = \begin{cases} 1 & \Delta c \leq 0 \\ e^{-\beta \Delta c / T} & 0 < \Delta c \end{cases}$$

To demonstrate, if the new cost is less than or equal to the old cost, the new cost will always be accepted. If on the other hand the new cost is greater, then we *might* pick it based on the formula shown above. This equation is inspired by the formula for the energy released by metal particles as they cool down from thermodynamics: $P(\Delta E) = e^{-\frac{\Delta E}{k \cdot T}}$. This process is known as annealing, hence the name of the algorithm! Borrowing this equation from physics turns out to be quite elegant, giving us a probability distribution known as the Boltzman distribution.

It is worth noting the different parameters that can be tuned, and the effectiveness of the algorithm depends on the choice of these parameters:

1. β - Normalising Constant The choice of this constant is dependant on the expected variation in the performance measure over the search space, If the chosen value of β is higher, the probability of accepting a solution is supposedly also higher in later iterations. In our use case, we can simply play around with this number and see if it changes anything!
2. T_0 - Initial Temperature This is simply the temperature we start with, and should be relatively close to one so that we accept a lot of new solutions at the start. For now, we will set $T_0 = 0.98$.
3. α - Temperature Scaling Factor As explained above, depending on the temperature reduction function chosen, α will reduce it at a different rate. Low α values restrict the search space faster, so we can choose $\alpha = 0.85$ for now.

The number of iterations before the temperature is updated can also be played around with, for now this will be set to 5. Also, the cutoff terminating temperature can also be set to allow the algorithm to search for longer.

The above should demonstrate the main weakness of simulated annealing: there are a lot of tuneable parameters that vastly influence the performance of the algorithm. If our input data is very sparse, the algorithm may perform much worse for certain use cases. Nonetheless, it is most definitely an improvement over the Hill Climbing algorithm as it does not increase time complexity or space complexity, but it does provide a more accurate output.

Below is the pseudocode that summarises the above discussion:

```

1 function acceptance_probability (
2     old_cost: number,
3     new_cost: number,
4     beta: number,
5     temp: number
6 ):
7     c = new_cost - old_cost
8
9     if c <= 0:
10         return 1
11     else:
12         return e**((-beta * c)/temp)
13 end function
14

```

```

15 function simulated_annealing (
16     candidate: path of nodes,
17     current_time: datetime,
18 ):
19     // parameters to fiddle with
20     temp = 0.98
21     min_temp = 0.00001
22     temp_change = 5
23     beta = 1.2
24     alpha = 0.85
25
26     old_cost = calculate_cost(candidate, current_time)
27
28     while temp > min_temp:
29         for n from 1 to temp_change:
30             u = random number from 1 to len(candidate) - 1 inclusive
31             v = random number from u to len(candidate) - 1 inclusive
32
33             new_tour = pairwise_swap(u, v, candidate)
34             new_cost = calculate_cost(new_tour, current_time)
35
36             ap = acceptance_probability(old_cost, new_cost, beta, temp)
37
38             if ap > random float from 0 to 1:
39                 candidate = new_tour
40                 old_cost = new_cost
41
42             temp *= alpha
43
44     return candidate
45 end function

```

Normalising Function Something that may have become apparent when viewing the above examples is how the paths generated by this approximate solution are somehow much shorter than those generated by Held-Karp. This is due to the fact that the implementation of Held-Karp is not restricted to only visiting each node once, whereas the approximate algorithms are. Due to this, we get some interesting behaviour that needs to be accounted for.

```

1 ['Brandon Park', 'Oakleigh', 'CGS WH', 'Wheelers Hill Library', 'Caulfield',
  'Flinders Street', 'Melbourne Central', 'Parliament', 'Glen Waverley',
  'Chadstone', 'Brighton Beach', 'Camberwell', 'Mount Waverley', 'Richmond',
  'Brandon Park']

```

The above is a path generated by the Hill Climbing algorithm. The issue to note is that it advises the user to go from Glen Waverley to Chadstone, but there is no edge between them for this to happen. Since the algorithms have been using Dijkstra's to go to any other node, it has in essence been treating our tour as a complete graph, even though it is not. As such, the edges in between these locations need to be added in again.

This is quite simple to do, and is similar to the `calculate_cost`, except the paths are added instead of the costs.

```

1 function normalise_path (

```

```

2   path: path of nodes,
3   current_time: datetime
4 ):
5   return_path = []
6   time = current_time
7
8   for i from 0 to len(path) - 1:
9       djk = fetch_djk(path[i], path[i + 1], current_time)
10      time += djk['cost'] number of minutes
11      // this is to prevent the last and first item double up
12      return_path += everything in djk['path'] except last item
13
14   return_route.add(last item in route)
15
16   return cost
17 end function

```

Final Solution

The problem these algorithms were set out to solve is a specific application of the TSP: how could the shortest closed walk be found that picks up all my friends as we travel around the city?

