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Exact physical model of magnetorheological damper

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ABSTRACT

This paper attempts to fill the gap in the literature by introducing and discussing an enhanced physical model of the MR damper. The essence of the presented model is to combine the effect of compressibility of the MR fluid enclosed in each chamber with the effect of blocking the flow between the chambers in the case of a low pressure difference. As it will be shown, the concurrence of both considered phenomena significantly affects mechanical behaviour of the damper, influences its dissipative characteristics, and in particular, it is the reason behind the distinctive ‘z-shaped’ force–velocity hysteresis loops observed in experiments. The paper presents explanation of the observed phenomena, detailed derivation of the thermodynamic equations governing response of the damper, their implementation for various constitutive models of the magnetorheological fluid and, finally, formulation of the corresponding reduced and parametric models. Experimental validation shows that proper identification of physical parameters of the proposed mathematical model yields the correct shapes of force–velocity hysteresis loops.

1. Introduction

Among many controllable devices applied in the systems of adaptive impact absorption [1–3], one of the most promising is semi-active magnetorheological damper [4]. The crucial problem in the optimal design and control of such devices is the accurate mathematical modelling of their response under external excitation. Since mechanical behaviour of the MR dampers is influenced by diverse thermodynamic and rheological phenomena, the corresponding mathematical models are relatively complex and their derivation is still the subject of intensive research efforts.

Two basic types of models of magnetorheological dampers are widely considered in the literature:

i) Parametric phenomenological models, which are typically based on Bingham plastic model [5,6], Bouc–Wen hysteretic model [7–10] or Duffing's equation [11]. Other types of models are based on sigmoid [12] or hyperbolic tangent function [13] or neural networks [14]. A very good review of parametric models can be found in [15].
ii) Physical models, which typically utilize a viscoplastic model of the MR fluid and the equations that govern its flow through an orifice [16].

Although a large number of various parametric models is already developed, many miscellaneous modifications of these models are permanently proposed. Most of the parametric models can properly capture the dynamic characteristics of the MR dampers, however their disadvantage is a large number of internal parameters, difficult and time consuming procedure of model tuning and the ambiguity of the obtained solution. On the other hand, the physical modelling of the MR dampers is

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much less developed. Physical models often utilize simplified assumptions such as the incompressible Bingham or Herschel–Bulkley plastic models of the fluid and the steady Poiseuille flow through the orifice. It seems that such constructed physical models cannot accurately explain all dissipative properties obtained in experimental tests.

This paper attempts to fill the gap in the literature by introducing and discussing an enhanced physical model of the MR damper. The essence of the presented model is to combine the effect of compressibility of the MR fluid enclosed in each chamber with the effect of blocking the flow between the chambers in the case of a low pressure difference. As it will be shown, the concurrence of both considered phenomena significantly affects mechanical behaviour of the damper, influences its dissipative characteristics, and in particular, it is the reason behind the distinctive 'z-shaped' force–velocity hysteresis loops observed in experiments.

The main objective of the paper is twofold. At first, it is to derive the mathematical equations of the exact physical model of the MR damper, to reveal their specific forms corresponding to various constitutive equations of the fluid, and to present different options of their possible simplifications. Secondly, it is to show that after measurement or identification of physical parameters the proposed mathematical model yields the correct shapes of force–velocity hysteresis loops.

The development of the parameter identification procedure and an exact comparison of the numerical and experimental results is out of scope of the current stage of research. Consequently, the model is based on the exact geometry of the damper and the assumption of a quadratic dependence of the yield stress level and apparent viscosity on the applied current. The coefficients are manually adjusted to fit the conducted experiment. The research is concluded by proving that a proper selection of these parameters within their physical range allows to achieve a satisfactory qualitative correspondence between the numerical and the experimental results for a selected set of working conditions.

The considerations start with a presentation of the results obtained during experimental testing of the damper and with their basic explanation. The subsequent section is aimed at a detailed derivation of the mathematical model, which utilizes an analytical model of the viscous flow and fundamental laws of thermodynamics in order to obtain a convenient form of the equations that describe the balance of the fluid volume and the balance of the fluid energy. The consecutive part presents the governing equations and the corresponding numerical results for various constitutive models of the magnetorheological fluid that involve diverse treatments of compressibility. Finally, in the last section, a reduced model is proposed, where the generated force is expressed analytically in terms of the kinematic excitation and a special form of the corresponding parametric model. The basic form of the governing equations based on fluid decomposition theory and their implementation for two constitutive models of the fluid were presented in our previous conference publication [17].

2. The experiment and its basic explanation

A series of dynamic tests was carried out on a magnetorheological fluid damper MRD1003-5 produced by the LORD company. The damper is designed for adaptive systems of vibration damping of driver's seats in heavy vehicles and it provides the maximal damping force of 2224 N at the piston velocity of 50 mm/s. The schematic view of the tested damper, which can be regarded as an example of a typical MR damper, is presented in Fig. 1b.

The damper consists of two chambers divided by a piston that contains a magnetic circuit. The movement of the piston induces the MR fluid flow through the channel in the presence of the magnetic field, which is perpendicular to the direction of the flow. Additionally, the damper is equipped with a gas spring located at the bottom of the lower chamber, which provides an initial pressure of approximately 2 MPa.

During the tests the damper was subjected to harmonic and linearly varying kinematic excitation with different displacement amplitudes and velocities by means of an MTS hydraulic testing system, consisting of FlexTest GT controller and a linear actuator type 242.01 (Fig. 1a). The experimental stand enabled measurements of force, displacement, velocity of the piston and the temperature of the damper housing, as well as recording analog control signals (voltage and current) of the electronic power amplifier, which was driving the damper.

The tests were conducted for selected range of amplitudes and frequencies of the kinematic excitation. The excitation frequency range was between 0.5 Hz and 4 Hz, and the applied control current varied from 0 A to 1 A. The most important result of the experiment was the occurrence of force–velocity hysteresis loops of a characteristic shape with large internal region in the vicinity of zero velocity. The evolution of hysteresis loops obtained experimentally for different combinations

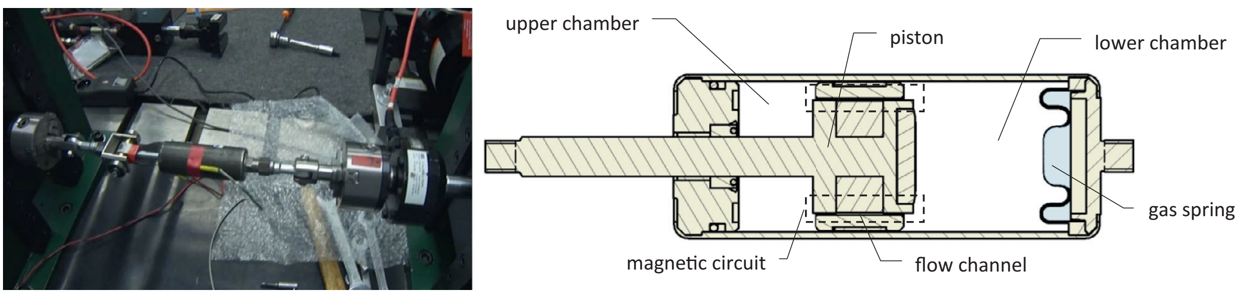


Fig. 1. (a) Experimental stand, (b) schematic model of the MR damper.

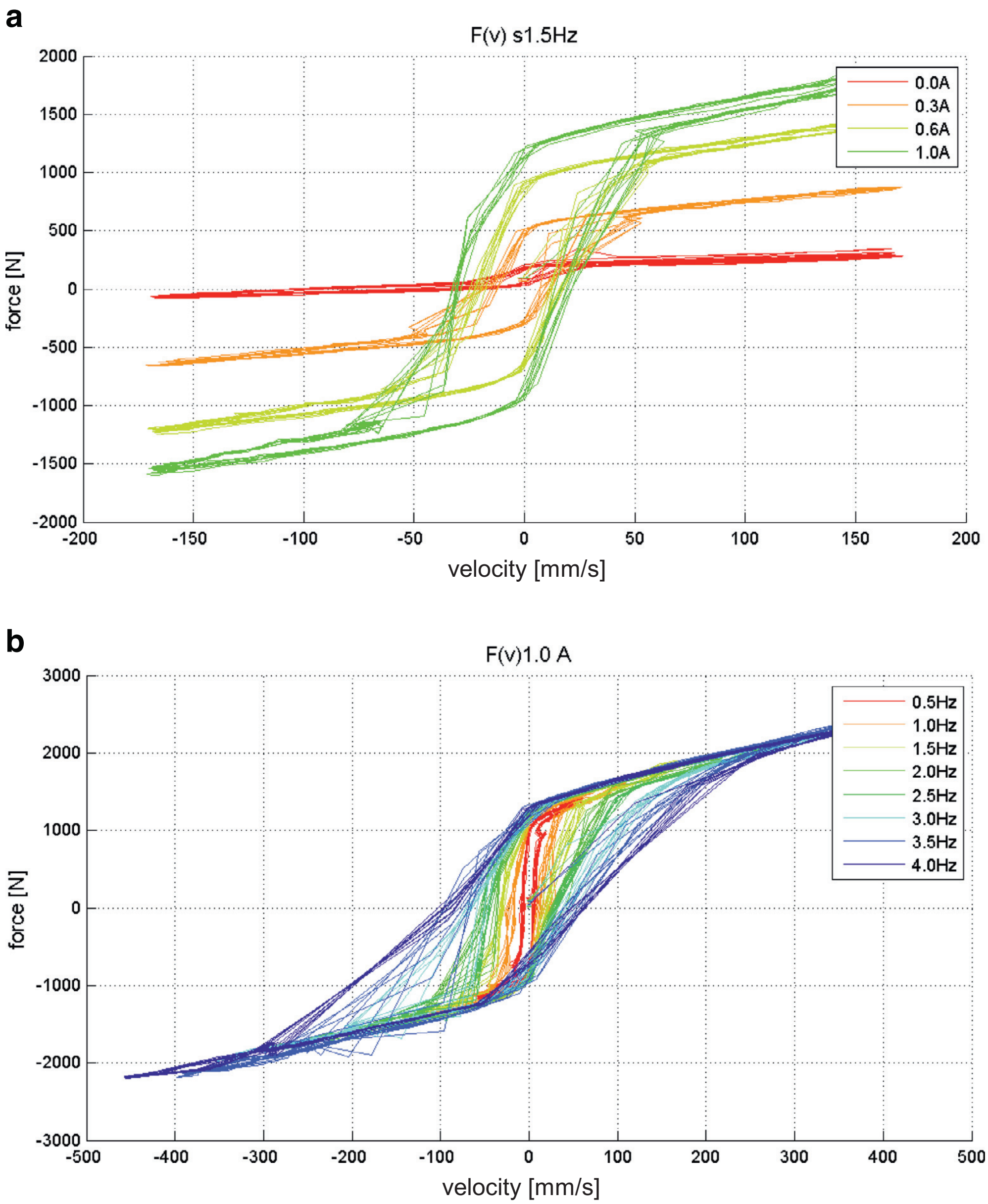


Fig. 2. Force–velocity hysteresis loops generated by MR damper subjected to sinusoidal kinematic excitation: (a) change in terms of applied current intensity; (b) change in terms of excitation frequency.

of excitation frequencies and control currents is presented in Fig. 2. These hysteresis loops were analysed in terms of driving current and parameters of the kinematic excitation (amplitude and frequency).

Conducted review of literature had revealed that force–velocity hysteresis loops of a similar shape can be relatively easily obtained with the use of various parametric models (e.g. the Bouc–Wen model or Duffing's equation) but not by using standard physical models of MR dampers. Consequently, an explanation of the physical phenomena that lead to the occurrence of the presented characteristic hysteresis loops requires an utterly new modelling approach.

Herein, the fundamental explanation is based on the analysis of other systems that are characterized by an analogous type of the dynamic response and by the force–velocity hysteresis loops of a similar shape. The simplest example are the systems that involve friction forces described by the bristle friction models (e.g. Dahl or LuGre model), see Fig. 3. In the bristle models, the friction force depends on both the relative velocity between contacting surfaces v and the local displacement of bristles z, which changes rapidly at the beginning of the relative motion in each direction and which further remains constant at a limit value.

Consequently, the formula for the friction force contains a classical term that indicates the velocity-dependent viscous force and an additional term that denotes the elasto-plastic force generated by the local displacement of bristles, which is

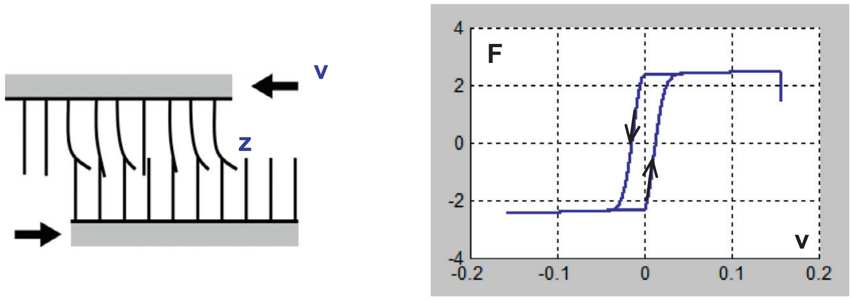


Fig. 3. Bristle friction models: (a) basic scheme, (b) obtained force–velocity hysteresis loops.

governed by the additional differential equation:

$$\begin{aligned} F_{fr} &= \sigma_1 v + \sigma_2 z \\ \dot{z} &= f(v, |v|, z). \end{aligned} \tag{1}$$

The evolution of the local deformation of bristles (as well as the corresponding differential equation) indicates that a non-zero elastic force appears when the relative velocity of the contacting surfaces changes direction. In the following stage of the motion, the elastic force rapidly decreases, changes sign and decreases further until the negative of its initial value is reached. The presence of such variable elastic force, which occurs exclusively at the vicinity of zero velocity, is the reason for the occurrence of the characteristic force–velocity hysteresis loops.

