pt: An R package for Prospect Theory
Version 1.0

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Abstract

The R package **pt** (prospect theory) provides computational tools to systematically illustrate the predictive results of prospect theory against other theories of risky decision making. The package has three main capabilities. First, it allows choices to be created, saved and visualised. Next, it allows the predictive results of expected and non-expected utility decision making theories to be compared for these choices. Different utility and probability weighting functional forms and parameterisations can be specified and visualised. Finally, it allows critical tests for decision making such as the Allais paradoxes to be examined. It can also help build insight to assist in the development of new theories of risky decision making.

Keywords: prospect theory, risky decision making, R

1 Introduction

Tversky and Kahneman (1992) introduced cumulative prospect theory as a theory of risky decision making (and decision making under uncertainty) that was able to explain empirical phenomena that the expected utility theory (Bernoulli, 1954; von Neumann and Morgenstern, 1947) was unable to account for, most notably the Allais (1953, 1979) paradoxes. (Following the convention in Wakker (2010) this theory will be referred to as prospect theory and the theory in Kahneman and Tversky (1979) as original prospect theory from now on.) Designed to address the shortcomings of original prospect theory, prospect theory has been highly influential in the field of decision making, and contributed to Kahneman being awarded the Nobel Prize for economics in 2002.

This article introduces the **pt** package which provides computational tools to explore prospect theory and alternative theories of risky decision making. The R statistical environment (R Core Team, 2013) was chosen to develop this software as R has become a popular standard for open source statistical computing and has a large and expanding user base. This paper is structured as follows. Section 2 reviews expected and non-expected utility theories of risky decision making. Section 3 describes the classes and functions available in the **pt** package (version 1.0). Section 4 describes the validity tests conducted for the package functions. Section 5 describes how the package can be used to explore key phenomena in risky decision making such as the Allais paradoxes. Section 6 shows how new critical tests of risky decision making can be explored using the package.

2 Theories of risky decision making

Many theories of risky decision making are related to each other through correspondence limits as shown in Figure 1. A straightforward way to understand these correspondences is to begin with the simplest theory (expected value or EV) and generalise to increasingly elaborate theories that provide more comprehensive accounts of risky decision making phenomena. The generalisations of EV can be through using different probability weighting functions or utility functions, or using different equational forms for combining the probability and objective consequence variables. Different psychological principles may underlie the introduction of these constructs.

Figure 1: Correspondence limits for theories of risky decision making.

2.1 Expected value (EV)

The basic unit of risky decision making is the gamble. A gamble $G = (x_1, p_1; x_2, p_2; \ldots, x_n, p_n)$ is a set of n objective consequences x_i with the probability of each objective consequence being p_i and $\sum_{i=1}^{n} p_i = 1$. The individual outcomes $\{x_i, p_i\}$ are considered to be mutually exclusive and exhaustive. Given two or more gambles to choose from, it is assumed a decision maker will prefer the gamble that has the highest utility. The EV of a gamble G is the long-run mean, given by

$$EV(G) = \sum_{i=1}^{n} p_i x_i. \tag{1}$$

Empirical studies provide evidence that decision makers do not maximise expected value when faced with individual choices but the principle becomes increasingly salient when decision makers are faced with a large (> 10) (Montgomery and Adelbratt, 1982) number of repeated choices (Lichtenstein et al., 1969; Li, 2003; Colbert et al., 2009). Nearly all decision making research has focussed on single choice rather than repeated choice situations, the latter being more common in game theoretic research (e.g. repeated Prisoner's dilemmas) (although see Post et al., 2008 for a study of path-dependence on decision making). The theories discussed in this paper have been applied to single choice decision situations. An open research question is how to accommodate these theories under a temporal framework where the effects of past and current decisions on future decisions can be predicted, incorporating the effects of learning (see Problem 6 Hastie, 2001, p. 665).

2.2 Expected utility (EU)

An analysis of the St. Petersburg paradox led Bernoulli (1954) to develop EU. An axiomatic formulation of EU assuming completeness, transitivity, continuity and independence was subsequently developed (von Neumann and Morgenstern, 1947, p. 26). The EU of a gamble G is given by

$$EU(G) = \sum_{i=1}^{n} p_i u(x_i). \tag{2}$$

If $u(x_i) = x_i$ then the EU reduces to EV.

2.3 Subjective weighted utility (SWU)

Edwards (1954, 1962) proposed a more general form to allow for probability weightings $w(p_i)$ defined by

$$SWU(G) = \sum_{i=1}^{n} w(p_i) u(x_i), \tag{3}$$

with SWU reducing to EU when $w(p_i) = p_i$.
SWU has a similar form to subjective expected utility Savage (1954), given by

$$SEU(G) = \sum_{i=1}^{n} S(p_i) u(x_i). \tag{4}$$

However, the $S(p_i)$ in SEU represents a subjective probability rather a probability weighting $w(p_i)$. When $u(x_i) = x_i$ SEU reduces to subjective expected value theory (Preston and Baratta, 1948), given by

$$SEV(G) = \sum_{i=1}^{n} S(p_i) x_i. \tag{5}$$

2.4 Rank-dependent utility (RDU)

Quiggin (1982, 1985, 1993) used ranks to explain the Allais paradoxes without transparently violating first-order stochastic dominance. First gamble outcomes are arranged from the best outcome to the worst. A rank is the probability of a gamble yielding an outcome better than a worse outcome. Rather than use probability weightings of individual probabilities, RDU first computes the differences in ranks, and applies a probability weighting to this difference. For strictly positive gambles $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$, the RDU of a gamble G is given by

$$RDU(G) = \sum_{i=1}^{n} \left[w^+\left(p_i + \ldots + p_1 \right) - w^+\left(p_{i-1} + \ldots + p_1 \right) \right] u(x_i)$$