The initial approach to solve this problem used the concepts of dynamic programming to recursively split up the larger problem into smaller overlapping subproblems. Unfortunately, because the number of subpaths increases exponentially as the size of the visit set increased, it was demonstrated that even though an exact algorithm may provide an optimal solution, intractable problems like the TSP may require a better time complexity in a trade off for accuracy.

The approaches for the approximate solutions have followed two main phases: - Generate a possible candidate solution. - Improve the candidate using some optimisation algorithm.

The Nearest Neighbour heuristic was used to generate the initial candidate, simply travelling to the closest node remaining in the visit set until a closed walk has been achieved. This was then later improved upon by processing this candidate through both the Hill Climbing and Simulated Annealing algorithms.

In regards to the performance of Simulated Annealing (SA) vs Hill Climbing (HC), it seems that the output of the former is heavily dependent on the parameters set. Whereas HC produced results in a relatively large range, SA could be tuned to consistently provide the same “good” results every time or if the parameters were not optimal, a completely rubbish result every time.

For example, with $T_0 = 0.98, \beta = 4, \alpha = 0.9$ and the 5 iterations before updating the temperature, SA consistently produced a hamiltonian path that would take 254 minutes to traverse. HC was more inconsistent, outputting 274 initially, 281 next and struck gold with the last try with 237. Surprisingly though, the difference between Hill Climbing and Simulated Annealing doesn't seem to be vast for this particular input graph, and SA can simply be viewed as a more tuneable and adjustable version of HC to be able to produce a more consistent result.

When this was changed to simply be the visit set that the friends reside at, the output for both HC and SA was as follows:

```
1 Final candidate cost is 143.0
2 Final candidate path is ['Brandon Park', 'Wheelers Hill Library', 'CGS WH', 'Glen
    Waverley', 'Mount Waverley', 'Richmond', 'Camberwell', 'Richmond', 'Flinders
    Street', 'Caulfield', 'Oakleigh', 'Brandon Park']
```

Nonetheless, neither of them are able to find the true optimal path that Held-Karp creates:

```
1 Final candidate cost is 130.0
2 Final candidate path is ['Brandon Park', 'Wheelers Hill Library', 'CGS WH', 'Glen
    Waverley', 'Mount Waverley', 'Richmond', 'Flinders Street', 'Caulfield',
    'Oakleigh', 'Richmond', 'Camberwell', 'Richmond', 'Oakleigh', 'Brandon Park']
```

This could simply be because 2-opt does not provide the required permutations to be able to reach the optimal path, but still demonstrates the required tradeoff between approximate solutions and exact algorithms, a tradeoff of time vs accuracy.

Comparison of Solutions

Design Features As discussed above, Held-Karp (the exact algorithm) used the principle of dynamic programming to split the larger problem into instances of the similar overlapping subproblems that can be solved recursively. By utilising the fact that every subpath of a path of minimum distance is itself of minimum distance, we can recursively reduce the size of the visit set by one and solve for the

smaller cases. In this case, due to the TSP’s intractability, this only decreases the time complexity from factorial to exponential, saving time by ensuring that subpaths are not re-calculated.

On the other hand, the combination of algorithms that produce the approximate solutions operate based off a variety of design principles.

The initial candidate solution generated by the NN Heuristic uses a greedy design pattern to find a possible path. This design pattern does not work with many problems (including the TSP) because sometimes things have to get worse for an overall better result.

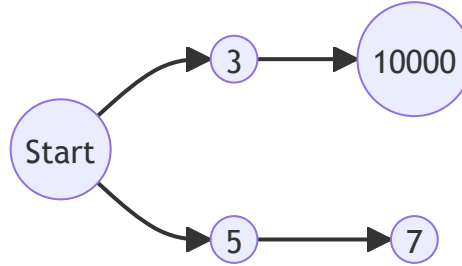


Figure 4: Demonstration of Why Greedy Algorithms Fail

Demonstrated above, the greedy design feature would select “3” as it is the best option visible at the time, but will end up selecting a far worse solution that could easily be avoided with some intuition for what comes afterwards.

Nonetheless, the greedy design pattern in the NN heuristic generally produces a somewhat viable candidate, that is then improved upon by certain Generate and Test algorithms.