It can be expected that a similar process of generation of the elastic force occurs in the MR damper. Consequently, the developed model takes into account the effect of blocking the flow in the case of critical pressure difference and compressibility of the MR fluid enclosed in each chamber. Such assumptions have the following implications. At the end of the stroke, when the shaft velocity approaches zero, the pressures are equalized until the pressure difference decreases to the critical value and the flow is blocked. Starting from this instant, exclusively the elastic force caused by compressibility of the fluid is generated. In the further stage of the process, a backward movement of the piston and small compressibility of the fluid results in a rapid decrease of the elastic force, a sign change and a further decrease until the critical pressure difference is reached again. The entire process influences the dissipative characteristics of the damper, and in particular, it is the reason of generation of the characteristic "z-shaped" hysteresis loops, which resemble the ones obtained from bristle friction models.

3. Thermodynamic modelling of MR damper

The derived mathematical model of the damper will be focused on exact constitutive modelling of the MR fluid and on modelling of the thermodynamic processes that arise in the MR fluid during operation of the damper. The gas cushion located in one of the chambers will be modelled as a nonlinear spring and it will not be subjected to a detailed thermodynamic analysis. Finally, the model will assume uniformity of the magnetic field in the entire orifice and its constant value during the entire process, which indicates that the passive mode of damper operation is considered.

As it is well known, modelling of the thermo-mechanical problems is based on three fundamental principles: conservation of mass, conservation of momentum and conservation of energy, which are expressed as partial differential equations and supplemented by a proper constitutive relations. In the classical models of two-chamber hydraulic, pneumatic or magneto-rheological dampers subjected to a relatively slow excitation, a reasonable assumption is homogeneity of the parameters of the fluid enclosed in each chamber. In such situation the original set of PDEs (usually simplified to a stationary case) is solved exclusively for the valve region in order to determine the mass flow rate through the valve in terms of the parameters of the fluid in both chambers. The remaining part of the model is simplified to an initial-value problem that involves:

- two ordinary differential equations that govern the balance of the MR fluid mass in each chamber,
- a differential or algebraic equation that govern the equilibrium of the piston (dynamic or static case),
- two ordinary differential equations that govern the balance of the MR fluid energy in each chamber.

Summation and integration of the equations of mass balance for two chambers of the damper leads to the obvious condition of conservation of the total mass of the fluid enclosed in the damper and eliminates one differential equation. Similarly, summation of equations of energy balance for each chamber allows to obtain global balance of energy of the fluid enclosed inside the damper. Another element of the model is the constitutive equation that defines the relation between parameters of the fluid (its pressure, temperature, mass and volume). It can be expressed either by an algebraic equation of state or, alternatively, by definitions of the coefficients of compressibility and thermal expansion. In the most straightforward approach, the analytical model of the flow and the constitutive equations are introduced into the balance equations in order to obtain the thermodynamic model of the damper expressed in terms of the displacement of the piston, the pressures and temperatures of the fluid in each chamber. The final form of the governing equations and their coupling strongly depends

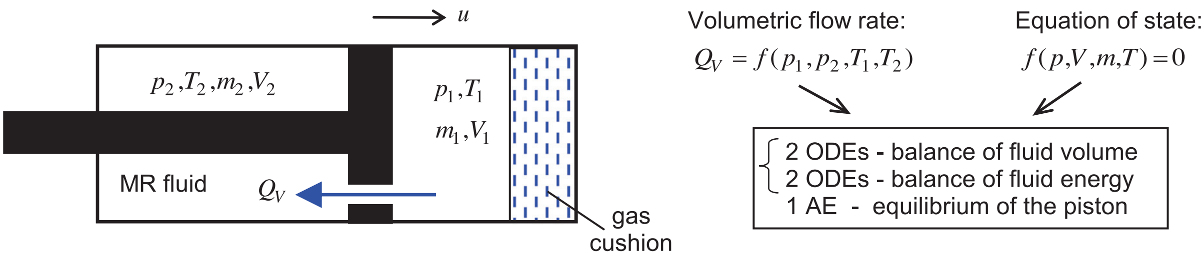


Fig. 4. Considered model of a double-chamber MR damper.

on the assumed model of the flow and the applied constitutive relations, especially on the fluid compressibility and on the thermal expansion.

In this paper, the derivation of the model of the damper will follow the above described classical methodology by using a constitutive model that assumes small compressibility of the magneto-rheological fluid. Compressibility and thermal expansion of the fluid will be neglected at the stage of solving the stationary partial differential equations that define the valve flow and the corresponding calculation of the volumetric flow rate of the fluid through the valve. On the other hand, both the compressibility and thermal expansion of the fluid will be taken into account in the derivation of the thermodynamic balances that govern the response of the fluid enclosed in each chamber of the absorber. Despite the application of various assumptions at subsequent stages of derivation, the consistency of the entire model will be provided.

The balance equations will be derived in a slightly different manner than in the classical approach, since the balance of the fluid mass will be replaced by the balance of the fluid volume. From the thermodynamic point of view, such an approach is fully justified because the volume is an equally important extensive thermodynamic quantity and the balance of fluid volume can be considered as equivalent to the balance of fluid mass. Moreover, in the considered case of a kinematic excitation, the entire considered problem will be partially decoupled, i.e. the equations that govern the balance of volume and the balance of energy will allow the thermodynamic parameters of gas to be determined without considering the equation of piston equilibrium. Finally, the proposed model will be composed of:

- two differential equations that will govern the balance of the MR fluid volume for each chamber,
- two differential equations that will govern the balance of the MR fluid energy for each chamber,
- a decoupled algebraic equation that will govern the equilibrium of the piston.

At first, the derivation of the thermodynamic model of the damper will be presented by using a straightforward methodology, the so called *global approach*. In this method, the analytical model of the valve flow and the global properties of the MR fluid (its compressibility and thermal expansion) will be introduced into the equations of volume and energy balance in order to obtain the model of the damper in the form of four differential equations involving the unknown pressures and temperatures of the fluid in each chamber. It will be shown that the method requires the volumetric inflow rate determined for compressible MR fluid and that it cannot be directly used with the standard model of incompressible flow.

Secondly, the so called decomposition approach will be introduced, where the MR fluid enclosed in each chamber will be decomposed into the incompressible viscous fluid and the compressible inviscid fluid with thermal expansion. Such approach will yield an analytical formula that will define the total volumetric flow rate of the MR fluid as the sum of the volumetric flow rate of the viscous fluid and the volumetric flow rate of the compressible fluid. Moreover, all the derived balance equations will be expressed in terms of the local thermodynamic properties of the component fluids. Solution of these equations will allow to determine pressures and temperatures of the MR fluid in both chambers, the total force generated by the damper and the shape of the corresponding force–velocity hysteresis loops.

3.1. Model of the fluid flow

The first step of modelling is the solution of the problem of the MR fluid flow through the valve and the calculation of the corresponding volumetric flow rate in terms of the parameters of the fluid in both chambers of the damper. This stage is conducted in a classical way and standard assumptions for the modelling of the magneto-rheological fluid are applied. The MR fluid is considered to be the incompressible Bingham viscous fluid [18]. Consequently, the general flow equations are reduced to the incompressibility equation, the classical momentum equation and the decoupled energy conservation equation. Since the objective of the model is to calculate the fluid velocity **v**, only two former equations are required:

$$\begin{aligned} &\nabla \cdot \mathbf{v} = 0 \\ &\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}. \end{aligned} \tag{2}$$

In addition, we apply the Bingham constitutive equation, where the dependence between the fluid stress tensor $\boldsymbol{\sigma}$ and the strain rate tensor $\mathbf{D}$ takes the form:

$$\begin{aligned}
&\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \\
&|\boldsymbol{\tau}| \le \boldsymbol{\tau}_0 \quad \rightarrow \quad \mathbf{D} = 0 \\
&|\boldsymbol{\tau}| > \boldsymbol{\tau}_0 \quad \rightarrow \quad \boldsymbol{\tau} = \pm\boldsymbol{\tau}_0 + 2\mu\,\mathbf{D}.
\end{aligned} \tag{3}$$

In the above equations $\boldsymbol{\tau}_0$ indicates the limit value of the shear stress, which in general depends on the local value of the applied magnetic field $\boldsymbol{\tau}_0 = \boldsymbol{\tau}_0(x, y)$. For the case of a two-dimensional stationary flow in a channel with a constant magnetic field, the above problem can be solved fully analytically and the solution is well known in the literature. The essence of the method is the assumption of a constant pressure gradient along the channel and the calculation of the distribution of the shear stresses in the fluid. Comparison of these stresses with the limit shear stress allows two situations to be distinguished:

- for pressure gradient above a certain limit: the flow field can be divided into two external regions of viscous flow and an internal region of the "plug flow" (flow with a zero strain rate),
- for pressure gradient below this limit: the flow of fluid does not occur.

The obtained fluid velocities are functions of the pressure gradient and they depend on the position across the width of the channel. The volumetric flow rate of the fluid Q_V^f is calculated by an integration of the fluid velocity over the cross section of the channel, while the mass flow rate Q_m^f is obtained simply by an additional multiplication by the fluid density:

$$Q_V^f(\Delta p) = 2\int_{dA_1} v^{visc}(y)dA_1 + \int_{dA_2} v^{plug} dA_2 \text{ and } Q_m^f = \rho_f Q_V^f, \tag{4}$$

where A_1 and A_2 are the areas of the viscous flow and the plug flow, respectively. The final definition of the flow rate takes the form:

$$\begin{aligned}
&Q_V^f = f(\Delta p) \quad \text{for} \quad \Delta p \ge \Delta p^{crit} \\
&Q_V^f = 0 \quad \text{for} \quad \Delta p < \Delta p^{crit}.
\end{aligned} \tag{5}$$

The exact formulae will not be stated since they seem too detailed for the assumed high level of generality. Nevertheless, the important conclusion is that the volumetric and mass flow rates of an incompressible MR fluid can be expressed analytically in terms of the pressure gradient and that for a low value of the pressure difference the flow does not occur. Although the temperature of the fluid can be computed from the decoupled equation of energy balance, it is not required for further calculations.

3.2. Balance of fluid volume

The equations of volume balance are usually used for the modelling of dampers based on incompressible viscous fluids, since in such a case their structure closely resembles the equations of mass balance, their interpretation is straightforward, and moreover, after summation and integration they directly lead to simple models where the total generated force is a sum of the elastic and viscous forces. Here, the equations of volume balance will be generalized and applied to the MR fluid characterized by compressibility and thermal expansion. Using the balance of volume instead of the balance of mass will facilitate a direct application of the exact definitions the coefficients of compressibility and thermal expansion, as well as a direct transformation of the derived equations into the classical model by assuming incompressibility of the fluid.

In the proposed approach, the volume of the fluid V will be considered to be the thermodynamic potential of fluid pressure p, temperature T and mass m:

$$V = V(p, T, m). \tag{6}$$

Thus, the total differential of the fluid volume can be expressed as a sum of the partial derivatives with respect to the subsequent thermodynamic parameters:

$$dV = \frac{\partial V}{\partial p}dp + \frac{\partial V}{\partial T}dT + \frac{\partial V}{\partial m}dm. \tag{7a}$$

By performing differentiation with respect to time and by changing the differentials into time derivatives we obtain:

$$\dot{V} = \frac{\partial V}{\partial p}\dot{p} + \frac{\partial V}{\partial T}\dot{T} + \frac{\partial V}{\partial m}\dot{m}. \tag{7b}$$

The above equation indicates that the total change of the fluid volume (e.g. calculated on the basis of piston displacement) equals to the sum of the volume changes caused by the change of the particular thermodynamic parameters. The above equation of volume balance is useful for determination of fluid parameters when derivatives of volume with respect pressure, temperature and mass can be calculated, i.e. when the constitutive relations are known. The constitutive relations can be expressed in the form of an algebraic equation of state (which involves either mass and velocity of the fluid or its density):

$$f(p,T,m,V)=0 \text{ or } f(p,T,\rho)=0, \tag{8}$$

or, alternatively, by definitions of the coefficients of fluid compressibility, thermal expansion and introduced by the authors 'mass expansion':

$$\beta=-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T,\ \alpha=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \text{ and } \gamma=\frac{1}{V}\left(\frac{\partial V}{\partial m}\right)_{p,T}=\frac{1}{m}. \tag{9}$$

Note that the last quantity is independent of the equation of state and that it is introduced only to facilitate a coherent formulation of the following equations. By using the above definitions, the equation of volume balance can be rewritten in a concise form:

$$\dot{V}+\beta V\dot{p}-\alpha V\dot{T}=\gamma V\dot{m}. \tag{10a}$$

Further, the term at the right-hand side can be identified as the mass inflow rate divided by density, which indicates the volumetric inflow rate of the fluid:

$$\dot{V}+\beta V\dot{p}-\alpha V\dot{T}=\frac{Q_m}{\rho} \text{ and } \dot{V}+\beta V\dot{p}-\alpha V\dot{T}=Q_V. \tag{10b,c}$$

It should be stressed that the above equations refer to a compressible MR fluid. In particular, Eq. (10c) can be directly applied only if the definitions of the coefficients of compressibility and thermal expansion of the MR fluid in terms of pressure and temperature ($\beta(p, T)$ and $\alpha(p, T)$) are known, and moreover, if the analytical formulae defining the volumetric inflow rate of the MR fluid through the valve Q_V can be determined. In such case the equations that govern the balance of volume of the MR fluid enclosed in both chambers take the form:

$$\begin{aligned}&\dot{V}_1+\beta_1V_1\dot{p}_1-\alpha_1V_1\dot{T}_1=Q_V^1\\&\dot{V}_2+\beta_2V_2\dot{p}_2-\alpha_2V_2\dot{T}_2=Q_V^2.\end{aligned} \tag{11a}$$

The method described above will be further called the *global approach* to modelling of the MR damper since it requires global properties of the MR fluid and a global model of the fluid flow. In such an approach the volumetric inflow rates to both chambers $Q_V^{1/2}$ have to be calculated from the model of compressible viscous flow, which is more complicated than the previously introduced classical incompressible model and which usually cannot be solved analytically. Leaving the above problem yet unresolved, let us note that the resulting from the compressible model volumetric inflow rates to both chambers are not equal since they depend on the densities of the fluid in both chambers. The condition of conservation of mass of the flowing fluid allows the R.H.S. of Eq. (11a) to be written by using the volumetric inflow rate to the first chamber and the ratio of the fluid densities:

$$\begin{aligned}&\dot{V}_1+\beta_1V_1\dot{p}_1-\alpha_1V_1\dot{T}_1=Q_V^1\\&\dot{V}_2+\beta_2V_2\dot{p}_2-\alpha_2V_2\dot{T}_2=-Q_V^1\frac{\rho_1}{\rho_2}.\end{aligned} \tag{11b}$$

It is important to notice that a direct application of the incompressible flow model and an introduction of equal total volumetric inflow rates at the right-hand sides of both above equations is definitely not correct (even if proposed by some authors). Such an approach violates the global balance of mass, disrupts the internal consistency of the mathematical model and leads in effect to incorrect, diverging shapes of the force–velocity hysteresis loops.