$$= \sum_{i=1}^{n} \left[w^+\left(\sum_{j=1}^{i} p_j \right) - w^+\left(\sum_{j=1}^{i-1} p_j \right) \right] u(x_i), \tag{6}$$

where $w^+(p_i)$ is a monotonically increasing function with boundary conditions $w^+(0) = 0$ and $w^+(1) = 1$. Similarly, for strictly negative gambles $x_1 \leq x_2 \leq \ldots \leq x_n \leq 0$, the RDU is given by

$$RDU(G) = \sum_{i=1}^{n} \left[w^-\left(p_i + \ldots + p_1 \right) - w^-\left(p_{i-1} + \ldots + p_1 \right) \right] u(x_i)$$

$$= \sum_{i=1}^{n} \left[w^-\left(\sum_{j=1}^{i} p_j \right) - w^-\left(\sum_{j=1}^{i-1} p_j \right) \right] u(x_i). \tag{7}$$

RDU reduces to EU when $w^{\pm}(p_i) = p_i$.

2.5 Prospect theory utility (PTU)

Tversky and Kahneman (1992) incorporated theoretical developments from Gilboa (1987) and Schmeidler (1989) to develop PT, providing the ability to model decision making under uncertainty. The outcomes in a gamble are ranked such that $x_1 \geq \cdots \geq x_k \geq 0 \geq x_{k+1} \geq \cdots \geq x_n$ and $0 \leq k \leq n$. The utility of a gamble G is the sum of two RDU terms and is given by

$$PTU(G) = \sum_{i=1}^{n} \left[w^+\left(p_i + \ldots + p_1 \right) - w^+\left(p_{i-1} + \ldots + p_1 \right) \right] u(x_i)$$

$$+ \sum_{j=n+1}^{m} \left[w^-\left(p_i + \ldots + p_{n+1} \right) - w^-\left(p_{i-1} + \ldots + p_{n+1} \right) \right] u(x_i)$$

$$= \sum_{i=1}^{n} \left[w^+\left(\sum_{j=1}^{i} p_j \right) - w^+\left(\sum_{j=1}^{i-1} p_j \right) \right] u(x_i)$$

$$+ \sum_{i=n+1}^{m} \left[w^-\left(\sum_{j=n+1}^{i} p_j \right) - w^-\left(\sum_{j=n+1}^{i-1} p_j \right) \right] u(x_j). \tag{8}$$

Under PT loss aversion is modelled by the assumption

$$u(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0 \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \tag{9}$$

where λ is the loss aversion coefficient (see Wakker (2008, p. 1336) for a review of the power utility family).

2.6 Subjectively weighted average utility (SWAU)

Birnbaum (1999, p. 35) describes the SWAU model (Karmarkar, 1978, 1979; Viscusi, 1989; Lattimore et al., 1992), where the relative weight of an outcome probability in a gamble G depends on the other outcome probabilities in the gamble. The SWAU is given by

$$SWAU(G) = \frac{\sum_{i=1}^{n} w(p_i) u(x_i)}{\sum_{i=1}^{n} w(p_i)}. \tag{10}$$

When $w(p_i) = p_i$, SWAU reduces to EU.

2.7 Rank-affected multiplicative weights utility (RAMU)

Given a set of ranked outcomes $x_1 \geq x_2 \geq \ldots \geq x_n$, the RAMU is given by Birnbaum (2008) as

$$RAMU(G) = \frac{\sum_{i=1}^{n} a(i, n, s_i) t(p_i) u(x_i)}{\sum_{i=1}^{n} a(i, n, s_i) t(p_i)}. \tag{11}$$

Here the $a(i, n, s_i)$ are the rank and augmented sign branch weights. The branch probability weighting function $t(p)$ is usually approximated by a power function $t(p) = p^\gamma$. If $\gamma = 1$ and $a(i, n, s_i) = 1$ then RAMU reduces to EU.

2.8 (Special) transfer of attention exchange utility (TAXU)

The general TAX formula is given by Birnbaum (2008) as

$$TAXU(G) = \frac{\sum_{i=1}^{n} t(p_i) u(x_i) + \sum_{i=1}^{n} \sum_{k=1}^{i} \left[u(x_i) - u(x_k) \right] \omega(p_i, p_k, n)}{\sum_{i=1}^{n} t(p_i)}. \tag{12}$$

If $\omega = 0$ then this reduces to SWAU. The special TAX case occurs with the assumption

$$\omega(p_i, p_k, n) = \begin{cases} \dfrac{\delta \cdot t(p_k)}{n+1}, & \text{if } \delta > 0 \\[3mm] \dfrac{\delta \cdot t(p_i)}{n+1}, & \text{if } \delta \leq 0. \end{cases} \tag{13}$$

If $t(p) = p$ and $\delta = 0$ then TAX reduces to EU. If $\delta = 0$ then TAX reduces to SWAU. PT, RAM and TAX are special cases of a general configural weighting model discussed in Birnbaum (2004, p. 91-92), given by

$$CWU(G) = \frac{\sum_{i=1}^{n} w(p_i, G) u(x_i)}{\sum_{i=1}^{n} w(p_i, G)}, \tag{14}$$

where the objective consequences are ranked $x_1 > x_2 > \ldots > x_n$ and $w(p_i, G)$ is the branch configural weight of objective consequence x_i. This theory in turn is a specific case of the configurally weighted, average configural value utility model (Birnbaum, 1999, p. 40) given by

$$CWACVU(G) = \frac{\sum_{i=1}^{n} w(x_i, G) u(x_i, G)}{\sum_{i=1}^{n} w(x_i, G)}. \tag{15}$$