One such algorithm is Hill Climbing, which refers to a type of local search optimisation technique that provides an iterative way to make incremental changes to a candidate and proceed if an improvement has been found.

Simulated Annealing expands upon this idea by using a probabilistic technique to decide if we accept an incremental change or not. Both these local search algorithms allow for an exploration of adjacent solutions that help find an improved solution in a tractable way.

The difference between the two approaches and their design patterns lies between the intended output. The dynamic programming approach guarantees a correct output, but since the requirements are slightly different for the approximate algorithms, a wider range of design techniques are available (such as using random probability or the Generate and Test pattern) that can get us closer to a better solution, even if it produces a non-deterministic non-optimal result.

Coherence Overall, Held-Karp is far more of a consistent and logical solution. Since the exact algorithm is inherently deterministic, it is always guaranteed to produce the same optimal result consistently. In contrast, the NN algorithm’s performance can vary widely depending on the arrangement of nodes and both the optimisation algorithms use probability to pick u and v values. Simulated annealing is also non-deterministic (\because probabilistic), meaning that it is nowhere near as consistent as as Held-Karp. That being said, Simulated Annealing does seem to converge consistently on the same or similar local optima based on its input parameters, so we can render it more coherent than Hill Climbing but much less so than Held-Karp.

The influence of this difference in consistency between the two approaches on the real world applications is key to deciding which approach is better. Exact algorithms would be preferred in scenarios where

predictability and repeatability are crucial. For example, in scientific research studies on geographical data that is static, the superior coherence of Held-Karp would mean that the study is repeatable and verifiable by peers. On the other hand, the lower consistency of SA and HC are not necessarily disadvantageous in real world applications, because they can provide more flexibility and adaptability. Instead of providing only one solution, they provide many good candidates that the user can consider between. This flexibility would be ideal for larger operations such as a logistics company, where the clients and pickup points are very actively changing, and alternative routes need to be provided in case the algorithm does not account for real world disturbances such as road closures.

Fitness for Problem In terms of fitness for the problem, it would be safe to say that the exact algorithm would be preferred for the initial problem described. Even though Held-Karp would have a larger space complexity (due to all the subpaths that need to be stored), a typical user's phone will have plenty of storage such that space should not be too much of an issue. The inefficient time complexity of the algorithm mostly relates to how it scales to larger visit set sizes, anything below $n = 14$ is barely noticeable to the typical user. Since most people will not be intending to travel in this fashion with such a large number of friends, it would likely be preferred to use the exact algorithm as it provides the optimal solution. This being said, Held-Karp is somewhat inflexible, especially when it comes to frequently changing data. As it only provides one path and one path only, it could be a bit of an issue when it does not account for certain data such as a bus replacement (very common around Victoria). As such, it might be best to use a combination of both in an application, defaulting to the modified Held-Karp but switching over to the approximate algorithms once $n > 13$ or more solutions are requested.

Efficiency & Time Complexity As established above, the improved Held-Karp algorithm maintains an exponential time complexity, similar enough to $O(2^n)$ that we can use this simplified version to come to more clear conclusions.

Going through the pseudocode for the approximate algorithms, the algorithm to find a candidate solution is run first. In this case, this would be the Nearest Neighbour heuristic, which runs the following code for every node in the visit set (of size n)

```
1 closest_node = fetch_djk(current_vertex, visit, current_time)
2 path.add(closest_node)
3 cost += closest_node.cost
4 visit.remove(closest_node)
5 current_vertex = closest_node
```

Since it runs Dijkstra's at every node, our time complexity for NN will just be $n \times \text{Dijkstra's Time Complexity}$. If we presume that the above optimisations for Dijkstra were effective then this would be at $O(L + R \log L)$ (the generally accepted time complexity for Dijkstra's using min heaps), but even if this was not the case, we would have a time complexity of $O(L^2)$. This provides an NN time complexity of $O(n \times L^2)$.

In terms of Hill Climbing, during every iteration i , the algorithm runs the following pseudocode:

```
1 cost = calculate_cost(candidate, current_time)
2 u = random number from 1 to len(candidate) - 1 inclusive
3 v = random number from u to len(candidate) - 1 inclusive
4
5 new_tour = pairwise_swap(u, v, candidate)
6 new_cost = calculate_cost(new_tour, current_time)
7
8 if new_cost <= cost:
9     // new cost is better/equal -> accept
10    Go again with the new tour
```

```

11 else:
12     // new cost is worse -> go again
13     Go again with the same tour

```

First, the cost of the candidate is evaluated. This requires us to run Dijkstra's on each node in the visit set again, but since the output of Dijkstra's is cached, this would actually only take $O(n)$ time. Next, a pairwise swap is done, which adds every node in the visit set to a new array in a differing order which is also in $O(n)$ time. Finally, the cost is calculated again, leaving us with a final total of $O(3n)$. Overall, this means that this process is done in linear time for i iterations, leaving a final time complexity of $O(i \times n)$.