In an attempt to overcome the above difficulties in determination of the volumetric inflow rates, an alternative method can be proposed for considering compressibility of the magnetorheological fluid, which is the so called *decomposition approach*. The analysed MR fluid will be assumed to be composed of two separate fluids characterized by different mechanical properties and constitutive equations:

- classical Bingham viscous fluid with zero compressibility and thermal expansion that constitutes the major part of the considered magnetorheological medium (the primary viscous fluid 'f'),
- an inviscid fluid with relatively large compressibility and thermal expansion coefficients, being a minor part of the considered magnetorheological medium (the secondary compressible fluid 'c').

At first, the above approach is justified in case of modelling MR dampers filled with fluid that contains gas bubbles. Secondly, it can be treated as an alternative method for the modelling of MR fluid compressibility, where the constitutive equations are not defined for the entire medium but, instead, separately for its particular components. Finally, the method will enable application of the classical incompressible model of viscous flow for the computation of the total volumetric inflow rate of the compressible MR fluid.

The important assumption is that the incompressible viscous fluid constitutes the dominating part of the considered medium, while the compressible fluid comprises its small fraction. The relation between the volumes of both fluids is specified at arbitrary selected referential conditions defined by the referential density of the compressible fluid $\rho_c^{(ref)}$ and it is described by coefficient k being a small constant. It directly results in the analogous dependence between the masses of both fluids in the referential conditions:

$$V_c^{(ref)} = kV_f^{(ref)} \quad \text{and} \quad m_c^{(ref)} = k\frac{\rho_c^{(ref)}}{\rho_f}m_f^{(ref)}. \tag{12a,b}$$

Note that the second equation can be related to the initial state of the fluid, which can significantly differ from the referential state. Moreover, by expressing the initial masses of the fluids in terms of the corresponding volumes and densities, the dependence between the initial volumes of both component fluids is obtained:

$$m_c^{(0)} = k\frac{\rho_c^{(ref)}}{\rho_f}m_f^{(0)} \quad \text{and} \quad V_c^{(0)} = k\frac{\rho_c^{(ref)}}{\rho_c^{(0)}}V_f^{(0)}. \tag{12c,d}$$

It will be further analysed how the above relations evolve during the operation of the damper that involves flow of the fluid between the chambers. Due to the prevailing mass and volume of the viscous fluid, the main equation governing the flow of the entire medium will be the equation of the incompressible viscous flow. Thus, a natural assumption is that the mass inflow rate of the compressible fluid Q_m^c is proportional to the mass inflow rate of the viscous fluid Q_m^f and that the proportionality factor is exactly the same as in the initial state. The assumption allows the relation between the volumetric inflow rates of the compressible and the viscous fluid (Q_V^c and Q_V^f) to be found, which appears to be influenced by the actual density of the compressible fluid $\rho_c(t)$. The relations between mass and volumetric inflow rates take the form:

$$Q_m^c = k\frac{\rho_c^{(ref)}}{\rho_f}Q_m^f \quad \text{and} \quad Q_V^c = k\frac{\rho_c^{(ref)}}{\rho_c(t)}Q_V^f. \tag{12e,f}$$

Eq. (12c,d) that define the initial state and Eq. (12e,f) that define the flow can be used to determine the relations between the actual masses and the volumes of the component fluids during operation of the damper. It can be concluded that the initial ratio of masses is preserved during the entire process, i.e., the mass of the compressible fluid enclosed in each chamber m_c is proportional to the mass of the viscous fluid m_f with the same proportionality factor. In turn, the ratio of the actual volume of the compressible fluid V_c to the actual volume of the viscous fluid V_f is affected by the change of the density of the compressible fluid $\rho_c(t)$, which is caused by the variation of its pressure and temperature:

$$m_c = k\frac{\rho_c^{(ref)}}{\rho_f}m_f \quad \text{and} \quad V_c = k\frac{\rho_c^{(ref)}}{\rho_c(t)}V_f. \tag{12g,h}$$

Eventually, the formulae derived above allow the volume of the viscous fluid and the volume of the compressible fluid to be determined in terms of the total volume of the MR fluid in each chamber:

$$V_f = \left(1 + k\frac{\rho_c^{(ref)}}{\rho_c}\right)^{-1} V \quad \text{and} \quad V_c = \left(1 + \frac{1}{k}\frac{\rho_c}{\rho_c^{(ref)}}\right)^{-1} V. \tag{12i,j}$$

The corresponding masses of the component fluids can be calculated through multiplication by their densities.

Having derived the relations between the flow rates and the volumes of both component fluids, the equation of volume balance can be easily formulated by using the decomposition approach. The total volume of the considered magnetorheological fluid V is defined as a sum of the volume of the primary viscous fluid V_f and the volume of the secondary compressible fluid V_c:

$$V = V_f + V_c \quad \text{where}: \ V_f = V_f(m_f) \quad \text{and} \quad V_c = V_c(p, T, m_c). \tag{13}$$

A substitution of the above definitions into the general equation of volume balance (Eq. 7b) yields:

$$\dot{V} = \frac{\partial V_f}{\partial m_f}\dot{m}_f + \frac{\partial V_c}{\partial p}\dot{p} + \frac{\partial V_c}{\partial T}\dot{T} + \frac{\partial V_c}{\partial m_c}\dot{m}_c. \tag{14}$$

By grouping the terms related to the fluid transfer (that involves the mass derivative) and by using the definitions of the coefficients of compressibility, thermal expansion and mass expansion, the equation that govern the volume balance is obtained:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = \gamma_f V_f \dot{m}_f + \gamma_c V_c \dot{m}_c. \tag{15a}$$

Further, the terms on the right-hand side are identified as the mass inflow rates divided by the density, or alternatively, as the volumetric inflow rates:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = \frac{Q_m^f}{\rho_f} + \frac{Q_m^c}{\rho_c} \text{ and } \dot{V} + \beta_c V_c \dot{p} = Q_V^f + Q_V^f. \tag{15b,c}$$

Eq. (15) are fully equivalent to Eq. (10) that govern the model formulated in the global approach. Now, the right hand side clearly indicates that the total volumetric inflow of the magneto-rheological fluid is the sum of the inflow rate of the primary incompressible viscous fluid Q_V^f and the volumetric inflow rate of the secondary compressible fluid Q_V^c. In addition, one more, alternative form of the volume balance is obtained by using the chain rule of differentiation for the time derivative of mass in the last term of Eq. (14) and by expressing it in terms of volume of the viscous fluid $V_f$and its volumetric flow rate Q_V^f:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = Q_V^f \frac{\partial V}{\partial V_f}. \tag{15d}$$

By using in Eq. (15c) the definition of volume of the compressible fluid (Eq. 12j) and the definition of the volumetric inflow rate of the compressible fluid (Eq. 12f), or alternatively, by using in Eq. (15d) two definitions of volume of the compressible fluid (Eq. (12j) and Eq. (12h)), the final form of the equation is obtained that govern the balance of the MR fluid volumes in the decomposition approach. The equations for two chambers of the damper read:

$$\begin{aligned} &\dot{V}_1 + \frac{k}{\rho_1^c/\rho_c^{(ref)} + k}\left(\beta_1^c \dot{p}_1 - \alpha_1^c \dot{T}_1\right) V_1 = Q_V^f \left(1 + k\frac{\rho_c^{(ref)}}{\rho_1^c}\right) \\ &\dot{V}_2 + \frac{k}{\rho_2^c/\rho_c^{(ref)} + k}\left(\beta_2^c \dot{p}_2 - \alpha_2^c \dot{T}_2\right) V_2 = -Q_V^f \left(1 + k\frac{\rho_c^{(ref)}}{\rho_2^c}\right). \end{aligned} \tag{16}$$

In contrast to the global approach, the left-hand side of the above equations is expressed in terms of the parameters of the secondary compressible fluid, namely its coefficient of compressibility $\beta_{1/2}^c$, the coefficient of thermal expansion $\alpha_{1/2}^c$ and the actual density $\rho_{1/2}^c$. Although the right-hand side (the total volumetric inflow rate of MR fluid) is expressed in terms of the inflow rate of the incompressible viscous fluid, it is also affected by the actual density of the secondary compressible fluid, which clearly reveals the contribution of the compressible part of the medium. Note that for the primary viscous fluid the volumetric inflow rates (as well as the corresponding mass inflow rates) are equal for both chambers and that they differ only by a sign. In turn, for the secondary compressible fluid the volumetric inflow rates are different, however the mass inflow rates obtained by multiplication by density are literally identical. Thus, total balance of fluid mass is satisfied and the application of the standard incompressible flow model does not lead to a contradiction.

The additional part of the model is the formulae that define the actual volumes of the MR fluid in both chambers. In the considered case of a kinematic excitation, the volume of the MR fluid in the chamber without gas cushion V_2 is directly defined by the actual volume of the chamber $V_2^{ch}(u)$, which is a function of the shaft displacement, while the volume of the MR fluid in the chamber with the gas cushion V_1 has to be determined from an additional algebraic equation, which takes into account the change of the chamber volume $V_1^{ch}(u)$ and the change of the volume of the gas cushion $V_g(p_1)$:

$$V_1 + V_g(p_1) = V_1^{ch}(u), \;\; V_2 = V_2^{ch}(u). \tag{17}$$

Eqs. (16 and 17), complemented by the definition of the inflow rate (Eq. 5), constitute a closed mathematical model of the damper and allow its response to a kinematic excitation to be determined exclusively in the case when the thermal expansion coefficient of the MR fluid equals to zero, i.e., when the equation of the thermodynamic energy balance is decoupled from the model and the fluid pressures are the only unknowns of the problem.

The remaining problem is to find the dependence between the global thermodynamic coefficients of the compressible viscous MR fluid and the coefficients of each component fluid. For the compressibility coefficient, such relation takes the form:

$$\beta = -\frac{1}{V_f + V_c}\left(\frac{\partial V_c}{\partial p}\right)_T = \frac{V_c}{V_f + V_c}\beta_c = \left(1 + \frac{1}{k}\frac{\rho_c}{\rho_c^{(ref)}}\right)^{-1}\beta_c. \tag{18a}$$

Similarly, for the thermal expansion coefficient,

$$\alpha = \frac{1}{V_f + V_c}\left(\frac{\partial V_c}{\partial T}\right)_p = \frac{V_c}{V_f + V_c}\alpha_c = \left(1 + \frac{1}{k}\frac{\rho_c}{\rho_c^{(ref)}}\right)^{-1}\alpha_c. \tag{18b}$$

Note that the global coefficients for the entire medium depend on the actual density of the compressible fluid, which is affected by the actual pressure and temperature. Moreover, the actual density of the MR fluid can be calculated by using the general definition and relations between the masses and volumes of both fluids (Eq. 12):

$$\rho = \frac{m_f + m_c}{V_f + V_c} = \frac{\rho_f + k\rho_c^{(ref)}}{\rho_c + k\rho_c^{(ref)}}\rho_c. \tag{18c}$$

By introducing the above definitions of the global coefficients of compressibility and thermal expansion, the definitions of the total volumetric flow rate and the definitions of the MR fluid density into the previously derived equation of volume balance in the global approach (Eq. 11b), the balance equations formulated in the decomposition approach (Eq. 16) are obtained. Such an equivalence proves the correctness of the applied methodology and the internal consistency of the proposed mathematical model.

Concluding the section that concerns the balance of fluid volume, it can be stated that Eq. (11) is convenient for application when the global constitutive relations of the magnetorheological medium are known and when the volumetric flow rates are determined by using the exact model of the *compressible viscous flow*. In turn, Eq. (16) is more suitable when various models of compressibility and thermal expansion are considered and when an analytical model of the *incompressible viscous flow* is intended to be applied.

3.3. Balance of fluid energy

The second group of the governing equations concerns the balance of energy of the magnetorheological fluid enclosed in each chamber of the damper. The general thermodynamic equation of energy balance combines the energy transferred to the fluid in the form of heat δQ and submitted enthalpy $dm\bar{H}$ with the internal energy of the fluid dU and the work done by the fluid δW:

$$\delta Q + dm\bar{H} - d(m\overline{U}) - \delta W = 0. \tag{19}$$

Hereafter, the process is assumed to be adiabatic and the term indicating the energy transferred in the form of heat is omitted. Further, the classical definitions of the differential of internal energy, the differential of enthalpy and the incremental work will be applied. The definition of change of internal energy can be derived by using the fundamental thermodynamic relation:

$$dU = TdS - pdV. \tag{20a}$$

By expressing the differential of entropy dS in terms of change of temperature and volume, by using Maxwell relation to replace the entropy derivative by the pressure derivative, and finally, by applying the definition of the heat capacity C_V, it can be obtained:

$$dU = C_V dT + \left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right]dV. \tag{20b}$$

The above formula can be expressed in terms of change of temperature and change of pressure by using the general definitions of the coefficient of heat capacity at constant pressure C_p, the coefficient of thermal expansion α and the coefficient of compressibility β:

$$dU = (C_p - \alpha pV)\,dT + (\beta p - \alpha T)\,Vdp. \tag{20c}$$

Further, the definition of the specific energy and its differential can be directly obtained:

$$\overline{U} = \left(c_p - \alpha p\frac{V}{m}\right)T + (\beta p - \alpha T)\,\frac{V}{m}p,\;\; d\overline{U} = \left(c_p - \alpha p\frac{V}{m}\right)dT + (\beta p - \alpha T)\,\frac{V}{m}dp. \tag{20d,e}$$

In the next step, the definition of the differential increase of enthalpy can be determined by using the classical relation:

$$dH = dU + d(pV). \tag{21a}$$

By using the previously derived definition of the increase of internal energy (Eq. 20b), the increase of enthalpy can be defined in terms of change of temperature and change of pressure:

$$dH = C_p dT + \left[V - T\left(\frac{\partial V}{\partial T}\right)_p\right]dp. \tag{21b}$$

Further, the above definition can be directly expressed in terms of the coefficient of thermal expansion:

$$dH = C_p dT + (1 - \alpha T)Vdp, \tag{21c}$$

which allows to find the following definition of the specific enthalpy:

$$\overline{H} = c_p T + (1 - \alpha T)\frac{V}{m}p. \tag{21d}$$

Finally, the last component of the equation of energy balance is the incremental work done by the fluid, which can be directly expressed in terms of fluid pressure and change of volume:

$$\delta W = pdV. \tag{22}$$

Substituting the definitions of internal energy, enthalpy and work (Eq. 20c,d, Eq. 21d, Eq. 22) into the equation of energy balance (Eq. 19), performing differentiation with respect to time and identifying the terms that indicate the volumetric flow rate through the valve, one can obtain the most general form of the equation that governs the energy balance:

$$\dot{m}c_pT_{(v)} + Q_V p_{(v)} - Q_V\alpha T_{(v)}p_{(v)} = \dot{m}\left(c_pT - \alpha p\frac{V}{m}T + \beta p\frac{V}{m}p - \alpha T\frac{V}{m}p\right) + mc_p\dot{T} - \alpha pV\dot{T} + \beta pV\dot{p} - \alpha TV\dot{p} + p\dot{V}. \tag{23a}$$

In the above equation, the index v indicates the parameters of the fluid flowing through the valve, which are not necessarily equal to the parameters of the fluid in the considered chamber. The mass of the fluid that appears in Eq. (23a) can be expressed in terms of other thermodynamic parameters by using the equation of state. In turn, the time derivative of the fluid mass can be determined either by differentiation of the equation of state or by multiplication of the corresponding volumetric inflow rate by the fluid density.

