Derive the far-field radiation pattern for a combination of a dipole antenna placed at the focus of a parabolic dish. Solve the problem for a 1-D case, and assume that the dipole illuminates the whole aperture of the dish uniformly.

We know that the far-field radiation pattern, $F(\theta)$ is given by the fourier transform of the aperture distribution, $I(x_\lambda)$ -

$$F(\theta) = \int_{-\infty}^{\infty} I(x_\lambda) \cdot e^{-i2\pi x_\lambda \cdot \theta} dx_\lambda \tag{1}$$

where $x_\lambda = \frac{x}{\lambda}$. In this case of a uniform distribution $I(x_\lambda)$ will be given by $I(x_\lambda) = \begin{cases} A & |x_\lambda| \leq \frac{D}{2\lambda} \\ 0 & |x_\lambda| \geq \frac{D}{2\lambda} \end{cases}$ where D is the size of the aperture and A is the constant magnitude of its distribution. From this we can reduce (1) to -

$$F(\theta) = \int_{-\frac{D}{2\lambda}}^{\frac{D}{2\lambda}} A \cdot e^{-i2\pi x_\lambda \cdot \theta} dx_\lambda \tag{2}$$

Integrating the exponential further gives -

$$F(\theta) = \frac{-A}{2\pi i \cdot \theta} \cdot e^{-i2\pi x_\lambda \cdot \theta} \Bigg|_{-\frac{D}{2\lambda}}^{\frac{D}{2\lambda}} = \frac{-A}{2\pi i \cdot \theta} \cdot (e^{-i\frac{\pi D\theta}{\lambda}} - e^{i\frac{\pi D\theta}{\lambda}}) \tag{3}$$

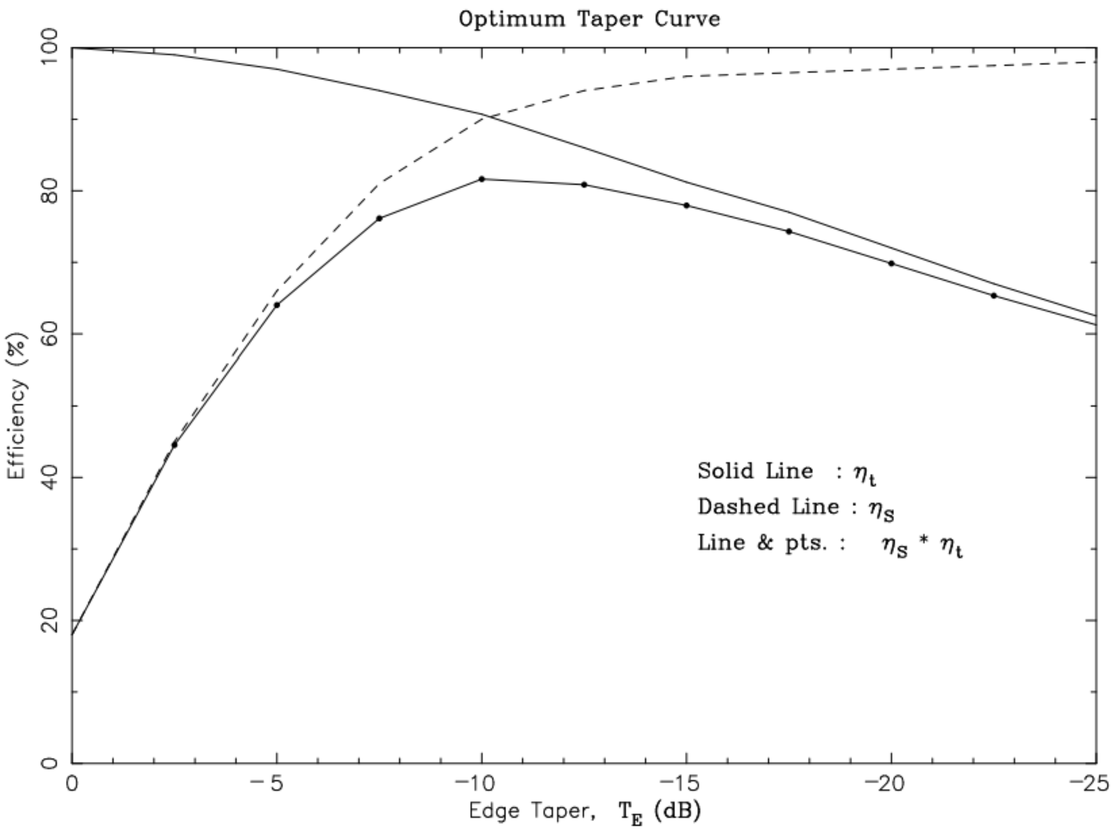
Since $e^{i\theta} = \cos\theta + i\sin\theta$, (3) yields -

$$F(\theta) = \frac{DA}{\lambda} \cdot \frac{\sin(\frac{\pi D\theta}{\lambda})}{\frac{\pi D\theta}{\lambda}} \tag{4}$$

which is nothing but a scaled sinc function.

Explain how astronomers arrive at an optimum range of taper for the aperture illumination.

The aperture efficiency η_a (ratio of effective collecting area to total physical area) is dependent on 2 important factors - the illumination efficiency η_t (measure of the non-uniformity/taper of the field on the dish) and the spillover efficiency η_s (portion of the feed power intercepted by the dish). However these 2 factors have complementary trends with the amount of edge taper (the fall-off of the power at the edge of the dish relative to the center). Lower edge taper implies larger η_t but results in more spillover ie smaller η_s and vice versa. The trends of η_t and η_s with the amount of edge taper being opposite, their product will be very low at the extremes and maximum at some middle value. Below is the plot of both efficiencies with amount of taper [1] -



As we can see, the maximum value the aperture efficiency is obtained around -11 dB of edge taper.