Simulated Annealing has the exact same time complexity as Hill Climbing because the only major difference is if a candidate solution is accepted or not and this is done in $O(1)$ time because the time complexity of selecting a random number is $O(1)$.

This leaves us with a final time complexity of $O(n \times L^2 + i \times n) = O(n(L^2 + i))$.

Tractability & Implications

As discussed above, the time complexity for the exact algorithm is effectively $O(2^n)$ and the time complexity for the approximate algorithms is $O(n(L^2 + i))$ where n is the size of the visit set, L is the number of landmarks in the graph overall and i is the amount of times that the optimisation algorithm will iterate. i will typically be a constant and can therefore be ignored and for the same input graph (this assumption was made for the simplification of Held-Karp too) L^2 will be constant as well.

In effect, this means that for the same input graph, the time complexities we are looking at are $O(2^n)$ vs $O(n)$ as the visit set size increases by a constant factor. The vast difference between these two time complexities shows how easily approximate solutions can be derived in polynomial time, which helps make this problem more tractable. Namely, this demonstrates that the problem of finding a solution to the TSP within a set factor of the optimal solution is a tractable one, even if finding the *actual* optimal solution is not.

This has many implications for the real world applications of the broader version of this problem. Though the discussion above concluded that the exact algorithm would be superior for the initial specified problem, the tradeoff of lower accuracy for an improved time complexity can be beneficial to many use-cases. Below is a list of applications that would be better suited to either type of algorithm:

Exact Algorithm:

- An exact algorithm would be well suited to static non-changing data where time is not much of a concern but the best solution is required. In a scenario where large freeways need to be built to visit a few key cities, the geographical data remains mostly static since the overall terrain does not change suddenly, but an inefficient solution could cost millions. Similarly, in wartime where tunnels and bunker networks need to be built that connect everyone to a few key locations, a few extra kilometres could result in hundreds of lost lives. In cases like this, provided that the number of key locations is sufficiently small, users would likely not mind waiting for a more optimal output.

Approximate Algorithm:

- As discussed previously, an approximate algorithm would be very well suited to logistics/trucking companies that have to move a lot of shipments and goods across the country fast. The nature of real world companies means that clients would appear and disappear on a daily basis, and there are always new locations to be delivered to or picked up from. Since the input graph is dynamically changing, an exact solution would be very quickly out of date and an $O(n)$ time

complexity would be preferred over the intractable $O(2^n)$ complexity since the amount of pickup points would simply be so large.

- An approximate algorithm would be well suited to data routing, specifically peer to peer networks that want to connect a large group of people. For example, a P2P video conferencing call would need to find a sufficiently small closed walk to ensure that the call has minimal delay. Since the input data for this case would be constantly changing (people leaving and joining with variable bandwidths), it would need to be run very often, and an intractable solution would not suffice.

This being said, most applications would be better suited to a combination of both. With a small number of nodes in the visit set, the intractability of finding an exact solution is not much of an issue, as the speeds are virtually instant anyways, but anything above about 15 to 20 nodes will render the computational time to be prohibitive. As such, for most real world applications, it makes more sense to use a combination of both the algorithms and switch over once the input size has exceeded the practical time constraints a layman user would expect. Such is the case with the initial solution, as described above.

Appendix

Initial Pseudocode

The following is the final pseudocode reiterated from the previous 2 parts, namely for convenience while analysing, since multiple modifications were made to the initial pseudocode.