Obviously, a complete model of the MR damper contains equations that govern energy balance for the fluid enclosed in both chambers. Eq. (23a) governs the chamber with fluid inflow. In turn, the equation that governs the chamber with fluid outflow has a simplified form due to the equality of parameters of the fluid transferred through the valve and the parameters of the fluid inside the chamber:

$$Q_V p = \dot{m}\left(-\alpha p \frac{V}{m} T + \beta p \frac{V}{m} p\right) + m c_p \dot{T} - \alpha p V \dot{T} + \beta p V \dot{p} - \alpha T V \dot{p} + p\dot{V}. \tag{23b}$$

The partial balance of fluid energy for the chamber with fluid outflow can be obtained by multiplication of the corresponding balance of fluid volume by the value of pressure:

$$Q_V p = p\dot{V} + p\beta V \dot{p} - p\alpha V \dot{T}. \tag{23c}$$

By subtracting Eq. (23c) from Eq. (23b), the simplest possible form of the equation of the energy balance for the chamber with fluid outflow is obtained:

$$\dot{m}\left(-\alpha p \frac{V}{m} T + \beta p \frac{V}{m} p\right) + m c_p \dot{T} - \alpha T V \dot{p} = 0. \tag{23d}$$

Although Eq. (23d) is much simpler than Eq. (23a) a typical periodic kinematic excitation causes that distinction between inflow and outflow chamber is only temporary and both equations have to be applied in an alternating manner. Thus, using the general form of energy balance for both chambers can be a more convenient option. Furthermore, summation of the equations that govern the energy balance for the fluid in each chamber leads to the equation that governs the global energy balance for the entire fluid enclosed in the damper. Although the considered flow is viscous and during the valve flow kinetic energy is transferred into heat, the total enthalpy of the transferred fluid remains constant. Consequently, summation of energy balances eliminates the enthalpy terms, and the resulting equation indicates the equivalence of the increase of the internal energy and the work done on the fluid. In the proposed model, the global equation of energy balance can replace the energy balance for an arbitrarily selected chamber.

Further, let us consider the equation of energy balance in the previously introduced *decomposition approach* where the considered magnetorheological medium is assumed to be composed of a primary incompressible viscous fluid and a secondary compressible fluid. Derivation of the equations of energy balance for such a compound fluid is straightforward, i.e. all terms of Eq. (19) are defined separately for the primary viscous and the secondary compressible fluid, and moreover, the corresponding definitions of enthalpy and internal energy for both fluids are appropriately simplified. The general equation of energy balance in the decomposition approach takes the form:

$$\begin{aligned}&\left(\dot{m}_f c_p^f T_{(v)} + Q_V^f p_v\right) + \left(\dot{m}_c c_p^c T_{(v)} + Q_V^c p_{(v)} - Q_V^c \alpha_c T_{(v)} p_{(v)}\right) = \left(\dot{m}_f c_p^f T + m_f c_p^f \dot{T}\right)\\ &\quad + \dot{m}_c\left(c_p^c T - \alpha_c p \frac{V_c}{m_c} T + \beta_c p \frac{V_c}{m_c} p - \alpha_c T \frac{V_c}{m_c} p\right) + \left(m_c c_p^c \dot{T} - \alpha_c p V_c \dot{T} + \beta_c p V_c \dot{p} - \alpha_c T V_c \dot{p}\right) + p\dot{V}_f + p\dot{V}_c.\end{aligned} \tag{24a}$$

In the above equation, the index v indicates the parameters of fluid flowing through the valve, while Q_V^f and Q_V^c denote the volumetric inflow rates of the viscous fluid and the compressible fluid. The volumes of both fluids and the volumetric inflow rates that appear in Eq. (24a) can be determined using the general relations defined in Eq. (12). In turn, the masses of the fluids and the mass derivatives can be computed by the multiplication of the corresponding volumes and the volumetric inflow rates by the densities. Simplification of Eq. (24a) in the case of the chamber with fluid outflow is performed in a similar manner as in the case of a homogeneous medium, and it yields:

$$Q_V^f p_{(v)} + Q_V^c p_{(v)} = m_f c_p^f \dot{T} + \dot{m}_c\left(-\alpha_c p \frac{V_c}{m_c} T + \beta_c p \frac{V_c}{m_c} p\right) + \left(m_c c_p^c \dot{T} - \alpha_c p V_c \dot{T} + \beta_c p V_c \dot{p} - \alpha_c T V_c \dot{p}\right) + p\dot{V}_f + p\dot{V}_c. \tag{24b}$$

Moreover, the equation of partial energy balance obtained by multiplication of the volume balance by pressure reads:

$$Q_V^f p_{(v)} + Q_V^c p_{(v)} = p\dot{V}_f + p\dot{V}_c + p\beta_c V_c \dot{p} - p\alpha_c V_c \dot{T}. \tag{24c}$$

Subtraction of the above equation from the equation that describe the energy balance for the outflow chamber causes elimination of selected terms related to the viscous and compressible fluid:

$$m_c c_p^c \dot{T} + m_f c_p^f \dot{T} + \dot{m}_c\left(-\alpha_c p \frac{V_c}{m_c} T + \beta_c p \frac{V_c}{m_c} p\right) - \alpha_c T V_c \dot{p} = 0. \tag{24d}$$

Similarly as in the case of a homogenous medium, the complete model of the damper contains two equations that govern the balance of fluid energy: (i) two equations in the general form (Eq. 24a) or (ii) one equation in the general form (Eq. 24a) and one equation in the simplified form (Eq. 24b or 24d) or (iii) combination of the equation in the general form and a sum of the equations for two chambers.

The remaining task is to express the global value of the heat capacity coefficient in terms of the quantities related to both component fluids. According to the general definition, and normalizing with respect to the fluid masses, one can obtain:

$$c_p = \frac{m_f c_p^f + m_c c_p^c}{m_f + m_c} = \frac{\rho_f c_p^f + k\rho_c^{(ref)} c_p^c}{\rho_f + k\rho_c^{(ref)}}. \tag{25}$$

By introducing the above global coefficient of thermal expansion, the previously derived global coefficients of compressibility and thermal expansion (Eq. 18a,b), the definitions of the total volumetric flow rate and the definitions of the MR fluid density (Eq. 18c) into the equations of energy balance in the global approach (Eq. 23), the equations of energy balance are obtained as formulated in the decomposition approach (Eq. 24). This again confirms the consistency and equivalence of both proposed formulations.

3.4. Definition of force generated by the damper

The last element of the mathematical model is the definition of the force generated by the damper as a response to the applied kinematic excitation. Since the derived model was focused on exact modelling of thermodynamic processes in the MR fluid, only the forces exerted by the MR fluid on the piston will be taken into account. Other types of forces, such as the friction forces generated by the sealing or the delimiting forces generated at the end of the stroke, will be neglected. Eventually, the formula that defines the generated force takes the following simple form:

$$F_{damper} = p_1 A_1 - p_2 A_2. \tag{26}$$

In contrast to the classical definition of force generated by hydro-pneumatic dampers, the above definition does not contain separate terms that denote hydraulic and pneumatic forces. The above definition of force will be used to obtain the force–velocity hysteresis loops that correspond to the applied kinematic excitation.

Concluding the proposed in this section approach for thermodynamic modelling of the magnetorheological dampers, it can be stated that the model is always composed of a definition of the volumetric flow rate through the valve, two differential equations that govern the balance of fluid volume in each chamber, two differential equations that govern the balance of fluid energy in each chamber, and finally, a definition of the generated force. The equations of volume and energy balance, which are crucial for computation of pressures and temperatures inside the chambers, can be formulated either by using the standard ‘global approach’ or the proposed ‘decomposition approach’.

The *global approach* can be applied exclusively if the global constitutive equations for the compressible MR fluid are known and the analytical model of the compressible viscous flow is assumed. The model formulated in the global approach contains analytical definitions of the volumetric inflow rates of the compressible fluid to both chambers, the equations of volume balance (Eq. 11) and the equations of energy balance (Eq. 23). The definitions of the compressibility coefficient, the thermal expansion coefficient and the density, which arise in the governing equations, can be determined from the global constitutive relations of the MR fluid. The system of governing equations can be solved in order to find the unknown pressures and temperatures of the fluid in both chambers of the damper.

In turn, in the proposed *decomposition approach* the constitutive equations and the flow models are defined separately for the primary viscous fluid and the secondary compressible fluid. The model formulated in the decomposition approach contains the standard definition of the volumetric inflow rate of the incompressible viscous fluid, the relations between volumes and volumetric flow rates of both fluids (Eq. 12), the equations of volume balance (Eq. 16) and the equations of energy balance (Eq. 24). The definitions of the thermal expansion coefficient, the compressibility coefficient and the density, which arise in the governing equations, can be determined from the constitutive equation of the secondary compressible fluid. Eventually, the system of governing equations can be solved in order to find the unknown pressures and temperatures of the fluid, similarly as in the previous case.

4. Model implementation for various constitutive models of MR fluid

In the current section the proposed methodology of thermodynamic modelling of magnetorheological dampers will be applied for several constitutive models of the secondary compressible fluid, i.e.:

- the model of an ideal gas with a general pressure and temperature dependence,
- the model of a compressible fluid with constant compressibility and thermal expansion (and its approximation by the model of linear elasticity with linear thermal expansion),
- the model of a compressible fluid resulting in constant values of the global coefficients of compressibility and thermal expansion of the MR fluid.

In each case the corresponding form of the governing equations will be derived, the values of global coefficients for the entire MR fluid will be determined, and moreover, the shapes of the force–velocity hysteresis loops resulting from the solution of the numerical model will be presented.

4.1. Compressible fluid modelled as an ideal gas

In the first example, the secondary compressible fluid will be modelled using one of the most classical models in thermodynamics—the model of an ideal gas. Such a model will be described by the well-known equation of state where pressure, temperature, volume and mass (or alternatively density) of the secondary compressible fluid are linked by the relation:

$$pV_c = m_c RT \;\text{ or }\; p = \rho_c RT, \tag{27a}$$

and R is the gas constant. By considering the equation of state for the initial state of the gas, the gas constant can be eliminated, which leads to the definition of the actual volume (or alternatively density) of the fluid in the form:

$$V_c = V_c^{(0)} \frac{m_c}{m_c^{(0)}} \frac{p_0}{T_0} \frac{T}{p} \;\text{ or }\; \rho_c = \rho_0 \frac{T_0}{p_0} \frac{p}{T}, \tag{27b}$$

which is convenient for application in the derived equations of volume and energy balance. The above equation of state can be used to determine the values of the compressibility coefficient, the thermal expansion coefficient and the mass expansion coefficient, according to their general definitions (cf. Eq. 9):

$$\beta_c = \frac{1}{p}, \quad \alpha_c = \frac{1}{T} \;\text{ and }\; \gamma_c = \frac{1}{m}. \tag{27c}$$

The definition of the fluid density, as well as the definitions of the thermodynamic coefficients, will be both used in the further derivation of the model in the decomposition approach. In the preliminary stage of the analysis, the relation between the volumetric inflow rates of the compressible and viscous fluids will be found. For simplicity, it will be assumed that the constant k that defines the relation of volumes in arbitrary selected referential conditions (cf. Eq. 12a) is defined for the initial state of the fluid, which is exactly the same for both chambers, $\rho_c^{(ref)}=\rho_c^{(0)}$. Accordingly, by using Eq. (12) the volumetric inflow rate of the secondary compressible fluid can be obtained in terms of the volumetric inflow rate of the primary viscous fluid:

$$Q_V^c = k\left(\frac{p_0}{T_0}\frac{T}{p}\right) Q_V^f. \tag{28a}$$

In turn, Eq. (12i,j) can be used to define the volumes of both fluids in terms of the total volume of the considered medium:

$$V_f = \left(1 + k\frac{p_0}{T_0}\frac{T}{p}\right)^{-1} V \;\text{ and }\; V_c = \left(1 + \frac{1}{k}\frac{T_0}{p_0}\frac{p}{T}\right)^{-1} V. \tag{28b,c}$$

