Assignment - 2

Q1

Prior Belief :- Prior Belief was 1%.

Prior → $P(\text{expert}) = 0.01$

Likelihood :- $P(3 \text{ bull-eyes} \mid \text{expert})$

if he was expert → $P(\cancel{\text{expert}}\ \text{hit}) = 0.7$ given

5 bull eyes will fllow binomial distribution with $p = 0.7$

$$p(x) = \binom{5}{x}(1-p)^{5-x} p^{x} \quad \text{where } p = 0.7$$

$$p(3) = \binom{5}{3}(1-p)^{2} p^{3}$$

$$= 10 \times (0.7)^3 \times (0.3)^2 = 0.3087$$

$P(3 \text{ bull eyes} \mid \text{not expert})$

$P(\text{not expert} \to \text{hit}) = 0.1$ {P of average}

Ans → $\binom{5}{3}(1-0.1)^2(0.1)^3$

$$= 10 \times (0.9)^2 (0.1)^3 = 0.0081$$

Bayesian update :- $P(E \mid 3 \text{ bull-eye in } 5)$

$$= \frac{P(3 \text{ bulleye in } 5 \mid \text{Expert}) \times P(\text{Expert})}{P(3 \text{ bull eye in } 5)}$$

$$= \frac{0.3087 \times 0.01}{0.3087 \times 0.01 + 0.0081 \times 0.99} = \frac{0.003087}{0.011106}$$

$= 0.277$

$\approx 27.8\%$

(a) $27.8\% \approx 28\% = 0.28$

(b) Our prior belief was that Alex was only 1% good $= 0.01$ as he claims but after the data we can say our posterior claim is 0.28 or 28% that he is an expert

(c) if $P(\text{expert}) = 0.2$

$$\text{Posterior claim} = \frac{0.3087 \times 0.2}{0.3087 \times 0.2 + 0.0081 \times 0.8}$$

$$= \frac{0.06174}{0.06822} = 0.905$$

then our posterior claim would be 90.5%

Q2 $T \sim Exp(\lambda)$

$$P(t_i \mid T \geq 10) = \frac{P(t_i)}{P(T \geq 10)}$$

$$P(t_i) = \lambda e^{-\lambda t}$$

$$P(T \geq 10) = 1 - P(T < 10)$$

$$= 1 - \int_0^{10} \lambda e^{-\lambda t} \, dt$$

$$P(T \geq 10) = e^{-10\lambda}$$

$$P(t_i \mid T \geq 10) = \frac{\lambda e^{-\lambda t}}{e^{-10\lambda}} = \lambda e^{-\lambda(t-10)}$$

Likehood function

$$L(\lambda \mid t) = \prod_{i=1}^{n} \lambda e^{-\lambda(t_i - 10)}$$

$$\cancel{\log L =} \; L = (\lambda)^n \cdot e^{-\lambda\left(\sum_{i=1}^{n} t_i - 10n\right)}$$

$$\log L = n \log \lambda - \lambda\left(\sum_{i=1}^{n} t_i - 10n\right)$$

$$\frac{d \log L}{d\lambda} = \frac{n}{\lambda} + 10n - \sum_{i=1}^{n} t_i$$

$$\sum_{i=1}^{n} t_i = n\left(\frac{1}{\lambda} + 10\right)$$

$$\lambda = \frac{\cancel{\mu_{t_i}} \; 1}{\mu_{t_i} - 10} \qquad \mu_{t_i} = \frac{\sum_{i=1}^{n} t_i}{n}$$