Let A = starting vertex Let B = ending vertex Let $S = \{P, Q, R\}$ or any other vertices to be visited along the way. Let $C \in S$ (random node in S)

```
1 function main(  
2     friends: dictionary,  
3     landmarks: dictionary,  
4     routes: dictionary,  
5     timetable: dictionary  
6 ):  
7     // global variable declarations  
8     concession: bool = Ask the user "Do you posses a concession card?"  
9     holiday: bool = Ask the user "Is today a weekend or a holiday?"  
10    user_name: string = Ask the user to select a friend from friends dictionary  
11    selected_time = Ask the user what time they are leaving  
12  
13    cached_djk: dictionary = empty dictionary  
14    edge_lookup_matrix: matrix = |V| x |V| matrix that stores a list of edges in  
15        each entry  
16  
17    // get distance of all friends from landmarks  
18    friend_distances: dictionary = calculate_nodes(friends, landmarks)  
19    visit_set: set = set of all closest nodes from friend_distances  
20    people_at_nodes: dictionary = all friends sorted into keys of which nodes they  
21        are closest to, from visit_set  
22  
23    home: string = closest node of user_name  
24  
25    print all friends, where they live closest to and how far away  
26  
27    print out friends that would take more than 20 minutes to walk (average human  
28        walking speed is 5.1 km/h)  
29  
30    hamiltonian_path = held_karp(home, home, visit_set, selected_time)  
31  
32    print how much the trip would cost and how long it would take  
33  
34    print the path of the hamiltonian_path  
35 end function
```

```
1 function calculate_nodes (  
2     friend_data: dictionary,  
3     node_data: dictionary  
4 ):  
5     for friend in friend_data:
```

```

6     home: tuple = friend['home']
7     // initial min vals that will be set to smallest iterated distance
8     min: float = infinity
9     min_node: node = null
10
11     for node in node_data:
12         location: tuple = node['coordinates']
13         // find real life distance (functional abstraction)
14         distance: float = latlong_distance(home, location)
15         if distance < min:
16             min = distance
17             min_node = node
18
19     distance_dict[friend]['min_node'] = min_node
20     distance_dict[friend]['distance'] = min
21 end function

```

```

1 function held_karp (
2     start: node,
3     end: node,
4     visit: set<node>,
5     current_time: datetime
6 ):
7     if visit.size = 0:
8         djvk = fetch_djvk(start, end, current_time)
9         return djvk['cost']
10    else:
11        min = infinity
12        For node C in set S:
13            sub_path = held_karp(start, C, (set \ C), current_time)
14            djvk = fetch_djvk(C, end, current_time + toMinutes(sub_path['cost']))
15            cost = sub_path['cost'] + djvk['cost']
16            if cost < min:
17                min = cost
18        return min
19 end function

```

```

1 function dijkstras (
2     start: node,
3     current_time: datetime
4 ):
5     // Set all node distance to infinity
6     for node in graph:
7         distance[node] = infinity
8         predecessor[node] = null
9         unexplored_list.add(node)
10
11     // starting distance has to be 0

```

```

12 distance[start] = 0
13
14 // while more to still explore
15 while unexplored_list is not empty:
16     min_node = unexplored node with min cost
17     unexplored_list.remove(min_node)
18
19     // go through every neighbour and relax
20     for each neighbour of min_node:
21         current_dist = distance[min_node] + dist(min_node, neighbour,
22             current_time + to_minutes(distance[min_node]))
23         // a shorter path has been found to the neighbour -> relax value
24         if current_dist < distance[neighbour]:
25             distance[neighbour] = current_dist
26             predecessor[neighbour] = min_node
27
28     return {
29         'distances': distance,
30         'predecessors': predecessor,
31     }
32 end function

```

```

1 cached_djk = dictionary of node -> dict
2
3 function fetch_djk (
4     start: node,
5     end: node,
6     current_time: datetime,
7 ):
8     name = start + '@' + current_time
9
10    if cached_djk[name] does not exists:
11        cached_djk[name] = dijkstras(start, current_time)
12
13    djk = cached_djk[name]
14    # reconstructs the path
15    path = [end] as queue
16    while path.back != start:
17        path.enqueue(djk['predecessors'][path.back])
18
19    return {
20        'distance': djk['distances'][end],
21        'path': path
22    }
23 end function

```

```

1 function dist (
2     start: node,

```

```

3     end: node,
4     current_time: datetime
5 ):
6     // if the start and end node are the same, it takes no time to get there
7     if start = end:
8         return 0
9     else if edges = null:
10         // if no edge exists between nodes
11         return infinity
12
13     edges = edge_lookup_matrix[start][end]
14     distances = []
15
16     // go over each possible edge between nodes (multiple possible)
17     for edge in edges:
18         line = edge.line
19         // next time bus/train will be at node (functional abstraction)
20         next_time = soonest_time_at_node(timetable, line, start, current_time)
21         wait_time = next_time - current_time
22         distances.add(edge.weight + wait_time)
23
24     return min(distances)
25 end function

```

Modified Exact Algorithm Pseudocode

Below is the final pseudocode for the exact algorithm, based on Held-Karp.

