Assignment - 1

Q1 $Y = |Z|$

$Z \sim N(0,1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{z}{1}\right)^2 \frac{1}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$Y = |Z|$

$F_Y(y) = P(Y < y)$

$= P(|Z| < y)$

$= P(-y < Z < y)$

$F_Y(y) = P(Z < y) - P(Z < -y)$

~~diff both sides wrt y~~

~~$f_Y(y) =$~~

$F_Y(y) = F_Z(y) - F_Z(-y)$

diff both sides wrt y

$f_Y(y) = f_Z(y) + f_Z(-y)$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}}$

~~$E(Y)$~~ $E[Y] = E[|Z|]$

$= \begin{cases} E(Z) & Z > 0 \\ E(-Z) & Z < 0 \end{cases}$

$= \begin{cases} \int_0^\infty z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ \int_{-\infty}^0 -z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \end{cases} = 2\int_0^\infty z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$= \sqrt{\frac{2}{\pi}}$

$$E[Y^2] = E[|Z|^2] = E[Z^2]$$

$$E[Z^2] = Var(Z) + (E[Z])^2$$

$$E[Z^2] = 1$$

$$E(Y^2) = E[Z^2] = 1$$

$$Var(Y) = E[Y^2] - [E[Y]]^2$$

$$= 1 - \frac{2}{\pi}$$

$$E[Y^3] = E[|Z|^3] = E[Z^2 \cdot |Z|] = \begin{cases} E[Z^3] & Z > 0 \\ E[-Z^3] & Z < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} \int_0^\infty z^3 e^{-z^2/2} \, dz \\ \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^0 z^3 e^{-z^2/2} \, dz \end{cases}$$

$$= 2\sqrt{\frac{1}{2\pi}} \int_0^\infty z \cdot z^2 e^{-z^2/2} \, dz \qquad \frac{z^2}{2} = u \Rightarrow zdz = du$$

$$= 4\sqrt{\frac{1}{2\pi}} \int_0^\infty u e^{-u} \, du = 2\sqrt{\frac{2}{\pi}} \, \Gamma(2) = 2\sqrt{\frac{2}{\pi}}$$

$\cancel{E(x g(x))}$

$\cancel{Q_1}$ $X \sim Pois(\lambda)$

Q_2 (a) $p(x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}$

$$E[x g(x)] = \sum_{x=\cancel{0}1}^{n} \frac{e^{-\lambda} \lambda^{x}}{x!} \cdot x \cdot g(x)$$

$$E[x g(x)] = \sum_{x=\cancel{0}1}^{n} \frac{e^{-\lambda} \lambda^{x}}{(x-1)!} \cdot g(x)$$

$$= \sum_{x=1}^{n} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} \cdot g(x) \cdot \lambda$$

$\cancel{= E[g}$ $x-1=y \Rightarrow x=y+1$

$$= \sum_{y=0}^{n} \frac{e^{-\lambda} \lambda^{y}}{y!} g(y+1) \cdot \lambda$$

$$E[x g(x)] = \lambda E[g(x+1)]$$

(b) $E[X] = \lambda$, $Var(X) = \lambda$ $\Big\} \rightarrow E[X^2] = \lambda^2 + \lambda$

$g(x) = x^2$

$$E(X^3) = \lambda E[(X+1)^2]$$

$$= \lambda [E(X^2) + 2E(X) + E(1)]$$

$$= \lambda(\lambda^2 + \lambda + 2\lambda + 1]$$

$$E[X^3] = \lambda^3 + 3\lambda^2 + \lambda$$

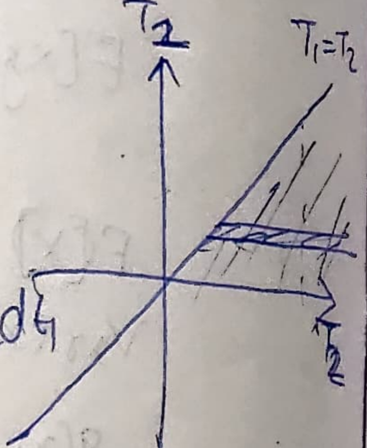
3 $T_1 \sim Exp(\lambda_1)$

$T_2 \sim Exp(\lambda_2)$

$f_{T_1}(t_2) = \lambda_1 e^{-\lambda_1 t_1}$

$f_{T_2}(t_2) = \lambda_2 e^{-\lambda_2 t_2}$

$$P(T_1 < T_2) = \int_0^\infty \int_{t_1}^\infty f_{T_1,T_2}(t_1,t_2)\, dt_2\, dt_1$$



