Practical sampling instruments record voltage values over a small but finite time and not instantaneously. How is the interpolated signal affected by this?

Let T(t) be the continuous analog signal before sampling and S(t) the sampled signal with a sampling interval τ. Then we can write $S(t) = T(t) \cdot D_\tau(t)$ where $D_\tau(t)$ is a dirac comb with τ as the spacing between the individual delta functions.

Then by the convolution theorem the fourier transform of S(t), F(S) will be the convolution of $F(T)$ with $F(D_\tau)$. The fourier transform of T(t) will simply be the spectrum of T, $f(\nu)$ ie the frequency components of T. We assume the spectrum range of our T to be from 0 to $\Delta\nu$. Further the fourier transform of a Dirac comb is also a dirac comb with inverse spacing, so $F(D_\tau(t)) = D_{\frac{1}{\tau}}$. Hence -

$$F(S(t)) = f(\nu) * D_{\frac{1}{\tau}} \tag{1}$$

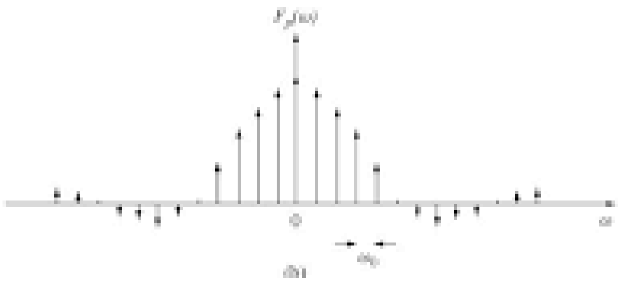
Now convolution of $f(\nu)$ with $D_{\frac{1}{\tau}}$ means a sum of infinite copies of $f(\nu)$ which are displaced relative to each other by $\frac{1}{\tau}$. We know that $f(\nu)$ is non-zero everywhere except $[-\Delta\nu, \Delta\nu]$. So if the sampling frequency $f_s = \frac{1}{\tau} \geq 2\Delta\nu$, then F(S(t)) would simply look like infinite copies of $f(\nu)$ laid side by side. Now if we multiply F(S(t)) with a rectangular function of width $\Delta\nu$ then we get as an output our spectrum $f(\nu)$.

In other words, in time domain all this corresponds to convolving our S(t) with a suitable sinc function (achieved by a low-pass filter), which will be given by the fourier transform of the above rectangular function of width $\Delta\nu$. This will give the interpolated continuous time signal from our digital one.

However practical sampling instruments dont record values instantaneously but during a finite time. This means that our sampled signal S(t) is now instead given by $S(t) = T(t) \cdot D'_\tau(t)$ where $D'_\tau(t)$ is an infinite train of finite width rectangular functions separated by τ. Then F(S) will be -

$$F(S(t)) = f(\nu) * F(D'_\tau) \tag{2}$$

Now the fourier transform of D' comes out to be the dirac comb $D_{\frac{1}{\tau}}$ **but** with the *heights of the delta functions modulated by a sinc function*, as shown in the figure [1]. Convolution of $f(\nu)$ with this function would again yield infinite copies of $f(\nu)$ laid side by side but successive copies away from the origin would get scaled down with the sinc. But in the case where $f(\nu)$ is centered on the origin, we can still retrieve its exact theoretical form by again passing S(t) through a low-pass filter.



However if the bandwidth of our signal is not centered on 0, then the copy of $F(\nu)$ in the above convolution that will be centered on 0 will be much scaled down. Using a low pass filter in this case would yield a very weak interpolated signal.

How do the detector and integrator change the spectrum of the signal?

Square-law detector

Let the voltage signal just before square-law detection be composed of frequencies running from $[a, b]$, the bandwidth being $b - a = \Delta\nu$. Then the signal can be thought of as a sum of many sinusoidal signals having frequencies in this range. Consider any two of these frequencies ω_1 and ω_2. So the signal looks like $s(t) = c_1 \sin\omega_1 t + c_2 \sin\omega_2 t + ...$

The result of square law detection is that this sum is getting squared. So the new signal becomes (ignoring the amplitudes c_1 and c_2 from now)-

$$sin^2(\omega_1 t) + sin^2(\omega_2 t) + 2sin(\omega_1 t)sin(\omega_2 t)$$

$$\Rightarrow 1 - \frac{1}{2}(\cos(2\omega_1) + \cos(2\omega_2)) + \cos(\omega_1 - \omega_2)t - \cos(\omega_1 + \omega_2)t$$

using trigonometric identities. We have a constant value which represents a signal of non-zero power and 0 frequency. From ω_1 and ω_2, we get the frequencies $2\omega_1$, $2\omega_2$, $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$.

[1]Source - www.sonoma.edu

These 4 frequencies will be generated by every pair of frequencies in the range $[a, b]$ from the original signal. Out of all the frequencies generated from detection, $2b$ would be the highest. And all the frequencies between 0 and 2b will be generated by detection. Hence the range of frequencies of the signal after detection is $[0, 2b]$. While we ignored the amplitude of the individual frequencies, it can be seen that the amplitudes of the resulting frequencies get changed in a non-linear way so the shape of the spectrum gets totally changed.

Integrator

Again consider the input of the integrator as a sum of many sinusoidal components. Let the integration time be τ. Then we are summing all the values of the signal in time bins of size τ. For frequency components less than $1/\tau$, all the values in each bin will have a phase difference less than 2π so these frequencies remain preserved. But for frequencies f greater than $1/\tau$ each integration bin will have at least one complete cycle. Adding values of a complete cycle yields 0, hence only the portion $\tau \,\%\, \frac{1}{f}$ would be added in each bin. This would cut all the oscillations in the input faster than $1/\tau$ and reduce these f to frequencies lower than $1/\tau$. Hence the range of frequencies of the output of the integrator will be $[0, \frac{1}{\tau}]$.