Let A = starting vertex Let B = ending vertex Let $S = \{P, Q, R\}$ or any other vertices to be visited along the way. Let $C \in S$ (random node in S)

```
1 function main(
2     friends: dictionary,
3     landmarks: dictionary,
4     routes: dictionary,
5     timetable: dictionary
6 ):
7     // global variable declarations
8     concession: bool = Ask the user "Do you posses a concession card?"
9     holiday: bool = Ask the user "Is today a weekend or a holiday?"
10    user_name: string = Ask the user to select a friend from friends dictionary
11    selected_time = Ask the user what time they are leaving
12
13    cached_djk: dictionary = empty dictionary
14    edge_lookup_matrix: matrix = |V| x |V| matrix that stores a list of edges in
        each entry
15
16    // get distance of all friends from landmarks
17    friend_distances: dictionary = calculate_nodes(friends, landmarks)
18    visit_set: set = set of all closest nodes from friend_distances
19    people_at_nodes: dictionary = all friends sorted into keys of which nodes they
        are closest to, from visit_set
20
21    home: string = closest node of user_name
22
23    print all friends, where they live closest to and how far away
24
25    print out friends that would take more than 20 minutes to walk (average human
        walking speed is 5.1 km/h)
26
27    hamiltonian_path = fetch_hk(home, home, visit_set, selected_time)
28
29    print how much the trip would cost and how long it would take
30
31    print the path of the hamiltonian_path
32 end function
```

```
1 function calculate_nodes (
2     friend_data: dictionary,
3     node_data: dictionary
4 ):
5     for friend in friend_data:
6         home: tuple = friend['home']
7         // initial min vals that will be set to smallest iterated distance
8         min: float = infinity
```



```

9      min_node: node = null
10
11      for node in node_data:
12          location: tuple = node['coordinates']
13          // find real life distance (functional abstraction)
14          distance: float = latlong_distance(home, location)
15          if distance < min:
16              min = distance
17              min_node = node
18
19          distance_dict[friend]['min_node'] = min_node
20          distance_dict[friend]['distance'] = min
21 end function

```

```

1 function held_karp (
2     start: node,
3     end: node,
4     visit: set<node>,
5     current_time: datetime
6 ):
7     if visit.size = 0:
8         djk = fetch_djk(start, end, current_time)
9         return djk['cost']
10    else:
11        min = infinity
12        For node C in set S:
13            sub_path = fetch_hk(start, C, (set \ C), current_time)
14            djk = fetch_djk(C, end, current_time + toMinutes(sub_path['cost']))
15            cost = sub_path['cost'] + djk['cost']
16            if cost < min:
17                min = cost
18        return min
19 end function

```

```

1 cached_hk = dictionary of list -> dict
2
3 function fetch_hk (
4     start: node,
5     end: node,
6     visit: set of nodes,
7     current_time: datetime,
8 ):
9     // unique identifier
10    name = start + '-' + end + visit set + '@' + current_time
11    if cached_hk[name] does not exists:
12        cached_hk[name] = held_karp(start, end, visit, current_time)
13    return cached_hk[name]
14 end function

```

```

1 function dijkstras (
2     start: node,
3     current_time: datetime
4 ):
5     unexplored = empty min priority queue of nodes based on distance
6
7     // Set all node distance to infinity
8     for node in graph:
9         distance[node] = infinity
10        predecessor[node] = null
11        unexplored.add(node)
12
13    // starting distance has to be 0
14    distance[start] = 0
15
16    // while more to still explore
17    while unexplored is not empty:
18        min_node = unexplored.minimum_node()
19        unexplored.remove(min_node)
20
21        // go through every neighbour and relax
22        for each neighbour of min_node:
23            current_dist = distance[min_node] + dist(min_node, neighbour,
24                current_time + to_minutes(distance[min_node]))
25            // a shorter path has been found to the neighbour -> relax value
26            if current_dist < distance[neighbour]:
27                distance[neighbour] = current_dist
28                predecessor[neighbour] = min_node
29
30    return {
31        'distances': distance,
32        'predecessors': predecessor,
33    }
34 end function

```

```

1 cached_djk = dictionary of node -> dict
2
3 function fetch_djk (
4     start: node,
5     end: node,
6     current_time: datetime,
7 ):
8     name = start + '@' + current_time
9
10    if cached_djk[name] does not exists:
11        cached_djk[name] = dijkstras(start, current_time)
12
13    djk = cached_djk[name]

```

```

14     # reconstructs the path
15     path = [end] as queue
16     while path.back != start:
17         path.enqueue(djk['predecessors'][path.back])
18
19     return {
20         'distance': djk['distances'][end],
21         'path': path
22     }
23 end function

```

```

1 function dist (
2     start: node,
3     end: node,
4     current_time: datetime
5 ):
6     // if the start and end node are the same, it takes no time to get there
7     if start = end:
8         return 0
9     else if edges = null:
10         // if no edge exists between nodes
11         return infinity
12
13     edges = edge_lookup_matrix[start][end]
14     distances = []
15
16     // go over each possible edge between nodes (multiple possible)
17     for edge in edges:
18         wait_time = wait time from data (precomputed)
19         distances.add(edge.weight + wait_time)
20
21     return min(distances)
22 end function

```

Approximate Algorithm Pseudocode

Below is the final pseudocode for the approximate algorithm, using Simulated Annealing.