Configural value models occur when $w(x_i, G) = w(p_i)$ in Eq. (15) and are given by

$$CVU(G) = \frac{\sum_{i=1}^{n} w(p_i) u(x_i, G)}{\sum_{i=1}^{n} w(p_i)}. \tag{16}$$

The lottery-dependent utility of Becker and Sarin (1987) occurs when $w(p_i) = p_i$ in Eq. (16) and is given by

$$LDU(G) = \frac{\sum_{i=1}^{n} p_i u(x_i, G)}{\sum_{i=1}^{n} p_i}. \tag{17}$$

Weighted averaging models occur when $u(x_i, G) = u(x_i)$ in Eq. (15) and are given by

$$WAU(G) = \frac{\sum_{i=1}^{n} w(x_i, G) u(x_i)}{\sum_{i=1}^{n} w(x_i, G)}. \tag{18}$$

Differentially weighted averaging models occur when $w(x_i, G) = w(x_i, p_i)$ and $u(x_i, G) = u(x_i)$ in Eq. (18):

$$DWAU(G) = \frac{\sum_{i=1}^{n} w(x_i, G) u(x_i)}{\sum_{i=1}^{n} w(x_i, G)}. \tag{19}$$

The constant weighting or SWAU model of Eq. (10) occurs when $w(x_i, G) = w(x_i, p_i) = w(p_i)$ in (19).

2.9 Prospective reference theory utility (PRTU)

The Viscusi (1989) PRT is the sum of a weighted EU term and a weighted average of the outcome utilities and is given by

$$PRTU(G) = \gamma \sum_{i=1}^{n} p_i u(x_i) + (1 - \gamma) \sum_{i=1}^{n} \frac{u(x_i)}{n}. \tag{20}$$

PRT is a special case of RAM, when $a(i, n) = 1 \forall i, n$ and $\delta(p) = \gamma p + \frac{(1-\gamma)}{n}$. PRT is also a special case of TAX, when $\delta = 0$ and $t(p) = \gamma p + \frac{(1-\gamma)}{n}$. When $\gamma = 1$, PRT reduces to EU. In PRT, the probability weightings are unaffected by rank.

2.10 (Lower) gains decomposition utility (GDU)

Luce (2000, p. 202) devised a process to iteratively decompose a multi-branch gamble into a series of binary gambles (see also Marley and Luce (2001, 2005); Luce and Marley (2005)), and provided an induction formula to compute the utility of a gamble $G = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$ with n outcomes as

$$GDU(G) = \sum_{j=0}^{n-1} GDU(x_1, p_1; \ldots; x_{n-j}, p_{n-j}) \left[1 - w \left(\frac{\sum_{i=1}^{n-j-1} p_i}{\sum_{i=1}^{n-j} p_i} \right) \right] \prod_{i=0}^{j-1} w \left(\frac{\sum_{k=1}^{n-i-1} p_k}{\sum_{k=1}^{n-i} p_k} \right). \tag{21}$$

For binary gambles, this induction formula reduces to

$$GDU(G) = w(p_1) u(x_1) + w(1 - p_1) u(x_2), \tag{22}$$

which is equivalent to binary RDU (Marley and Luce, 2001, p. 143). RDU is a special case of GDU where the GDU weights take on a specific form.

As the **pt** package has an object-oriented design, it would be straightfoward to incorporate other risky decision making theories for comparison.

3 The R package pt

This package contains six classes: `Utility`, `ProbWeight`, `Outcome`, `Gamble`, `Gambles` and `Choices`. Only `Utility`, `ProbWeight` and `Choices` are directly modifiable. These classes have been designed to capture risky decision making structure. `Outcome`, `Gamble`, `Gambles` and `Choices` are arranged hierarchically. At the highest level sits the `Choices` class, which is a container for the `Gambles` class. A `Choices` object can be used to compare the results of different decision making scenarios. For instance one can use a `Choices` object to explore the Allais paradoxes. The paradoxes involve two separate situations, with each situation involving a decision between gambles. The `Gambles` class is a container for `Gamble` class objects. The `Gambles` class can be used as an input for calculating the utilities and certainty equivalents of gambles under different decision making theories. Each `Gamble` class object in turn contains a vector of `Outcome` class objects. Each `Outcome` class object stores information such as probabilities, objective outcomes and calculational variables for risky decision making theories.

The `Utility` and `ProbWeight` classes generate utility and probability weighting objects with different functional forms and parameterisations. These objects can be used to explore the parameter spaces of different risky decision making theories.

`Choices` class objects can be created dynamically or read from external text files. They can also be saved to external text files for refererence.

The package has extensive visualisation capabilities. `Choices` class objects can be visualised as decision nodes leading to one-stage decision trees.

Individual probability weighting functions and families of probability weighting functions can be plotted. Individual utility functions can also be plotted. Certainty equivalents and risk premiums can be plotted. Finally the probability simplex can be plotted along with indifference curves.

The decision making theories currently implemented are EV, EU, SWU, RDU, PT, SWAU, RAM, TAX, PRT and GDU.

Some key features of the package include the capability to input a practically unlimited number of choices, and also the capability to compute ranks of gamble outcomes. The computation of ranks has historically made it difficult for many people to work with PT, however these issues have been addressed in the **pt** package. At its most basic level, the **pt** package can be used to compute nothing more than the PT utilities for general gambles, alleviating a major problem for students of PT. Students can concentrate on the psychological concepts introduced by Kahneman and Tversky rather than being caught up in performing intricate mathematical calculations.

Finally the Birnbaum (1997) recipe for generating gambles violating stochastic dominance has been implemented.