The above relations allow the general equations that govern the balances of the fluid volume in the decomposition approach (Eq. 16) to be rewritten in the following form:

$$\begin{aligned} &\dot{V}_1 + \left(1 + \frac{1}{k}\frac{T_0}{p_0}\frac{p_1}{T_1}\right)^{-1}\left(\frac{\dot{p}_1}{p_1} - \frac{\dot{T}_1}{T_1}\right) V_1 = Q_V^f \left(1 + k\frac{p_0}{T_0}\frac{T_1}{p_1}\right) \\ &\dot{V}_2 + \left(1 + \frac{1}{k}\frac{T_0}{p_0}\frac{p_2}{T_2}\right)^{-1}\left(\frac{\dot{p}_2}{p_2} - \frac{\dot{T}_2}{T_2}\right) V_2 = -Q_V^f \left(1 + k\frac{p_0}{T_0}\frac{T_2}{p_2}\right). \end{aligned} \tag{29}$$

The volumetric inflow rate of the viscous fluid Q_V^f, which appears in the above equations, can be determined from the classical model of the incompressible Bingham viscous flow described in Sec. III.A, while the volumes of the MR fluid inside the chambers $V_{1/2}$ can be defined as a function of the actual volumes of the chambers and the actual pressure (cf. Eq. 17). Finally, both equations that govern the balance of volume can be expressed in terms of the imposed displacement of the piston, as well as the unknown pressures and temperatures of the fluid inside the chambers. In the special case, when the process is isothermal, and consequently, the thermal expansion coefficients are equal to zero, the equations of volume balance take the following forms:

$$\begin{aligned} &\dot{V}_1 + \frac{kp_0}{p_1 + kp_0}\left(\frac{\dot{p}_1}{p_1}\right) V_1 = Q_V^f \left(1 + k\frac{p_0}{p_1}\right) \\ &\dot{V}_2 + \frac{kp_0}{p_2 + kp_0}\left(\frac{\dot{p}_2}{p_2}\right) V_2 = -Q_V^f \left(1 + k\frac{p_0}{p_2}\right). \end{aligned} \tag{30}$$

In such a situation, the only unknowns in the equations of volume balance are the pressures of the MR fluid in each chamber, and they are sufficient to determine the mechanical response of the absorber. The final aspect of the considerations is the calculation of the global thermodynamic constants of the magnetorheological fluid. By using Eq. (18a) for the compressibility coefficient and Eq. (18b) for the thermal expansion coefficient, one can obtain:

$$\beta = k\left(1 + \frac{1}{k}\frac{T_0}{p_0}\frac{p}{T}\right)^{-1}\beta_c = k\left(\frac{\beta_c^{(0)}}{\alpha_c^{(0)}}\frac{\alpha_c}{\beta_c} + k\right)^{-1}\beta_c \cong k\frac{(\beta_c)^2}{\beta_c^{(0)}}\frac{\alpha_c^{(0)}}{\alpha_c}, \tag{31a}$$

and

$$\alpha = k\left(1 + \frac{1}{k}\frac{T_0}{p_0}\frac{p}{T}\right)^{-1}\alpha_c = k\left(\frac{\beta_c^{(0)}}{\alpha_c^{(0)}}\frac{\alpha_c}{\beta_c} + k\right)^{-1}\alpha_c \cong k\frac{\alpha_c^{(0)}}{\beta_c^{(0)}}\beta_c. \tag{31b}$$

In the proposed approach the global thermodynamic coefficients of the magnetorheological medium depend on the initial values of both local coefficients $\alpha_c^{(0)}$ and $\beta_c^{(0)}$, the initial volumetric fraction of the compressible fluid k and the actual values of both local coefficients α_c and β_c. In addition, in the case of the simplified versions of the above formulae, when the volume of the compressible fluid in the denominator is omitted, the global thermal expansion coefficient depends exclusively on the compressibility of the secondary fluid. Finally, the global density of the MR fluid can be determined either from Eq. (18c) or by using the above definitions of the global thermodynamic coefficients to reconstruct the definition of the density (cf. Sect. IVB)

The derivation of the equation that governs the thermodynamic balance of the MR fluid energy is performed by introducing the values of the thermodynamic coefficients into Eq. (24a), which yields:

$$\left(\dot{m}_f c_p^f T_{(v)} + Q_V^f p_v\right) + \dot{m}_c c_p^c T_{(v)} = \left(\dot{m}_f c_p^f T + m_f c_p^f \dot{T}\right) + \left(\dot{m}_c c_p^c T - \dot{m}_c p\frac{V_c}{m_c}\right) + \left(m_c c_p^c \dot{T} - \frac{pV_c}{T}\dot{T}\right) + p\dot{V}_f + p\dot{V}_c. \tag{32a}$$

By using the ideal gas law and the relation between the heat capacities at a constant volume and a constant pressure, one can recognize the classical terms that indicate enthalpy and internal energy of a viscous fluid and an ideal gas:

$$\left(\dot{m}_f c_p^f T_{(v)} + Q_V^f p_v\right) + \dot{m}_c c_p^c T_{(v)} = \left(\dot{m}_f c_p^f T + m_f c_p^f \dot{T}\right) + \left(\dot{m}_c c_V^c T + m_c c_V^c \dot{T}\right) + p\dot{V}_f + p\dot{V}_c. \tag{32b}$$

In both above equations the volumes of the fluids can be determined from Eq. (28b,c), while the corresponding masses can be obtained through a multiplication by the fluid density. The standard simplifications, which can be applied when the outflow chamber is considered (cf. general form defined by Eq. (24b) or direct transformation of Eq. (32b)), yield:

$$Q_V^f p + Q_V^c p = m_f c_p^f \dot{T} + m_c c_V^c \dot{T} + p\dot{V}_f + p\dot{V}_c. \tag{32c}$$

In turn, the equation of the partial energy balance obtained by multiplication of the equations of volume balances by pressure take the form (cf. Eq. 24c):

$$Q_V^f p + Q_V^c p = p\dot{V}_f + p\dot{V}_c + V_c\dot{p} - mR\dot{T}, \tag{32d}$$

and their subtraction from the equation of energy balance for the outflow chamber gives:

$$m_f c_p^f \dot{T} + m_c c_p^c \dot{T} - V_c\dot{p} = 0. \tag{32e}$$

Since the volume of the compressible fluid is assumed as relatively small, the value of the last term in the above equation is also small and thus the temperature in the outflow chamber remains approximately constant during the entire process. In turn, the temperature of the fluid in the inflow chamber increases as the result of work done on the system by the shaft movement. Finally, the general set of equations that govern the thermodynamic balance of energy for two chambers of the damper takes the form:

$$\begin{aligned}
&\left(\dot{m}_1^f c_p^f T_{(v)} + Q_V^f p_v\right) + \dot{m}_1^c c_p^c T_{(v)} = \left(\dot{m}_1^f c_p^f T_1 + m_1^f c_p^f \dot{T}_1\right) + \left(\dot{m}_1^c c_V^c T_1 + m_1^c c_V^c \dot{T}_1\right) + p_1\dot{V}_1^f + p_1\dot{V}_1^c \\
&\left(\dot{m}_2^f c_p^f T_{(v)} + Q_V^f p_v\right) + \dot{m}_2^c c_p^c T_{(v)} = \left(\dot{m}_2^f c_p^f T_2 + m_2^f c_p^f \dot{T}_2\right) + \left(\dot{m}_2^c c_V^c T_2 + m_2^c c_V^c \dot{T}_2\right) + p_2\dot{V}_2^f + p_2\dot{V}_2^c.
\end{aligned} \tag{33}$$

An alternative to using two equations of energy balance in the above general form is to replace one of them by the simplified equation of energy balance for the outflow chamber or to use the sum of two above equations, which is the global balance of energy of the fluid inside the damper.

Eventually, the proposed model of the MR damper with compressibility modelled by the ideal gas law is governed by the equations of volume and energy balance (Eqs. 29 and 33), which allow the pressures and temperatures of the fluid in both chambers to be determined, and the general definition of the total force generated by the damper (Eq. 26). The above model was implemented in MAPLE software and solved numerically with the use of the Runge–Kutta method. The solution of the above model for the case of a sinusoidal kinematic excitation reveals the occurrence of the characteristic shapes of the force–velocity hysteresis loops (Fig. 5), which were observed in the described experiment but cannot be obtained from the classical non-parametric models of the MR dampers.

4.2. Compressible fluid with constant values of compressibility and thermal expansion

In the next example the secondary compressible fluid will be modelled by another classical model described by constant values of the coefficients of compressibility and thermal expansion, which will be supplemented by the standard definition of the mass expansion coefficient:

$$\beta_c = \beta_0, \quad \alpha_c = \alpha_0, \quad \gamma_c = m_c^{-1}. \tag{34}$$

Due to the fact that the developed model of the MR damper utilizes both the definitions of thermodynamic coefficients and the equation of state, an additional task is to reconstruct the formulae that define the actual density of the compressible

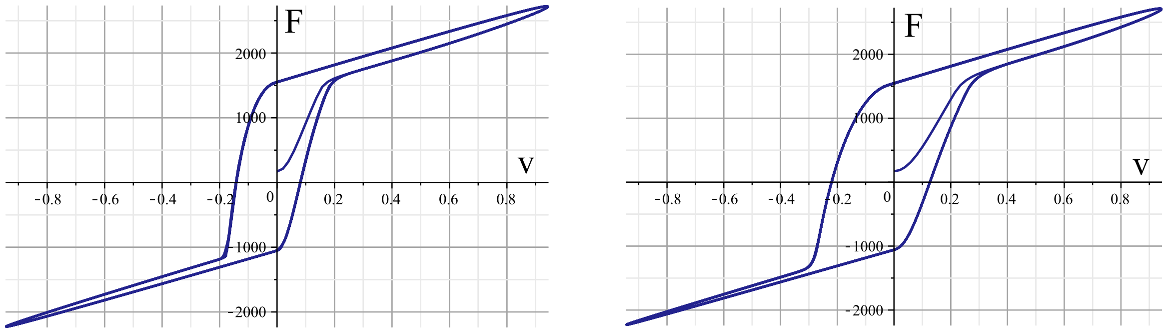


Fig. 5. Force–velocity hysteresis loops obtained from the model with a compressible fluid described by the ideal gas law: (a) $k=0.02$, (b) $k=0.05$.

fluid (the formulae analogous to Eq. 27b for an ideal gas). Recalling the standard definitions of the coefficients of compressibility, thermal expansion and mass expansion (Eq. 9), one can obtain the general formula that define the actual volume of the fluid in terms of its pressure, temperature and mass:

$$V = V_0 \frac{m}{m_0} \exp\left(-\int_{p_0}^{p} \beta(\bar{p}, T) d\bar{p} + \int_{T_0}^{T} \alpha(p, \bar{T}) d\bar{T} \right). \tag{35}$$

In the considered case of constant coefficients of compressibility and thermal expansion, an analytical calculation of the definite integrals in the exponent leads to the formula:

$$V_c = V_c^{(0)} \frac{m_c}{m_c^{(0)}} \, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}. \tag{36}$$

The above relation can be directly used to determine the actual density of the compressible fluid in terms of its initial density, the thermodynamic coefficients as well as the actual pressure and temperature of the fluid:

$$\rho_c = \rho_c^{(0)} \, e^{\beta_0(p-p_0)-\alpha_0(T-T_0)}. \tag{37}$$

Note that the above expression for the actual density of the fluid can be also derived directly from the definitions of the coefficients of compressibility and thermal expansion expressed by the derivatives of fluid density. Such definitions can be obtained from the standard volume-based definitions by applying the chain rule of differentiation to the derivatives of volume.

The knowledge of the compressibility coefficient, the thermal expansion coefficients (Eq. 34) and the definition of density of the compressible fluid (Eq. 37) enables taking the successive steps the from previous subsection and deriving the model of the MR damper in the decomposition approach. Primarily, the volumetric flow rate of the compressible fluid is determined:

$$Q_V^c = k \, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} Q_V^f, \tag{38}$$

as well as the expressions that link the volume of the primary viscous fluid and the secondary compressible fluid with the total volume of the MR fluid:

$$V_f = \frac{1}{1 + ke^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}} V \quad \text{and} \quad V_c = \frac{k}{k + e^{\beta_0(p-p_0)-\alpha_0(T-T_0)}} V. \tag{39}$$

For the simplicity of further considerations, the thermal expansion of the fluid will be neglected, and consequently, the equation of energy balance will be decoupled from the main part of the model. By introducing the values of the thermodynamic coefficients (Eq. 34), the definition of the volumetric inflow rate of the compressible fluid (Eq. 38) and the definition of its volume (Eq. 39b) into the general equations of volume balance, two equations are obtained:

$$\begin{aligned} \dot{V}_1 + \left(\frac{k}{k + e^{\beta_0(p_1-p_0)}} \right) \beta_0 \dot{p}_1 V_1 &= Q_V^f \left(1 + k \, e^{-\beta_0(p_1-p_0)} \right) \\ \dot{V}_2 + \left(\frac{k}{k + e^{\beta_0(p_2-p_0)}} \right) \beta_0 \dot{p}_2 V_2 &= -Q_V^f \left(1 + k \, e^{-\beta_0(p_2-p_0)} \right), \end{aligned} \tag{40}$$

which govern the change of fluid pressure in both chambers of the damper. Numerical solution of the above equations allows the analytical formula that defines the force generated by the damper to be applied and the corresponding shapes of the force–velocity hysteresis loops to be obtained. The analysis of the model will be accomplished with the calculation of the global coefficient of compressibility of the MR fluid:

$$\beta = \left(\frac{k}{k + e^{\beta_0(p-p_0)}} \right) \beta_0, \tag{41}$$

which can be directly used to formulate the equations of volume balance in the global approach and to prove their equivalence with the above derived equations that utilize the decomposition approach.