Let A = starting vertex Let B = ending vertex Let $S = \{P, Q, R\}$ or any other vertices to be visited along the way. Let $C \in S$ (random node in S)

```
1 function main(
2     friends: dictionary,
3     landmarks: dictionary,
4     routes: dictionary,
5     timetable: dictionary
6 ):
7     // global variable declarations
8     concession: bool = Ask the user "Do you posses a concession card?"
9     holiday: bool = Ask the user "Is today a weekend or a holiday?"
10    user_name: string = Ask the user to select a friend from friends dictionary
11    selected_time = Ask the user what time they are leaving
12
13    cached_djk: dictionary = empty dictionary
14    edge_lookup_matrix: matrix = |V| x |V| matrix that stores a list of edges in
        each entry
15
16    // get distance of all friends from landmarks
17    friend_distances: dictionary = calculate_nodes(friends, landmarks)
18    visit_set: set = set of all closest nodes from friend_distances
19    people_at_nodes: dictionary = all friends sorted into keys of which nodes they
        are closest to, from visit_set
20
21    home: string = closest node of user_name
22
23    print all friends, where they live closest to and how far away
24
25    print out friends that would take more than 20 minutes to walk (average human
        walking speed is 5.1 km/h)
26
27    candidate = candidate_solution(home, home, visit_set, selected_time)
28    hamiltonian_path = simulated_annealing(candidate['path'], selected_time)
29    // or hill_climbing(candidate['path'], selected_time)
30
31    hamiltonian_path['path'] = normalise_path(hamiltonian_path['path'],
        selected_time)
32
33    print how much the trip would cost and how long it would take
34
35    print the path of the hamiltonian_path
36 end function
```

```
1 function calculate_nodes (
2     friend_data: dictionary,
3     node_data: dictionary
```

```

4 ):
5     for friend in friend_data:
6         home: tuple = friend['home']
7         // initial min vals that will be set to smallest iterated distance
8         min: float = infinity
9         min_node: node = null
10
11         for node in node_data:
12             location: tuple = node['coordinates']
13             // find real life distance (functional abstraction)
14             distance: float = latlong_distance(home, location)
15             if distance < min:
16                 min = distance
17                 min_node = node
18
19         distance_dict[friend]['min_node'] = min_node
20         distance_dict[friend]['distance'] = min
21 end function

```

```

1 // creates a candidate solution using the NN Heuristic
2 function canditate_solution (
3     start: node,
4     end: node,
5     visit: set of nodes,
6     current_time: datetime,
7 ):
8     path = [start]
9     current_vertex = start
10    cost = 0
11
12    while len(visit) != 0:
13        closest_node = fetch_djk(current_vertex, visit, current_time)
14        path.add(closest_node)
15        cost += closest_node.cost
16        visit.remove(closest_node)
17        current_vertex = closest_node
18
19    // go back to the end node
20    closest_node = fetch_djk(current_vertex, end, current_time)
21    path.add(closest_node)
22    cost += closest_node.cost
23
24    return {'path': path, 'cost': cost}
25 end function

```

```

1 function pairwise_swap (
2     u: integer,
3     v: integer,

```

```

4   path: path of nodes
5 ):
6   new_tour = []
7
8   for i in [0, u]:
9       new_tour.add(path[i])
10  for i in [v, u):
11      new_tour.add(path[i])
12  for i in (v, len(path)]:
13      new_tour.add(path[i])
14
15  return new_tour
16 end function

```

```

1 function calculate_cost (
2     path: path of nodes,
3     current_time: datetime
4 ):
5     cost = 0
6     time = current_time
7
8     for i from 0 to len(path) - 1:
9         djk = fetch_djk(path[i], path[i + 1], current_time)
10        cost += djk['cost']
11        time += djk['cost'] number of minutes
12
13    return cost
14 end function