4 Validation tests

Published literature was used to validate the **pt** package. The calculations in Birnbaum (1999), Birnbaum et al. (1999), Wakker (2003), Birnbaum (2005b), Birnbaum (2007), Birnbaum and Bahra (2007), Birnbaum (2008), Birnbaum and Schmidt (2008), Birnbaum (2010) and Wakker (2010) were used to validate the methods. These examples have been included as tests in the **pt** package. Calculations were also cross checked for consistency against the basic RAM, TAX and PT calculators available at Birnbaum's website (`http://psych.fullerton.edu/mbirnbaum/calculators/`), Köbberling's basic PT calculator at Wakker's website `http://people.few.eur.nl/wakker/miscella/calculate.cpt.kobb/index.htm` and Sewell's basic PT calculators available at `http://prospect-theory.behaviouralfinance.net/`.

5 The Allais paradoxes

To illustrate the usage of the **pt** package the Allais paradoxes will be presented. Prospect theory gained widespread attention as it was the first theory to explain the Allais paradoxes. EU cannot explain these paradoxes. These paradoxes are examples of critical tests (Birnbaum, 2011) as no functional forms or parameterisations need to be estimated to show internal inconsistencies in EU.

5.1 Common consequence paradox

First load the **pt** library. In the remainder of the paper all R code is highlighted with a background.

```
library("pt")
```

Consider problems 1 and 2 from Kahneman and Tversky (1979, p. 265-266), representing the common consequence paradox. This can be modelled using **pt** as follows.

```
choice_ids <- c(1, 1, 1, 1, 2, 2, 2, 2)
gamble_ids <- c(1, 1, 1, 2, 1, 1, 2, 2)
outcome_ids <- c(1, 2, 3, 1, 1, 2, 1, 2)
objective_consequences <- c(2500, 2400, 0, 2400,
        2500, 0, 2400, 0)
probability_strings <- c("0.33", "0.66", "0.01", "1.0",
        "0.33", "0.67", "0.34", "0.66")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##   cid gid oid   pr   oc
## 1   1   1   1 0.33 2500
## 2   1   1   2 0.66 2400
```

```
## 3    1    1    3 0.01     0
## 4    1    2    1  1.0 2400
## 5    2    1    1 0.33 2500
## 6    2    1    2 0.67     0
## 7    2    2    1 0.34 2400
## 8    2    2    2 0.66     0

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.05,
        circle_radius=0.025, y_split_gap=0.1, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=11,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=11, label=c("A","B","C", "D"),
        label_font_colour=c("orange","magenta","green","blue"),
        label_font_size=c(11,11,11,11),
        label_positions=list(c(0.26,0.85),c(0.26,0.55),
                c(0.26,0.4),c(0.26,0.1)))
```
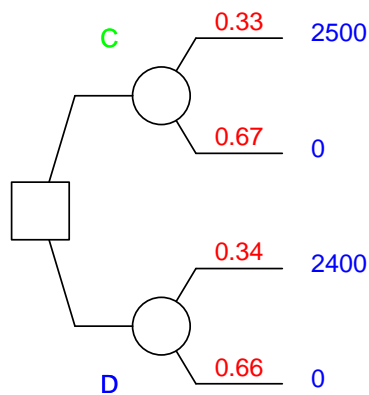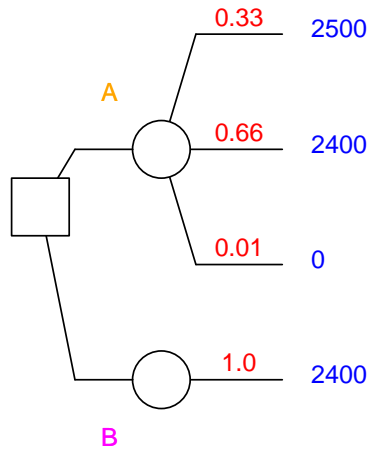

In a group of 72 subjects, 82% preferred $B > A$ and 83% preferred $C > D$, with 61% of the individuals making the majority (modal) choices across both conditions. Compare the EV for each gamble in the two choices.

```
compareEV(my_choices, digits=4)
```

```
##   cid gid   ev
## 1   1   1 2409
## 2   1   2 2400
## 3   2   1  825
## 4   2   2  816
```

Here `cid` represents the choice id (`cid` 1 represents the upper decision tree and `cid` 2 represents the lower decision tree), `gid` is the gamble id (in the upper decision tree `gid` 1 represents gamble A and `gid` 2 represents gamble B, while in the lower decision tree `gid` 1 represents gamble C and `gid` 2 represents gamble D), and `ev` is the expected value of each gamble. Gambles A and C have the highest EV, but empirically people are preferring the certainty of B to the uncertainty of A (the certainty effect). Compute the EU using a hypothetical concave utility function.

```
my_utility <- Utility(fun="power", par=c(alpha=0.63, beta=0.63, lambda=2.25))
compareEU(my_choices, utility=my_utility, digits=4)
```

```
##   cid gid   ev    eu    ce                  rp
## 1   1   1 2409 134.6  2395               14.34
## 2   1   2 2400 134.8  2400 0.0000000000009095
## 3   2   1  825 45.63 430.2               394.8
## 4   2   2  816 45.82   433                 383
```