The next step of consideration is a simple approximation of the above model, which will be obtained by an expansion of the formula that defines the volume of the compressible fluid (Eq. 36) into Taylor series. By taking into account only the two initial terms of the expansion, the following formula is obtained:

$$V_c = V_c^{(0)} \frac{m_c}{m_c^{(0)}} \left[1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)\right], \tag{42}$$

which defines the linear relation between the change of volume and the change of fluid pressure and temperature (a model of linear volumetric elasticity with linear thermal expansion). The above expression can be used to determine the actual density of the compressible fluid, which takes the form:

$$\rho_c = \rho_c^{(0)} \left[1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)\right]^{-1}. \tag{43}$$

Note that in the considered linearization case, the quantities β_0 and α_0 should no longer be treated as thermodynamic coefficients but just as scaling factors. In turn, the compressibility coefficient and the thermal expansion coefficient can be determined from their general definitions (Eq. 9) and they are defined by the formulae:

$$\beta_c = \frac{\beta_0}{1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)}, \quad \alpha_c = \frac{\alpha_0}{1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)}. \tag{44}$$

The thermodynamic quantities determined above are sufficient to formulate the model in the decomposition approach. Primarily, the relation between the volumetric flow rate of the compressible fluid and the total volumetric flow rate takes the form:

$$Q_V^c = k\left[1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)\right] Q_V^f, \tag{45}$$

while the relation between the volumes of the component fluids and the total volume of the MR fluid is given as:

$$V_f = \frac{1}{1 + k\left[1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)\right]} V \quad \text{and} \quad V_c = \frac{k}{k + \left[1 - \beta_0 (p - p_0) + \alpha_0 (T - T_0)\right]^{-1}} V. \tag{46}$$

In the special case when the thermal expansion of the compressible fluid is omitted, the equations that govern the balance of volume take the form:

$$\begin{aligned} \dot{V}_1 + \frac{k\beta_0}{k + \left[1 - \beta_0 (p_1 - p_0)\right]^{-1}} \dot{p}_1 V_1 &= Q_V^f \left[1 + k(1 - \beta_0 (p_1 - p_0))\right] \\ \dot{V}_2 + \frac{k\beta_0}{k + \left[1 - \beta_0 (p_2 - p_0)\right]^{-1}} \dot{p}_2 V_2 &= -Q_V^f \left[1 + k(1 - \beta_0 (p_2 - p_0))\right], \end{aligned} \tag{47}$$

and they can be used to determine the pressure of the fluid enclosed in both chambers of the damper and to calculate the value of the generated force. Finally, the coefficient of the compressibility for the entire considered MR fluid is defined by the formula:

$$\beta = \frac{k\beta_0}{k + \left[1 - \beta_0 (p - p_0)\right]^{-1}}, \tag{48}$$

which proves the equivalence of the model in the global approach and the decomposition approach. Note that by neglecting the volume of the compressible fluid in the denominator of Eq. (48) (as being relatively small in comparison to the volume of the viscous fluid), one obtains a constant value of the global coefficient of compressibility.

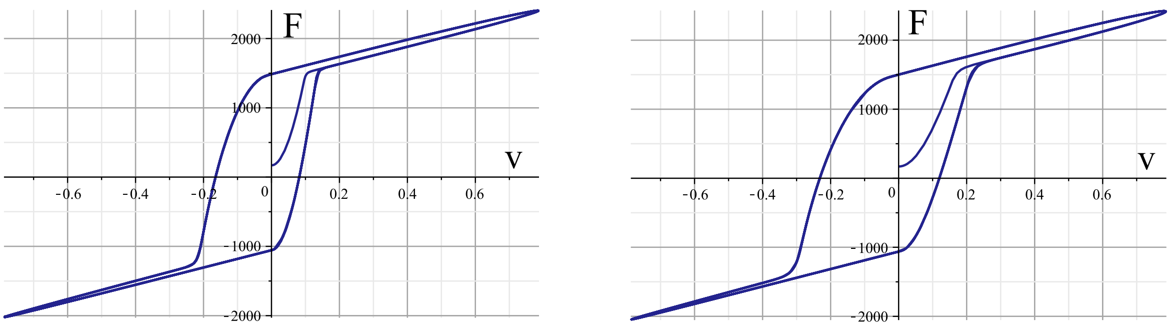


Fig. 6. Force–velocity hysteresis loops obtained from the model with compressible fluid of constant compressibility coefficient.

The model described in this section that utilizes compressible fluid characterized by constant thermodynamic coefficients and the models that utilize compressible fluid described by linear elasticity result both in an overall similar response of the damper subjected to a kinematic excitation. The results obtained from both these models confirm the occurrence of the characteristic shapes of the force–velocity hysteresis loops that resemble the ones obtained experimentally.

4.3. Compressible fluid resulting in constant values of global compressibility and thermal expansion of the MR fluid

The following example is motivated by the results obtained in the previous subsection, where an application of the linear elastic model of the secondary compressible fluid resulted in approximately constant values of the global coefficients of compressibility and thermal expansion of the MR fluid. Herein, the objective will be to find the constitutive model of the secondary fluid, which would yield exactly constant values of both global thermodynamic coefficients.

The first step will be the assumption of constant values of global compressibility and thermal expansion as well as the standard definition of the mass expansion coefficient:

$$\beta = \beta_0, \quad \alpha = \alpha_0, \quad \gamma = m^{-1}, \tag{49a}$$

and reconstruction of the corresponding constitutive equation for the entire medium. Similarly as in Sect. IVB, the calculation is based on Eq. (35), which allows the definition of the actual volume of the MR fluid and its actual density to be obtained:

$$V = V^{(0)} \frac{m}{m^{(0)}} \, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}, \quad \rho = \rho^{(0)} \, e^{\beta_0(p-p_0)-\alpha_0(T-T_0)}. \tag{49b,c}$$

In contrast to the previous section, the above formulae do not concern the compressible fluid itself, but the entire considered magnetorheological medium. Thus, they cannot be directly used to derive the exact model of the damper in the decomposition approach, which requires the local properties of the component fluids. The derivation of the local properties of the secondary compressible fluid will be based on the general definitions of the coefficients of global compressibility, thermal expansion and mass expansion (Eq. 9), expressed in terms of volumes of the components fluids. By using such definitions, a set of differential equations can be stated with the unknown volume of the compressible fluid expressed in terms of pressure, temperature and mass:

$$\begin{aligned}
&-\frac{1}{V_f(m_f)+V_c(p,T,m_c)}\left(\frac{\partial V_c(p,T,m_c)}{\partial p}\right) = \beta_0 \\
&\frac{1}{V_f(m_f)+V_c(p,T,m_c)}\left(\frac{\partial V_c(p,T,m_c)}{\partial T}\right) = \alpha_0 \\
&\frac{1}{V_f(m_f)+V_c(p,T,m_c)}\left(\frac{\partial V_f(m_f)}{\partial m}+\frac{\partial V_c(p,T,m_c)}{\partial m}\right) = \frac{1}{m}.
\end{aligned} \tag{50a,b,c}$$

The last of the above equations can be transformed to the form:

$$\frac{1}{V_f+V_c}\left(\frac{\partial\left(V_f+V_c\right)}{\partial V_c}\right)\left(\frac{\partial V_c}{\partial m_c}\right)\left(\frac{\partial m_c}{\partial m}\right) = \frac{1}{m}. \tag{50d}$$

Further, the linearity of the relationship between the volume of the compressible fluid and the total volume of the considered medium as well as similar dependence between their masses allows the derivatives in the second component of Eq. (50d) to be replaced by the corresponding ratios. As a result, the third differential equation assumes the following simple form:

$$\frac{1}{V_c(p,T,m_c)}\left(\frac{\partial V_c(p,T,m_c)}{\partial m_c}\right) = \frac{1}{m_c}. \tag{50e}$$

The system of differential equations (Eq. 50a,b,e) has to be complemented with a condition that defines the actual volume of the viscous fluid V_f and the initial condition that defines the initial volume of the compressible fluid:

$$V_f = V_f^{(0)} \frac{m_f}{m_f^{(0)}} = V_c^{(0)} \frac{1}{k} \frac{m_c}{m_c^{(0)}}, \quad V_c(p_0, T_0, m_c^{(0)}) = V_c^{(0)}. \tag{50f,g}$$

The formulated problem has an analytical solution, which defines the actual volume of the compressible fluid in terms of its initial state and the actual thermodynamic parameters:

$$V_c = V_c^{(0)} \frac{m_c}{m_c^{(0)}} \left[\frac{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1}{k}\right]. \tag{51}$$

The above expression can be directly used to determine the actual density of the compressible fluid:

$$\rho_c = \rho_c^{(0)} \frac{k}{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1}, \tag{52}$$

and further to calculate the corresponding definitions of the coefficients of compressibility and thermal expansion of the secondary compressible fluid:

$$\beta_c = \beta_0 \frac{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}}{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1} \quad \text{and} \quad \alpha_c = \alpha_0 \frac{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}}{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1}. \tag{53}$$

In the second stage of analysis, the steps from the previous subsections can be followed in order to derive the complete model of the damper. The relation between the volumetric inflow rate of the secondary compressible fluid and the primary viscous fluid takes the form:

$$Q_V^c = \left[(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1\right] Q_V^f, \tag{54a}$$

while the dependencies between the actual volume of the component fluid and the actual volume of the entire medium read:

$$V_f = \frac{1}{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}}\, V \quad \text{and} \quad V_c = \frac{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)} - 1}{(1+k)\, e^{-\beta_0(p-p_0)+\alpha_0(T-T_0)}}\, V. \tag{54b}$$

Finally, by applying the standard assumptions with the zero value of thermal expansion of the compressible fluid, a simple model of two equations that govern the balance of fluid volume is obtained:

$$\begin{aligned} &\dot{V}_1 + \beta_0 V_1 \dot{p}_1 = Q_V^f (1+k) e^{-\beta_0(p_1-p_0)} \\ &\dot{V}_2 + \beta_0 V_2 \dot{p}_2 = -Q_V^f (1+k) e^{-\beta_0(p_2-p_0)}. \end{aligned} \tag{55}$$

The above system of equations with multipliers on the right hand side equal to 1 is often used as a simplified balance of volume for compressible MR fluid of a constant global compressibility. Nevertheless, the presented derivation highlights two intrinsic novel aspects of the model. At first, the application of the decomposition method and the resulting multipliers on the r.h.s. of the above equations confirm that the volumetric flow rate Q_V^f can be determined from the model of incompressible viscous flow. Secondly, the strict derivation reveals that the above multipliers are required to ensure that the mass inflow rates to both chambers are equal and that the global balance of mass of the fluid enclosed in the entire system is not violated.

4.4. Experimental validation of the model

The ultimate verification of the numerical model is the comparison with experimental results. All the models presented above are capable of generating hysteresis loops that closely replicate the experimental data. Thus, the following comparison will be conducted exclusively for the last considered model, which involves compressible fluid and results in a constant value of global compressibility and thermal expansion of the MR fluid.

The derived physical model is based on two groups of physical parameters. The first group comprises geometrical properties of the damper such as the dimensions of both fluid chambers, the volume of the internal gas cushion and the geometry of the magnetic system (in particular the width of the orifice and the length of the magnetic poles). In turn, the second group includes the physical parameters of the magnetorheological fluid including its yield stress level and viscosity in terms of the applied current intensity. While the geometrical parameters are known in a straightforward manner from the damper design, the physical parameters of the fluid usually cannot be determined directly. Although the initial physical parameters of the MR fluid are given by the damper manufacturer, the fluid is usually subjected to ageing, sedimentation and deterioration, which substantially change the actual values of the yield stress level and the apparent viscosity. Thus, the actual values of fluid parameters have to be determined based on the experiment and using a dedicated identification procedure. Herein, the identification of the physical parameters was confined to the assumptions of the quadratic functions that describe the change of the yield stress level and the apparent viscosity in terms of the applied current and a manual adjustment of these parameters to fit the results of the conducted experiment.

The comparison was performed for a selected range of excitation frequencies and control currents. The selected force–displacement and force velocity hysteresis loops are presented in Fig. 7. The plots reveal a qualitative and quantitative agreement of the model and the experiment. The primary effects including blocking of the flow and fluid compressibility are well represented by the numerical model. The observed discrepancies between the plots can be attributed to the lack of exact identification of model parameters, measurements inaccuracies and secondary local effects that occur during the damper operation.

5. Reduced and parametric models of MR dampers

The disadvantage of the exact model of MR dampers derived above is the fact that it is defined as a set of nonlinear differential equations, which has to be solved numerically. Such a model can be too complex for engineering applications and therefore there is a strong need for derivation of a reduced model where the force generated by the damper is expressed directly in terms of kinematics of the piston. In turn, such a reduced model can be used as a basis to derive the corresponding parametric model, which can be applied when geometrical or physical properties of the damper are not known. Both reduced and parametric models are expected to generate desired characteristic shapes of the force–velocity hysteresis loops.

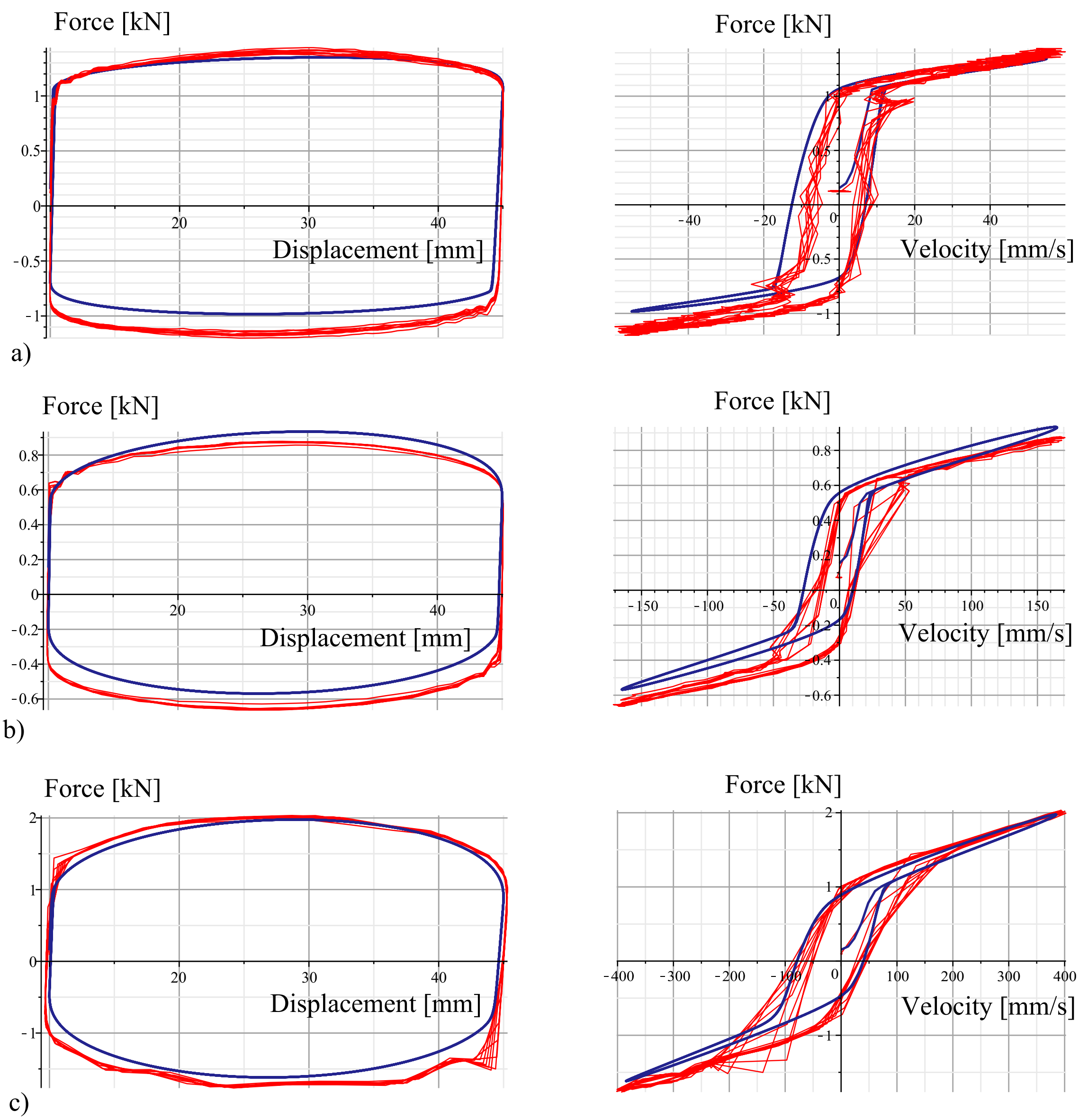


Fig. 7. Comparison of the hysteresis loops obtained from the numerical model and the experiment: (a) $f=0.5$ Hz, $I=1$ A, (b) $f=1.5$ Hz, $I=0.3$ A, (c) $f=3.5$ Hz, $I=0.6$ A.