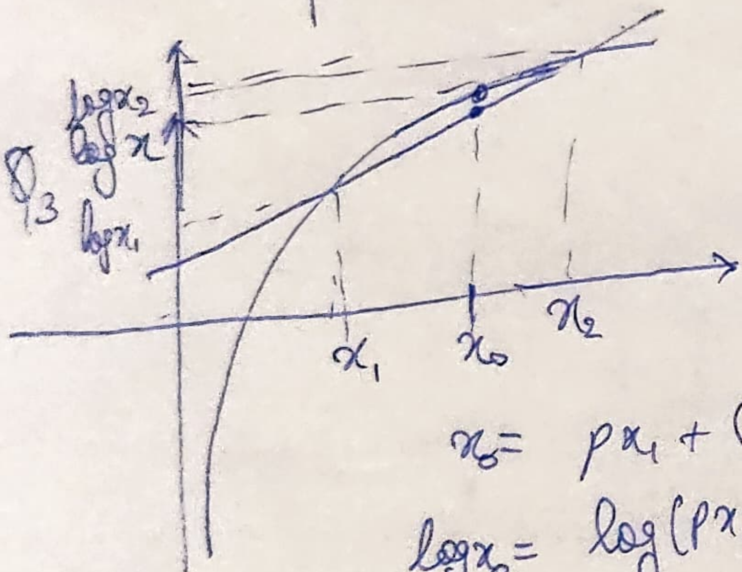
Ignoring truncation would give us estimate of $\lambda = \frac{1}{\mu + i}$

but by taking in account the truncation

~~λ wi~~ mean will be ~~d~~ subtracted by 10 and

gives us diff λ value.

Q3



$$x_0 = p x_1 + (1-p) x_2$$

$$\log x_0 = \log(p x_1 + (1-p) x_2)$$

Eqⁿ of secant $y = \frac{\log x_2 - \log x_1}{x_2 - x_1}(x - x_1) + \log x_1$

put $x \to x_0$

$$y = \frac{\log x_2 - \log x_1}{x_2 - x_1}(p x_1 + (1-p) x_2 - x_1) + \log x_1$$

$$y = \frac{\log x_2 - \log x_1}{\cancel{x_2 - x_1}} \cancel{(x_2 - x_1)}(1-p) + \log x_1$$

$$y = \log x_2 - p \log x_2 + p \log x_1$$

$$y = p \log x_1 + (1-p) \log x_2$$

we know

~~log x > y~~ $y \le \log x$

$$p\log x_1 + (1-p)\log x_2 \le \log\left[p x_1 + (1-p) x_2\right]$$

$$\cancel{E[G(x)] \le \log}$$

$$\Rightarrow E[\log(x)] \le \log\left[E(x)\right]$$

using this in

$$D_{KL}(P \| Q) = \sum_{i=1}^{K} p_i \log\left(\frac{p_i}{q_i}\right)$$

$$\sum_{i=1}^{K} p_i \log\left(\frac{q_i}{p_i}\right) \le \log\left(\sum_{i=1}^{K} \cancel{p_i} \cdot \frac{q_i}{\cancel{p_i}}\right)$$

$$\sum_{i=1}^{K} p_i \log\left(\frac{q_i}{p_i}\right) \le \log\left(\sum_{i=1}^{K} q_i\right)$$

$$\sum_{i=1}^{K} p_i \log\left(\frac{q_i}{p_i}\right) \le 0$$

$$\Rightarrow \boxed{\sum_{i=1}^{K} p_i \log\left(\frac{p_i}{q_i}\right) \ge 0}$$

for equality $p_i = q_i \Rightarrow P(x) = Q(x)$

$$q_{min} = \arg\min_q D_{KL}\left[P(x) \| Q(x)\right]$$

$$= \arg\min_q \sum_{i=1}^{K} p_i\left[\log(p_i) - \log q_i\right]$$

Since we are minimizing q, p_i is not necessary

$$q_{min} = \arg\min_q -\sum_{i=1}^{K} p_i \log q_i$$

which is same as definition of cross-entropy

Q4 $X \sim N(\mu, 10^2)$

$H_0 : \mu = 500$
$H_1 : \mu \neq 500$

$n = 16$

$\bar{X} = 504$ gm

$\sigma = 10$

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{504 - 500}{10/\sqrt{16}} = 1.6$$

$\alpha = 0.05$

$\alpha/2 = 0.025$

$Z_{\alpha/2} = 1.96$

$|T| < Z_{\alpha/2}$ we will accept H_0

Interpretations

- The evidence shows that our null hypothesis is very unlikely is the condition when we reject H_0

If evidence (test) show that our value is within condifence limit of our true mean we fail to reject H_0

- If sample size is increased we might reject H_0 because value of z-score increase as standard error $= \frac{\sigma}{\sqrt{n}}$ decreases