Since both are independent

$$= \int_0^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \cdot \lambda_2 e^{-\lambda_2 t_2}\, dt_2\, dt_1$$

$$= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2) t_1}\, dt_1$$

$$P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

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$W \rightarrow$ noise

$W \sim N(0, \sigma^2)$

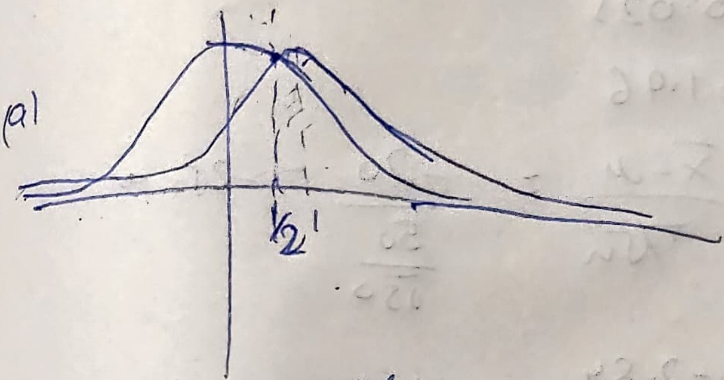
$$Y = X + W$$

$Y \rightarrow$ output ; $X \rightarrow$ input

if $X = 0$ $\quad Y_0 = W \sim N(0, \sigma^2)$

$X = 1$ $\quad Y_1 = 1 + W \sim N(1, \sigma^2)$

(a)



$Y > Y_2 \rightarrow$ yes

$Y < Y_2 \rightarrow$ no

$\cancel{P(Y > Y_2) = P(Y_0 > Y_2) + P(Y_1 > Y_2) = P(Y_0 Y_1) \cdot P(Y}$

$\cancel{\cong 0.69}$

$\cancel{= P\left(Z_0 > \frac{Y_2}{2\sigma^2}\right) \cdot P\left(Z_0 > \frac{-1}{2\sigma^2}\right)}$

$\cancel{= \left(1 - \phi\left(\frac{1}{2\sigma}\right)\right)\left(1 - \phi\right.}$

for Shree to understand message

$Y_0 < Y_2$ or $Y_1 > Y_2$

$$P(Y) = \frac{1}{2}\left(P(Y_0 < Y_2) + P(Y_1 > Y_2)\right)$$

$$= P\left(Z_0 < \frac{1}{2\sigma}\right) + P\left(Z_1 > -\frac{1}{2\sigma}\right)$$

$$P(Y) = \frac{1}{2}\left(\phi\left(\frac{1}{2\sigma}\right) + 1 - \phi\left(\frac{-1}{2\sigma}\right)\right) = \cancel{2}\,\phi\left(\frac{1}{2\sigma}\right)$$

(b) $\sigma \rightarrow$ very small $P(Y)$ will be high (Close to 1) & $\sigma \rightarrow$ very large $P(Y)$ will be ≈ 0.50

5 → Hypothesis

$H_0 :- \mu_0 = 800$

$H_1 :- \mu_0 \neq 800$

$n = 50$

$\mu = 780$

$\sigma = 50$

$\alpha = 0.05$

$\alpha/2 = 0.025$

$z_{\alpha/2} = 1.96$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{-20}{\frac{50}{\sqrt{50}}} = -2\sqrt{2}$$

$Z \approx -2.84$

$|Z| > z_{\alpha/2}$

We reject H_0