5.1. Derivation of a reduced model of the MR damper

The proposed reduced model involves various treatments of compressibility in two stages of work of the damper. The prerequisite for the formulation of the model will be the fact that compressibility of the fluid and its thermal expansion have significant influence on the processes arising inside the damper and that they affect the value of the generated force exclusively in the case when the flow of the fluid through the valve is blocked. In contrast, when the flow of the fluid is developed, the compressibility of the fluid and its thermal expansion seem to have a minor contribution to the operation of the damper. Consequently, the proposed reduced model will be based on the following assumptions:

- compressibility of the fluid and its thermal expansion will be taken into account when the flow is blocked,
- compressibility of the fluid and its thermal expansion will be totally neglected when the flow of the fluid occurs.

The equations of the reduced model will be derived by a direct simplification of the equations of the previously derived exact physical model.

During a major part of damper operation, when the flow through the valve occurs ($\Delta p > \Delta p_{crit}$), disregarding the thermal expansion coefficient of the fluid results in decoupling the equations of volume balance from the equations of energy balance. Moreover, neglecting the compressibility coefficient causes that the reduced model of the MR damper assumes the form of the classical model of a hydro-pneumatic damper, which contains two simple equations of volume balance:

$$\dot{V}_1 = Q_V^f(\Delta p), \quad \dot{V}_2 = -Q_V^f(\Delta p)$$

where: $V_1 + V_g(p_1) = V_1^{ch}(u)$ and

$$V_2 = V_2^{ch}(u), \tag{56a,b,c}$$

supplemented by the following general definition of the generated force:

$$F_{damper} = p_1 A_1 - p_2 A_2 = A_{shaft} p_1 + A_2 \Delta p. \tag{56d}$$

A summation and integration of the equations of volume balance enable finding the pressure p_1 in terms of the piston displacement, while the equation of volume balance for the chamber without the gas cushion allows the pressure difference Δp to be expressed in terms of the actual piston velocity. Finally, the force generated by the damper can be defined analytically as a sum of the pneumatic and hydraulic forces:

$$F_{damper} = A_{shaft}\, f_1(u) + A_2 f_2(\dot{u}), \tag{56e}$$

where f_1 and f_2 are known functions, which depend on parameters of the gas cushion and the derived model of the viscous flow, respectively. For example, in the special case when the adiabatic conditions for the gas cushion are assumed and the volumetric flow rate of the viscous fluid is defined as a linear function of the pressure difference (with the proportionality factor $1/C$), one obtains the formula:

$$F_{damper} = A_{shaft}\, p_1^0 \left(\frac{V_1^{g(0)}}{V_1^{g(0)} - A_{shaft}\, u} \right)^{\kappa} + \frac{(A_2)^2 \dot{u}}{C}, \tag{56f}$$

which defines the generated force in terms of known physical parameters and kinematics of the piston.

During the short periods of damper operation, when the valve flow is blocked ($\Delta p < \Delta p_{crit}$), the model is governed by two simplified equations of volume balance and two simplified equations of energy balance. Since the volume of the primary viscous fluid in each chamber does not change, the most convenient form of the equations of volume balance directly utilizes the volume of the compressible fluid (cf. Eq. 15):

$$\dot{V}_1^c + \left(\beta_1^c \dot{p}_1 - \alpha_1^c \dot{T}_1\right) V_1^c = 0$$
$$\dot{V}_2^c + \left(\beta_2^c \dot{p}_2 - \alpha_2^c \dot{T}_2\right) V_2^c = 0, \tag{57a,b}$$

where

$$V_1 + V_g(p_1) = V_1^{ch}(u),$$

and

$$V_2 = V_2^{ch}(u). \tag{57c,d}$$

Integration of the above simplified form of the equations of volume balance yields the following analytical formula that defines the actual volume of the compressible fluid:

$$V_c = V_c^{(0)} \exp\left(-\int_{p_0}^{p} \beta^c(\bar{p}, T) d\bar{p} + \int_{T_0}^{T} \alpha^c(p, \bar{T}) d\bar{T} \right), \tag{57e}$$

which, in fact, is equivalent to the general formula that defines the reconstructed equation of state (cf. Eq. 35) combined with the condition of constant mass of the fluid. In turn, by reducing the enthalpy term and omitting the derivatives of mass in the general equations of energy balance, one obtains:

$$0 = \left(m_1^f c_p^f + m_1^c c_p^c - \alpha_c p_1 V_c\right) \dot{T}_1 + (\beta_c p_1 - \alpha_c T_1) V_1^c \dot{p}_1 + p_1 \dot{V}_1^c$$
$$0 = \left(m_2^f c_p^f + m_2^c c_p^c - \alpha_c p_2 V_c\right) \dot{T}_2 + (\beta_c p_2 - \alpha_c T_2) V_2^c \dot{p}_2 + p_2 \dot{V}_2^c. \tag{57f,g}$$

In the considered case the equations of volume balance are coupled with the equations of energy balance, but the equations related to specific chambers are decoupled from each other, which significantly facilitates obtaining the solution. Although the above equations cannot be solved analytically in the general form presented above, an analytical solution can be derived for appropriately selected specific cases. As it can be expected, the crucial simplifying assumption is the same constitutive modelling of the compressible part of the MR fluid and the gas cushion.

Consider further an exemplary reduced model of the MR damper, in which the compressible part of the MR fluid is modelled as an ideal gas described by the classical equation of state and the standard values of the coefficients of compressibility and thermal expansion:

$$\beta_c = p^{-1}, \quad \alpha_c = T^{-1}. \tag{58}$$

Since the version of the model that corresponds to a large pressure difference and a developed valve flow does not depend on the model of compressibility, it will be still described by the general equation that defines the total generated force as a sum of the pneumatic and hydraulic forces (Eq. 56e). In contrast, the assumed model of compressibility will strongly affect the balance equations corresponding to blocked flow. For the assumed definitions of the thermodynamic coefficient, the integrals in Eq. (57e) can be calculated analytically, which indicates that the equations of volume balance are reduced to the equations of state assuming that the mass of the fluid is constant:

$$V_c^1 = \frac{m_1^{c(0)} R T_1}{p_1}, \quad V_c^2 = \frac{m_2^{c(0)} R T_2}{p_2}. \tag{59a,b}$$

In turn, the assumption of the adiabatic conditions for the gas cushion transforms Eq. (57c,d) into the form:

$$V_1^f + V_1^c + V_g^{(0)} \left(\frac{p_1^{(0)}}{p_1} \right)^{1/\kappa} = V_1^{ch}(u), \quad V_2^f + V_2^c = V_2^{ch}(u). \tag{59c,d}$$

By introduction of the assumed definitions of the thermodynamic coefficient and by applying the standard thermodynamic relations in the equations of energy balance, one obtains:

$$\begin{aligned} 0 &= \left(m_1^{f(0)} c_p^f + m_1^{c(0)} c_v^c \right) \dot{T}_1 + p_1 \dot{V}_1^c \\ 0 &= \left(m_2^{f(0)} c_p^f + m_2^{c(0)} c_v^c \right) \dot{T}_2 + p_2 \dot{V}_2^c. \end{aligned} \tag{59e,f}$$

In the above equation, the upper index ‘0’ indicates the state of the fluid at the beginning of the stage with blocked flow, i.e. at the time instant when the critical pressure difference is reached. By calculating the fluid temperatures from Eq. (59a,b) and by introducing them into Eq. (59e,f) the following set of differential equations is obtained:

$$\begin{aligned} 0 &= C_1\, \dot{p}_1 V_1^c \;+\; (C_1 + 1) p_1 \dot{V}_1^c \\ 0 &= C_2\, \dot{p}_2 V_2^c \;+\; (C_2 + 1) p_2 \dot{V}_2^c, \end{aligned}$$

where:

$$C_1 = \frac{m_1^{f(0)} c_p^f + m_1^{c(0)} c_v^c}{m_1^{c(0)} R}, \quad C_2 = \frac{m_2^{f(0)} c_p^f + m_2^{c(0)} c_v^c}{m_2^{c(0)} R}, \tag{60a,b}$$

which has the following analytical solution:

$$V_1^c = V_1^{c(0)} \left(\frac{p_1^{(0)}}{p_1} \right)^{\frac{C_1}{C_1+1}}, \quad V_2^c = V_2^{c(0)} \left(\frac{p_2^{(0)}}{p_2} \right)^{\frac{C_2}{C_2+1}}. \tag{60c,d}$$

Further, by substitution of the obtained results into Eq. (59c,d), a set of two independent algebraic equations is obtained, which allows the pressure of the fluid enclosed in both chambers of the damper to be computed. Although the pressure in the chamber without the gas cushion can be always determined, the equation for the chamber with the gas cushion is nonlinear and does not have an analytical solution. Therefore, the additional assumption introduced in order to obtain a fully analytical model of the damper will be that the heat capacity coefficient of the viscous fluid is zero:

$$c_p^f = 0. \tag{61}$$

The above assumption results in the modified values of the constants C_1 and C_2:

$$C_1 = (\kappa - 1)^{-1}, \quad C_2 = (\kappa - 1)^{-1}, \tag{62a,b}$$

in the modified definitions of the actual volume of the compressible fluid:

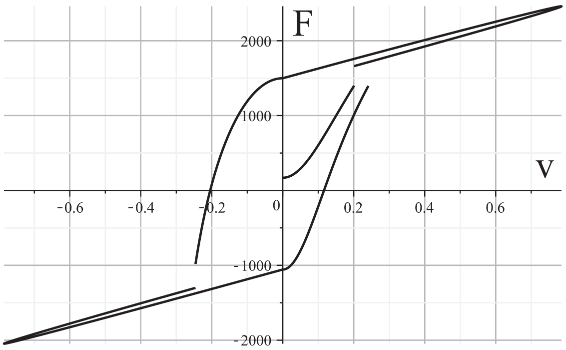
$$V_1^c = V_1^{c(0)} \left(\frac{p_1^{(0)}}{p_1} \right)^{\frac{1}{\kappa}}, \quad V_2^c = V_2^{c(0)} \left(\frac{p_2^{(0)}}{p_2} \right)^{\frac{1}{\kappa}}, \tag{62c,d}$$

and the following simple final form of the equations that govern the change of pressure in both chambers (cf. Eq. 59c,d):

$$\left(V_1^{c(0)} + V_g^{(0)} \right) \left(\frac{p_1^{(0)}}{p_1} \right)^{\frac{1}{\kappa}} = V_1^{ch}(u) - V_1^f, \quad V_2^{c(0)} \left(\frac{p_2^{(0)}}{p_2} \right)^{\frac{1}{\kappa}} = V_2^{ch}(u) - V_2^f. \tag{62e,f}$$

The above set of equations has a simple analytical solution:

$$p_1 = p_1^{(0)} \left(\frac{V_1^{c(0)} + V_g^{(0)}}{V_1^{ch}(u) - V_1^f} \right)^{\kappa}, \quad p_2 = p_2^{(0)} \left(\frac{V_2^{c(0)}}{V_2^{ch}(u) - V_2^f} \right)^{\kappa}, \tag{62g,h}$$



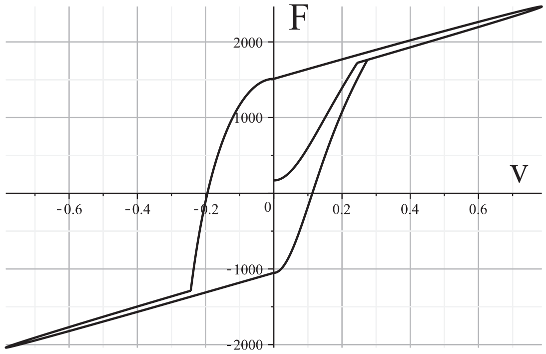


Fig. 8. Force–velocity hysteresis loops obtained from the reduced model of the MR damper: (a) discontinuous model, (b) artificially obtained continuous model.

and allows the final analytical formula that defines the force generated by the damper to be derived in terms of the actual displacement of the piston. Finally, the complete reduced model of the damper is composed of two analytical expressions:

$$F_{damper} = A_{shaft}\, p_1^{g(0)} \left(\frac{V_1^{g(0)}}{V_1^{g(0)} - A_{shaft}\, u} \right)^{\kappa} + \frac{(A_2)^2 \dot{u}}{C} \quad \text{for} \quad \Delta p \geq \Delta p_{crit}$$

$$F_{damper} = A_1 p_1^{(0)} \left(\frac{V_1^{c(0)} + V_g^{(0)}}{V_1^{ch}(u) - V_1^{f}} \right)^{\kappa} - A_2 p_2^{(0)} \left(\frac{V_2^{c(0)}}{V_2^{ch}(u) - V_2^{f}} \right)^{\kappa} \quad \text{for} \quad \Delta p < \Delta p_{crit}, \tag{63a,b}$$

which have to be applied alternately depending on the actual value of the pressure difference. In each equation, the initial values of the fluid parameters should be taken from the end of the preceding stage of the process. Note that further simplification of the model can be obtained by neglecting compressibility of the MR fluid enclosed in the chamber without the gas cushion (neglecting the quantity $V_1^{c(0)}$ in the numerator of the first term of Eq. (63b)).