EU predicts the preferences $B > A$ and $D > C$. Here `eu` is the expected utility of the gamble, `ce` is the certainty equivalent (quantifying the degree of risk aversion or seeking exhibited by a decision maker) and `rp` is the risk premium (the difference between the expected value and certainty equivalent). Intuitively, Yates (1990, p. 248) defines the risk premium as "the amount of expected value the decision maker is willing to "give up", perhaps to avoid the risk entailed by the prospect [gamble]". Risk averse decision makers have a positive risk premium, risk neutral decision makers have a risk premium of zero (they are indifferent to risk) and risk seeking decision makers have a negative risk premium. In EU risk aversion is modelled using concave utility functions, representing diminishing marginal utility. However, in other theories such as RDU, risk aversion can arise using convex utility functions and probability weightings (e.g. see Wakker (2010, p. 52-53, 71-3, 175-176)). With a hypothetical convex utility function, EU predicts the preferences $A > B$ and $C > D$.

```
my_utility <- Utility(fun="power", par=c(alpha=1.2, beta=1.2, lambda=2.25))
compareEU(my_choices, utility=my_utility, digits=4)
```

```
##   cid gid   ev    eu    ce     rp
## 1   1   1 2409 11458  2413 -4.129
## 2   1   2 2400 11383  2400      0
## 3   2   1  825  3945 992.4 -167.4
## 4   2   2  816  3870 976.7 -160.7
```

With a hypothetical linear utility function $u(x) = x$, EU reduces to EV.

```
my_utility <- Utility(fun="power", par=c(alpha=1.0, beta=1.0, lambda=1.0))
compareEU(my_choices, utility=my_utility, digits=4)
```

```
##   cid gid   ev   eu   ce rp
## 1   1   1 2409 2409 2409  0
## 2   1   2 2400 2400 2400  0
## 3   2   1  825  825  825  0
## 4   2   2  816  816  816  0
```

In choice 1 if decision makers are preferring $B > A$, this implies (applying the expected utility formula of Eq. (2)) that

$$
\begin{aligned}
EU(B) &> EU(A) \\
1.0 \times u(2400) &> 0.33 \times u(2500) + 0.66 \times u(2400) + 0.01 \times u(0) \\
0.34 \times u(2400) &> 0.33 \times u(2500) + 0.01 \times u(0),
\end{aligned}
\tag{23}
$$

subtracting $0.66 \times u(2400)$ from both sides of the inequality. In choice 2 if decision makers are preferring $C > D$, this implies that

$$EU(C) > EU(D)$$
$$0.33 \times u(2500) + 0.67 \times u(0) > 0.34 \times u(2400) + 0.66 \times u(0) \tag{24}$$
$$0.33 \times u(2500) + 0.01 \times u(0) > 0.34 \times u(2400),$$

subtracting $0.66 \times u(0)$ from both sides of the inequality. The inequalities are reversed across the two choice situations, no matter what functional form and parameterisation is used for the expected utility function u. Expected utility predicts that people will either choose $A > B$ and $C > D$ or choose $B > A$ and $D > C$ if the independence axiom holds. It does not predict the empirically observed pattern of choices. This pattern has been replicated in a number of studies (e.g. see MacCrimmon and Larsson, 1979). In contrast compute the PTU.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
linear_in_log_odds_prob_weight <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.61, beta=0.724))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=linear_in_log_odds_prob_weight,
        prob_weight_for_negative_outcomes=linear_in_log_odds_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid   ev    pt    ce                      rp
## 1   1   1 2409 881.3  2222                     187
## 2   1   2 2400 943.2  2400 -0.000000000001819
## 3   2   1  825 312.6 684.2                   140.8
## 4   2   2  816 307.2 670.9                   145.1
```

PT predicts the preferences $B > A$ and $C > D$, in line with the empirical data. Given there are three objective consequences in this example $\{0, 2400, 2500\}$, they can be displayed in the probability simplex (Marschak, 1950; Machina, 1987) as follows.

```
my_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))