```

```

1 function hill_climbing (
2     candidate: path of nodes,
3     current_time: datetime,
4     fail_count: int = 0
5 ):
6     if fail_count < 200:
7         cost = calculate_cost(candidate, current_time)
8         u = random number from 1 to len(candidate) - 1 inclusive
9         v = random number from u to len(candidate) - 1 inclusive
10
11        new_tour = pairwise_swap(u, v, candidate)
12        new_cost = calculate_cost(new_tour, current_time)
13
14        if new_cost <= cost:
15            // new cost is better/equal -> accept
16            return hill_climbing(new_tour, current_time, 0)
17        else:
18            // new cost is worse -> go again
19            return hill_climbing(candidate, current_time, fail_count + 1)

```

```

20     else:
21         return candidate
22 end function

```

```

1 function simulated_annealing (
2     candidate: path of nodes,
3     current_time: datetime,
4 ):
5     // parameters to fiddle with
6     temp = 0.98
7     min_temp = 0.00001
8     temp_change = 5
9     beta = 1.2
10    alpha = 0.85
11
12    old_cost = calculate_cost(candidate, current_time)
13
14    while temp > min_temp:
15        for n from 1 to temp_change:
16            u = random number from 1 to len(candidate) - 1 inclusive
17            v = random number from u to len(candidate) - 1 inclusive
18
19            new_tour = pairwise_swap(u, v, candidate)
20            new_cost = calculate_cost(new_tour, current_time)
21
22            ap = acceptance_probability(old_cost, new_cost, beta, temp)
23
24            if ap > random float from 0 to 1:
25                candidate = new_tour
26                old_cost = new_cost
27
28            temp *= alpha
29
30    return candidate
31 end function

```

```

1 function acceptance_probability (
2     old_cost: number,
3     new_cost: number,
4     beta: number,
5     temp: number
6 ):
7     c = new_cost - old_cost
8
9     if c <= 0:
10        return 1
11    else:
12        return e**((-beta * c)/temp)

```

```
13 end function
```

```
1 function dijkstras (  
2     start: node,  
3     current_time: datetime  
4 ):  
5     unexplored = empty min priority queue of nodes based on distance  
6  
7     // Set all node distance to infinity  
8     for node in graph:  
9         distance[node] = infinity  
10        predecessor[node] = null  
11        unexplored.add(node)  
12  
13    // starting distance has to be 0  
14    distance[start] = 0  
15  
16    // while more to still explore  
17    while unexplored is not empty:  
18        min_node = unexplored.minimum_node()  
19        unexplored.remove(min_node)  
20  
21        // go through every neighbour and relax  
22        for each neighbour of min_node:  
23            current_dist = distance[min_node] + dist(min_node, neighbour,  
24                current_time + to_minutes(distance[min_node]))  
25            // a shorter path has been found to the neighbour -> relax value  
26            if current_dist < distance[neighbour]:  
27                distance[neighbour] = current_dist  
28                predecessor[neighbour] = min_node  
29  
30    return {  
31        'distances': distance,  
32        'predecessors': predecessor,  
33    }  
end function
```

```
1 cached_djk = dictionary of node -> dict  
2  
3 function fetch_djk (  
4     start: node,  
5     end: node,  
6     current_time: datetime,  
7 ):  
8     name = start + '@' + current_time  
9  
10    if cached_djk[name] does not exists:  
11        cached_djk[name] = dijkstras(start, current_time)
```



```

12
13     djk = cached_djk[name]
14     # reconstructs the path
15     path = [end] as queue
16     while path.back != start:
17         path.enqueue(djk['predecessors'][path.back])
18
19     return {
20         'distance': djk['distances'][end],
21         'path': path
22     }
23 end function

```

```

1 function dist (
2     start: node,
3     end: node,
4     current_time: datetime
5 ):
6     // if the start and end node are the same, it takes no time to get there
7     if start = end:
8         return 0
9     else if edges = null:
10         // if no edge exists between nodes
11         return infinity
12
13     edges = edge_lookup_matrix[start][end]
14     distances = []
15
16     // go over each possible edge between nodes (multiple possible)
17     for edge in edges:
18         wait_time = wait time from data (precomputed)
19         distances.add(edge.weight + wait_time)
20
21     return min(distances)
22 end function

```

```

1 function normalise_path (
2     path: path of nodes,
3     current_time: datetime
4 ):
5     return_path = []
6     time = current_time
7
8     for i from 0 to len(path) - 1:
9         djk = fetch_djk(path[i], path[i + 1], current_time)
10        time += djk['cost'] number of minutes
11        // this is to prevent the last and first item double up
12        return_path += everything in djk['path'] except last item

```

```
13  
14     return_route.add(last item in route)  
15  
16     return cost  
17 end function
```