The reduced model derived above was implemented in MAPLE software and compared with the previously presented exact model (Fig. 7). A good qualitative correspondence of the shapes of the force–velocity hysteresis loops obtained from both models confirms the correctness of the assumptions applied in the reduced model, including the disregarding of compressibility and thermal expansion during the prevailing stage of the process when the flow of the MR fluid occurs.

A clearly visible disadvantage of the model is the discontinuity of the generated force after the transition from the stage of blocked flow to the stage of the developed flow (Fig. 8a). Such a discontinuity is caused by the fact that the definition of the force for the developed flow (Eq. 63a) enables introducing the initial pressure in the chamber with the gas cushion, while the value of the pressure difference (and thus the value of the pressure in the second chamber) is implicitly predetermined by the actual velocity of the piston. In contrast, since the equation for the case of blocked flow enables introducing both initial pressures, the discontinuity during the transition to blocked flow does not appear. The occurrence of the described discontinuity can be artificially overcome by extending the range of the validity of the formula that defines the blocked flow beyond the initially assumed range of pressure differences (Fig. 8b).

5.2. Derivation of the parametric model of the MR damper

The proposed reduced model of the MR damper can be successfully used as a so-called "physics-based" parametric model. In such a parametric model, the quantities that appear in Eq. (63a,b) are not based on the corresponding physical properties of the damper and features of the viscous flow, but instead, they are considered as parameters, which are tuned by using optimization procedures. Although such a parametric model has the advantage of being derived from the physical model, its strong drawback is the non-continuous, piecewise definition of the generated force and the requirement of simultaneous computation of the actual pressure difference, which used for switching between two equations (cf. Eq. 63a,b). Consequently, there appears a need for derivation of a simpler parametric model, which would be based on a continuous definition of the generated force expressed exclusively in terms of the kinematics of the piston.

Before deriving the physics-based parametric model of the MR damper proposed above, one can analyse the possibility of application of a simple parametric model based on the bristle friction models, which have been the starting point for the entire consideration. The application of the bristle friction models can be viewed in two manners. At first, they can be considered as models of a similar mathematical structure as the Bouc–Wen model and thus qualified as purely parametric. Secondly, they can be considered as models that enable simulation of the elastic force generated at a low range of velocities and consequently qualified (at least partially) as physical.

Recall the Dahl model [19], which is one of the simplest models in the group of the bristle friction models. It is governed by the following equations:

$$F_{damp} = \sigma_0 z + \sigma_1 v$$
$$\frac{dz}{dt} = v - \frac{\sigma_0}{F_c} z|v|. \tag{64a,b}$$

In the above model, the internal differential equation describes the evolution of the internal variable z in terms of the actual relative velocity of the contacting surfaces v. In fact, a similar mathematical structure is preserved by all Bouc–Wen models. The internal equation has an analytical solution expressed in terms of the relative displacement u, which takes the form:

$$z = \frac{F_c}{\sigma_0} - \left(\frac{F_c}{\sigma_0} - z_0\right) e^{-\frac{\sigma_0 (u-u_0)}{F_c}} \quad \text{for} \quad v > 0$$
$$z = -\frac{F_c}{\sigma_0} + \left(\frac{F_c}{\sigma_0} + z_0\right) e^{\frac{\sigma_0 (u-u_0)}{F_c}} \quad \text{for} \quad v > 0, \tag{64c}$$

and indicates that the internal variable contains a piecewise constant term and an exponential elastic term, which significantly influences the generated force directly after change of the relative velocity but which decreases and eventually vanishes during the further relative motion of the surfaces. Thus, in the considered model the hysteretic force is a sum of a piecewise constant force and an exponentially decreasing elastic force, which occurs exclusively after change of relative velocity, at the low velocity range. In addition, the classical viscous force holds in the entire velocity range. Concluding, two situations can be distinguished:

- the low velocity range, where the simulated force is the sum of a piecewise constant force, an exponential elastic force and a linear viscous force,
- the high velocity range, where the simulated force is the sum of a constant force and a linear viscous force.

Therefore, the assumption of the Dahl model (or any other model of the Bouc–Wen type) predefines the type of the generated forces and their constitutive definitions. Note that the forces simulated by the Dahl model do exactly correspond to the forces simulated by the proposed exact physical model of the MR damper, in which one can clearly distinguish:

- the low velocity range (that corresponds to low pressure difference), where exclusively a nonlinear elastic force is generated,
- the high velocity range (that corresponds to high pressure difference), where a nonlinear elastic force and a linear viscous force are generated.

The simple conclusion is that the Dahl model can be used to simulate different types of forces than the ones that physically occur in the MR damper and thus it will never provide the possibility of exact modelling of the MR damper. Despite such a discrepancy, the parameters of the Dahl model can be tuned to obtain the acceptable correspondence with the derived physical model of the magnetorheological damper (Fig. 9a).

An attempt of adjustment of the Dahl model and its application for the simulation of the MR damper will be based on supplementing the standard definition of the generated force with a constant term and an additional elastic term dependent on the actual relative displacement. In such a case, the equation that governs the modified Dahl model takes the form:

$$F_{damp} = \sigma + \sigma_0 z + \sigma_1 v + \sigma_2 u. \tag{65}$$

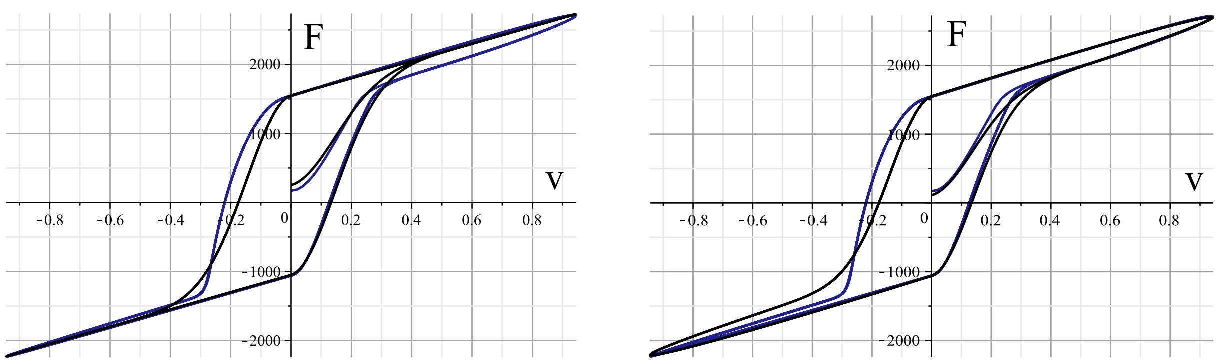


Fig. 9. Comparison of selected hysteresis loops obtained with physical models (blue lines) and: (a) standard Dahl model, (b) improved Dahl model involving the additional linear elastic term (black lines). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The additional elastic term causes that in the high velocity range the simulated force has a better correspondence with the physical force generated by the damper. Although the full equivalence of the simulated and the physical forces is still not obtained, their correlation is considerably improved. In the consequence, four parameters that arise in the modified Dahl model can be chosen in a way that the obtained hysteresis loop closely resembles the one achieved from the exact physical model that assumes compressibility of the MR fluid and blocking of the flow (Fig. 9b). In addition, extended version of modified Dahl model might include the possibility simulating various thickness of left and right, high-velocity hysteresis loop.

The proposed, utterly new type of "physics-based" parametric model combines the advantages of a standard parametric model of the Bouc–Wen type and the advantages of the previously derived reduced physical model. The basic features of this model are as follows:

1. Similarly as the Bouc–Wen models, the proposed model contains an additional differential equation that governs the change of the internal variable and which helps to describe the hysteretic behaviour of the system.
2. The determined evolution of the internal variable and of its first time derivative is used to define the periods of operation when particular types of the forces are generated.
3. The definitions of the forces simulated in certain periods of operation are directly based on the formulae introduced in the previously derived reduced physical model.

Consequently, the first step of model derivation is a proper definition of the subsequent stages of the process with the use of the internal equation, while the second step is a parametric definition of the simulated forces.

The applied internal differential equation is analogous as in the Dahl model (cf. e.g. 64b) and it describes the change of the internal variable z in terms of velocity of the applied kinematic excitation v:

$$\frac{dz}{dt} = v - P_0 z|v|. \tag{66}$$

The introduced parameter P_0 defines the width of the hysteresis loop in the plot of the internal variable in terms of the velocity $z(v)$. Solving the above differential equation allows the conditions to be formulated that define the occurrence of four subsequent branches of the hysteresis loop, which correspond to various conditions of the damper operation and to various types of the generated forces:

1. $v > 0$ and $\dot{z} > 0$: beginning of the forward stroke (left cylinder end), blocked flow, generation of the elastic force (right vertical branch of the hysteresis);
2. $v > 0$ and $\dot{z} = 0$: main part of the forward stroke, developed flow, generation of the viscous and elastic forces (top edge of the hysteresis);
3. $v < 0$ and $\dot{z} < 0$: beginning of the backstroke (right cylinder end), blocked flow, generation of the elastic force only (left vertical branch of the hysteresis);
4. $v < 0$ and $\dot{z} = 0$: main part of the backstroke, developed flow, generation of the viscous and elastic forces (bottom edge of the hysteresis).

In contrast to standard parametric models, in the proposed model the internal variable z will not be directly used in the definitions of the forces generated by the MR damper. Instead, for each of the damper operation modes distinguished above, one can employ the formulae based on the physical definitions of the forces that appear in the reduced physical model (cf. Eq. 63). However, the quantities that appear in the definitions of the generated forces will not be considered as physical parameters that are predetermined by the construction of the damper (as the chamber cross section) or that result from the previous stages of the process (as the initial chamber volume or the initial pressure). Instead, they will be treated as standard parameters, which have to be tuned during the comparison with the experiment with the use of an optimization procedure.

The separate definitions of the generated forces will be applied for the case of developed and blocked flow, as well as for the case of forward and backward movement of the piston. Consequently, four distinct definitions corresponding to four branches of the hysteresis loop can be distinguished. For the dominating mode of operation, when the valve flow is developed (cases 2 and 4), by recalling Eq. (63a) and substituting physical quantities by parameters P_1–P_{20}, one obtains:

$$\begin{aligned} F_{damper} &= P_1\left(\frac{P_2}{P_2 - P_3\,u}\right)^{\kappa} + \frac{(P_4)^2\dot{u}}{P_5} \quad \text{for} \quad v > 0 \quad \text{and} \quad \dot{z} > 0 \\ F_{damper} &= P_6\left(\frac{P_7}{P_7 - P_3\,u}\right)^{\kappa} + \frac{(P_4)^2\dot{u}}{P_5} \quad \text{for} \quad v > 0 \quad \text{and} \quad \dot{z} > 0. \end{aligned} \tag{67a,b}$$

Analogously, for the short periods of blocked flow, which occur at both ends of the damper stroke (cases 1 and 3), by recalling Eq. (63b), one gets the formulae:

$$F_{damper} = P_8\left(\frac{P_9}{P_{10} - P_{11}u}\right)^{\kappa} - P_{12}\left(\frac{P_{13}}{P_{14} + P_4 u}\right)^{\kappa} \quad \text{for} \quad v > 0 \quad \text{and} \quad \dot{z} > 0$$

$$F_{damper} = P_{15}\left(\frac{P_{16}}{P_{17} - P_{11}u}\right)^{\kappa} - P_{18}\left(\frac{P_{19}}{P_{20} + P_4 u}\right)^{\kappa} \quad \text{for} \quad v < 0 \quad \text{and} \quad \dot{z} < 0. \tag{68a,b}$$

Due to the fact that the model proposed above contains a large number of internal parameters, it possesses most of the drawbacks characteristic for parametric models. At first, tuning of these parameters and adjusting the model to experimental data can be difficult, since the optimization procedure may stall in local minima. Secondly, the solution of the optimization problem can be ambiguous and the determined parameters may not correspond to any physically meaningful properties of the system. Note that the model described above corresponds to the most general case, when none of the physical parameters of the damper is known. In practical situation at least some features of the analysed system (e.g., cross sections of both chambers or cross-section of the shaft) are known, which reduces the number of parameters to be identified, facilitates the optimization procedure and increases the effectiveness of the proposed model.

A proper adjustment of all parameters of the parametric model described by Eqs. (66–68) reveals their physical values, yields a full equivalence with the reduced physical model defined by Eqs. (63) and the force–velocity hysteresis loops that exactly match the shapes presented in Fig. 7.

6. Conclusions and final remarks

The presented exact physical model of the MR damper considerably differs from the classical models since it takes into account a combination of the effects of blocking the flow between the chambers in case of low pressure difference and the compressibility of the fluid enclosed in each chamber. As proved, taking into account both of these phenomena is required to correctly model the dissipative characteristics of the damper and to obtain the characteristic “z-shaped” force–velocity hysteresis loops. A convenient implementation of the model involves a decomposition of the compressible viscous fluid into a primary incompressible viscous fluid and a secondary fluid characterized by compressibility and thermal expansion. The proposed model can be implemented for various constitutive models of the compressible fluid including the ideal gas law, constant compressibility and linear volumetric elasticity. In addition, the derived reduced model involves a simplified treatment of compressibility, lends a critical view on classical parametric models and yields an utterly new type of a parametric model based on physical background. It is believed that the proposed approach will provide new insight on the problem of physical modelling of the MR dampers, facilitate in-depth understanding of their dissipative properties and contribute to their deliberate application in adaptive systems for impact absorption.

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