drawSimplex(x1=0, x2=2400, x3=2500,
        line_dot_density=3000,
        draw_ev_flag=TRUE, ev_colour="black",
        draw_pt_flag=TRUE, alpha=0.61, beta=0.724, pt_colour="red",
        draw_utility_flag=TRUE, utility=my_utility, eu_colour="purple",
        start_points=list(c(0.1,0.9),c(0.2,0.8),c(0.3,0.7),c(0.4,0.6),
                c(0.5,0.5),c(0.6,0.4),c(0.7,0.3),c(0.8,0.2),c(0.9,0.1)),
        labels=c("A","B","C","D","increasing preference"),
        label_positions=list(c(0.05,0.38),c(0.05,0.05),c(0.7,0.38),
                c(0.7,0.05),c(0.7,0.7)),
        label_colours=c("orange","magenta","green","blue","red"),
        label_font_sizes=c(12,12,12,12,16),
        label_font_faces=c("plain","plain","plain","plain","bold"),
        label_rotations=c(0,0,0,0,-45),
        circle_radii=c(0.01,0.01,0.01,0.01),
        circle_outline_colours=c("black","black","black","black"),
        circle_fill_colours=c("orange","purple","orange","purple"),
        circle_positions=list(c(0.01,0.33),c(0,0),
                c(0.67,0.33),c(0.66,0)),
        lines=list(c(0.01,0.33,0,0),c(0,0,0.66,0),
                c(0.66,0,0.67,0.33),c(0.67,0.33,0.01,0.33)),
        line_widths=c(1, 1, 1, 1),
        line_styles=c("dashed", "dashed", "dashed", "dashed"),
        line_colours=c("blue","blue","blue","blue"),
        arrows=list(c(0.8,0.5,0.5,0.8)),
        arrow_widths=c(2), arrow_styles=c("solid"), arrow_colours=c("red"))
```
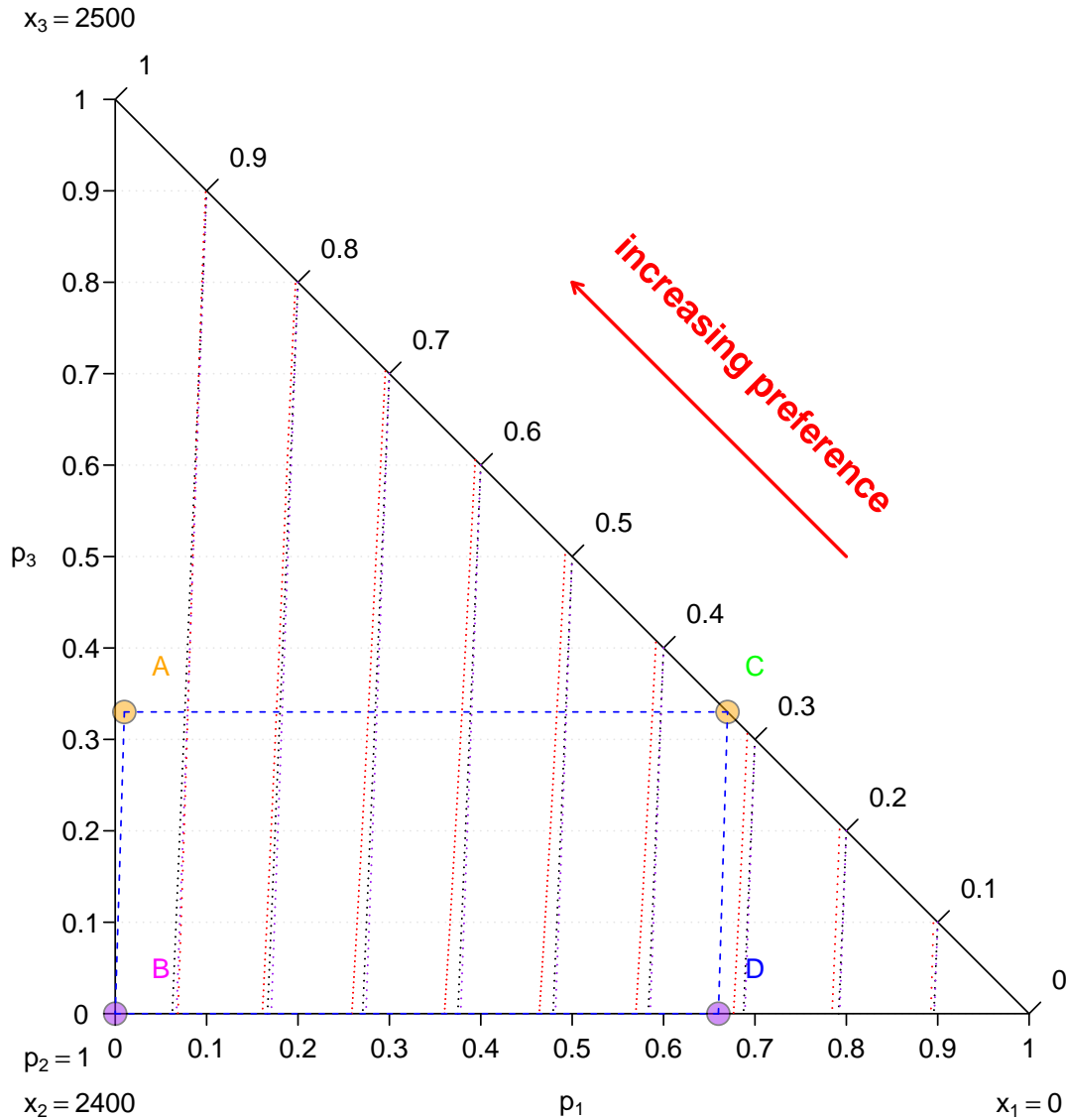
The iso-expected value lines are shown in black, the expected utility indifference curves (lines) in purple and the prospect theory indifference curves in red. The probability of x_2 is 1.0 at the bottom left hand corner. North-east movement along each iso-expected value line increases the probabilities of x_3 and x_1 relative to x_2. North-west movement increases the probability of the higher outcome x_3. The Allais paradoxes arise due to the assumption of linear probabilities in the EU formula Eq. (2). Linear probabilities generate indifference curves that are parallel lines in the triangle. Line slopes capture the degree of risk aversion/seeking. Risk neutrality for decision makers is represented by the iso-expected value lines. Risk averse expected utility decision makers have indifference lines that are steeper than the iso-expected value lines. Expected utility risk seekers have indifference lines that are flatter than the iso-expected value lines. In the first choice, decision makers exhibit risk aversion, opting for the sure gain, so the indifference line running through A is steep. In the second choice, decision makers lean more toward risk neutrality, opting for C, so the indifference line running through C is less steep than the line running through A. However in expected utility, lines with different steepnesses are impossible in the same triangle diagram. These paradoxes disappear when non-linear probabilities are allowed, as in theories such as PT, resulting in a "fanning out" of the indifference curves. This means that an indifference curve running through C can be flatter than one running through A.

Note that PT is not unique in predicting the empirically observed results. For instance TAX also predicts this pattern of decision making (using fewer parameters and a linear utility function).

```
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid   ev    tax    ce    rp
## 1   1   1 2409   1761  1761   648
## 2   1   2 2400   2400  2400     0
## 3   2   1  825  630.9 630.9 194.1
```
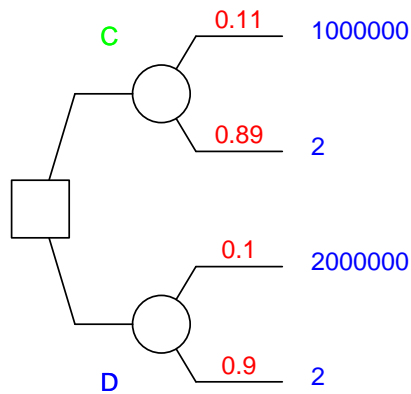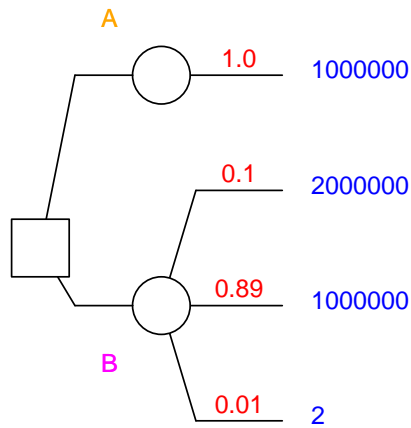
```
## 4    2    2   816 617.5 617.5 198.5
```

For illustrative purposes, the fanning out effect shows more clearly in the original formulation of the Allais common consequence paradox involving objective consequences of millions. Create choice problems 9.1 and 9.2 from Birnbaum (2008, Table 2 p. 480) to show the choices.