[1] Source - Chapter 19, Low Frequency Radio Astronomy a.k.a. blue-book, NCRA

Let V and σ_V be the voltage values and their standard deviation for a signal picked up by the receiver, and let P and σ_P be the power and its std deviation for the same signal. What are the relationships between these quantities? Quantitatively describe how σ_P is reduced by integration.

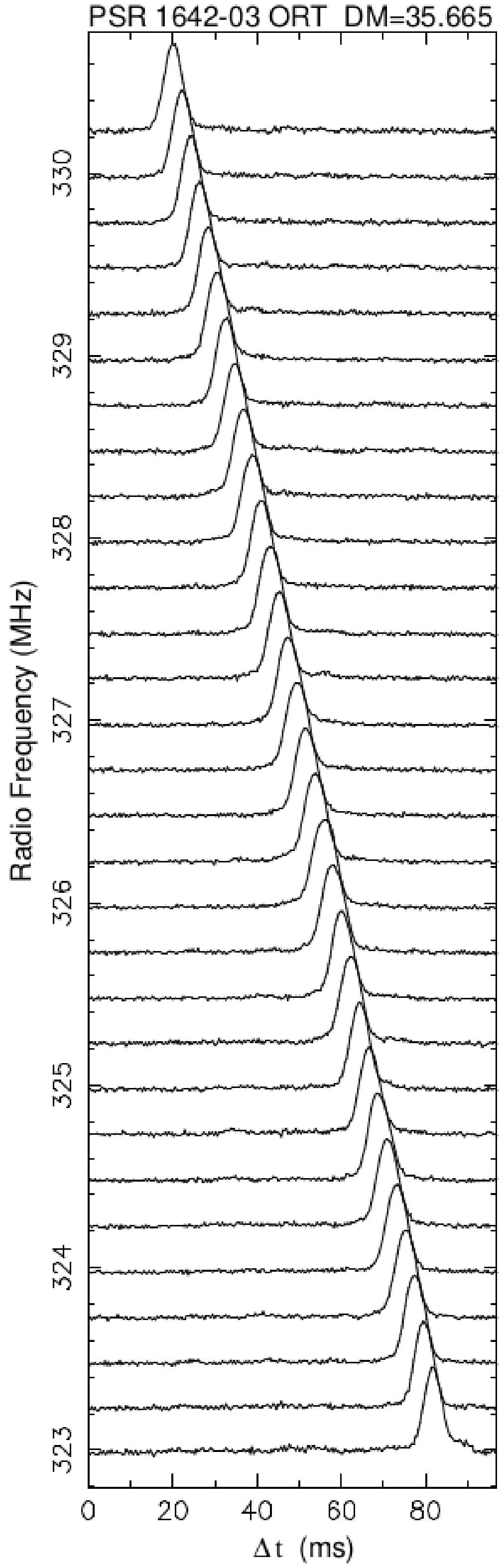
Due to square-law detection the random variable $P \sim V^2$ (ignoring proportionality constant). The voltage values V are assumed to have a gaussian distribution and they will have a mean value of 0. The variance of this distribution σ_V^2 is given by $E(V^2) - E^2(V)$ where $E(X)$ represents the expectation value of random variable X. $E(V)$ being zero $\sigma_V^2 = E(V^2) = E(P)$. $E(P)$ is simply the mean value of the power P. Hence $\sigma_V^2 = \mu_P$ – (1).

Now $\sigma_P^2 = E(P^2) - E^2(P) = E(V^4) - \sigma_V^4$. Knowing the gaussian distribution $E(V^4)$ can be calculated and it comes out to be $3\sigma_V^4$. So $\sigma_P = \sqrt{2}\sigma_V^2$ – (2). The relations (1) and (2) express the relationships between V, σ_V, P and σ_P.

The signal passing through detector and integrator will be digital, with a sampling frequency being equal to the Nyquist frequency ie $2\Delta\nu$ where $\Delta\nu$ is the bandwidth of the signal. If the detected signal is integrated over integration time τ then the number of distinct values being added are $N = \tau \cdot 2\Delta\nu$. We know that the sum of N gaussian random variables with mean μ and std deviation σ, also has a normal distribution with mean μ and std deviation $\sigma/\sqrt{N}$. So the std deviation of the integrated signal values σ'_P will be equal to $\sigma_P/\sqrt{N} = \sigma_P/\sqrt{2\Delta\nu\tau}$. Hence σ_P will get reduced and since it has a inverse relation with the SNR, it increases the SNR.

What is dispersion in the context of a pulsar signal?

Dispersion in the context of pulsars is similar to how a refractive medium like a glass prism causes the speed of light to have a frequency dependent decrease in the medium. The radiation from the pulsar travelling through the ionized plasma in the interstellar medium interacts with the ions (mainly electrons) which slows it down such that lower frequencies are slowed down more relative to the higher frequencies.



So even though all the frequencies leave the pulsar simultaneously, the higher frequencies will be intercepted by the telescope earlier than lower frequencies as shown in the image[2]. This results in the apparent broadening of the pulse of the pulsar signal.

[2]Source - Detection of radio emission from pulsars - A pulsar observation primer - Dipankar Bhattacharya