```r
choice_ids <- c(1, 1, 1, 1, 2, 2, 2, 2)
gamble_ids <- c(1, 2, 2, 2, 1, 1, 2, 2)
outcome_ids <- c(1, 1, 2, 3, 1, 2, 1, 2)
objective_consequences <- c(1000000, 2000000, 1000000, 2,
        1000000, 2, 2000000, 2)
probability_strings <- c("1.0", "0.1", "0.89", "0.01",
        "0.11", "0.89", "0.1", "0.9")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##   cid gid oid    pr        oc
## 1   1   1   1   1.0 1000000
## 2   1   2   1   0.1 2000000
## 3   1   2   2 0.89 1000000
## 4   1   2   3 0.01        2
## 5   2   1   1 0.11 1000000
## 6   2   1   2 0.89        2
## 7   2   2   1   0.1 2000000
## 8   2   2   2   0.9        2

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.05,
        circle_radius=0.025, y_split_gap=0.1, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=11,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=11, label=c("A","B","C", "D"),
        label_font_colour=c("orange","magenta","green","blue"),
        label_font_size=c(11,11,11,11),
        label_positions=list(c(0.26,0.95),c(0.26,0.65),
                c(0.26,0.4),c(0.26,0.1)))
```

A

1.0   1000000

0.1   2000000

0.89   1000000

B

0.01   2

C

0.11   1000000

0.89   2

0.1   2000000

0.9   2

D

Now show the simplex.

```
my_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))

drawSimplex(x1=2, x2=1000000, x3=2000000,
        line_dot_density=1000,
        draw_ev_flag=TRUE, ev_colour="black",
        draw_pt_flag=TRUE, alpha=0.61, beta=0.724, pt_colour="red",
        draw_utility_flag=TRUE, utility=my_utility, eu_colour="purple",
        start_points=list(c(0.1,0.9),c(0.2,0.8),c(0.3,0.7),c(0.4,0.6),
                c(0.5,0.5),c(0.6,0.4),c(0.7,0.3),c(0.8,0.2),c(0.9,0.1)),
        labels=c("A","B","C","D","increasing preference"),
        label_positions=list(c(0.05,0.02),c(0.05,0.12),c(0.92,0.02),
                c(0.95,0.10),c(0.7,0.7)),
        label_colours=c("orange","purple","orange","purple","red"),
        label_font_sizes=c(12,12,12,12,16),
        label_font_faces=c("plain","plain","plain","plain","bold"),
        label_rotations=c(0,0,0,0,-45),
        circle_radii=c(0.01,0.01,0.01,0.01),
        circle_outline_colours=c("black","black","black","black"),
        circle_fill_colours=c("orange","purple","orange","purple"),
        circle_positions=list(c(0,0),c(0.01,0.1),c(0.89,0),c(0.9,0.1)),
        lines=list(c(0,0,0.01,0.1),c(0.89,0,0.9,0.1),
                c(0.01,0.1,0.9,0.1),c(0,0,0.89,0)),
```

```
        line_widths=c(1, 1, 1, 1),
        line_styles=c("dashed", "dashed", "dashed", "dashed"),
        line_colours=c("blue","blue","blue","blue"),
        arrows=list(c(0.8,0.5,0.5,0.8)),
        arrow_widths=c(2), arrow_styles=c("solid"), arrow_colours=c("red"))
```
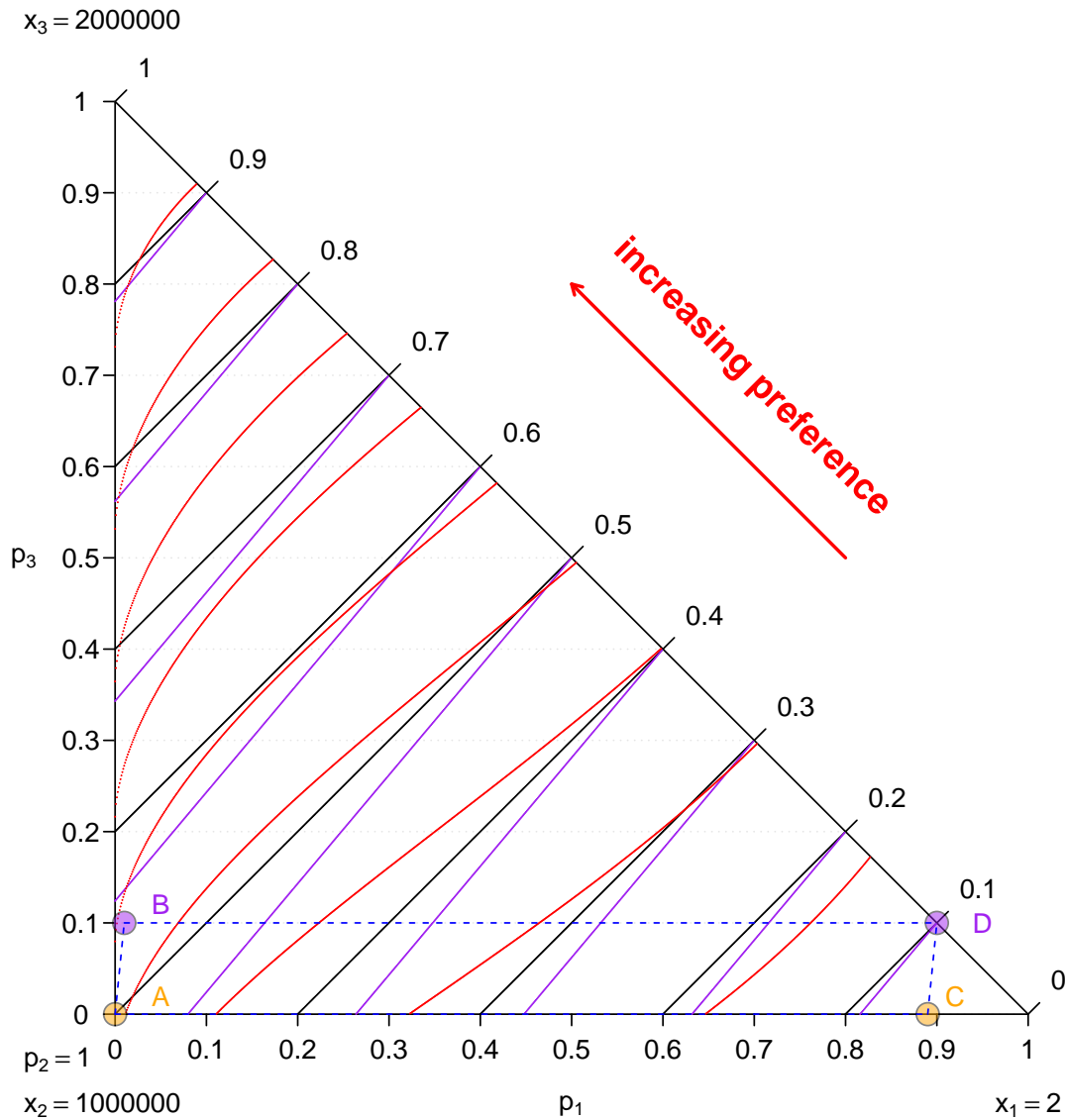

5.2 Common ratio paradox

Create choices problems 3 and 4 from Kahneman and Tversky (1979, p. 266) to illustrate the common ratio paradox.

```
choice_ids <- c(1, 1, 1, 2, 2, 2, 2)
gamble_ids <- c(1, 1, 2, 1, 1, 2, 2)
outcome_ids <- c(1, 2, 2, 1, 2, 1, 2)
objective_consequences <- c(4000, 0, 3000,
        4000, 0, 3000, 0)
probability_strings <- c("0.8", "0.2", "1.0",
        "0.2", "0.8", "0.25", "0.75")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices
```

```
##   cid gid oid   pr   oc
## 1   1   1   1  0.8 4000
## 2   1   1   1  0.2    0
## 3   1   2   2  1.0 3000
## 4   2   1   1  0.2 4000
## 5   2   1   2  0.8    0
## 6   2   2   1 0.25 3000
## 7   2   2   2 0.75    0

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.05,
        circle_radius=0.025, y_split_gap=0.1, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=11,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=11, label=c("A","B","C", "D"),
        label_font_colour=c("orange","magenta","green","blue"),
        label_font_size=c(11,11,11,11),
        label_positions=list(c(0.26,0.85),c(0.26,0.6),
                c(0.26,0.4),c(0.26,0.1)))
```
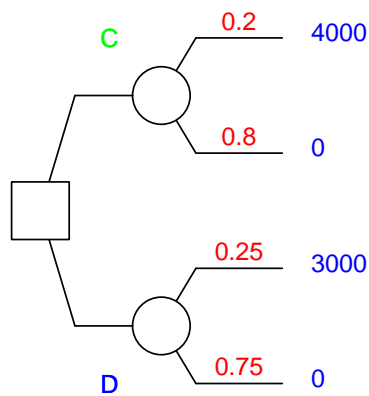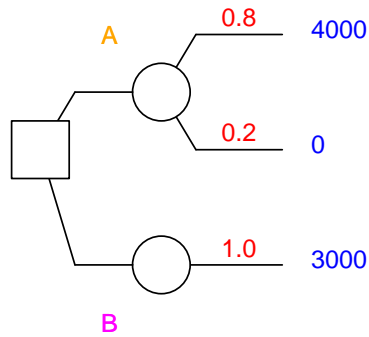

In a sample of 95 people, 80% preferred $B > A$ and 65% preferred $C > D$. EU predicts either the preferences $A > B$ and $C > D$ or the preferences $B > A$ and $D > C$. Following the reasoning of the previous common consequence example, if

decision makers are preferring $B > A$, this implies (again applying the expected utility formula of Eq. (2)) that

$$\begin{aligned}
EU(B) &> EU(A) \\
1.0 \times u(3000) &> 0.8 \times u(4000) + 0.2 \times u(0) \\
\frac{u(3000)}{u(4000)} &> 0.8 \times + \frac{0.2 \times u(0)}{u(4000)} \\
\frac{u(3000)}{u(4000)} &> 0.8,
\end{aligned} \tag{25}$$

dividing both sides of the inequality by $u(4000)$ and setting $u(0) = 0$. If decision makers are preferring $C > D$, this implies that

$$\begin{aligned}
EU(C) &> EU(D) \\
0.2 \times u(4000) + 0.8 \times u(0) &> 0.25 \times u(3000) + 0.75 \times u(0) \\
0.2 + \frac{0.8 \times u(0)}{u(4000)} &> \frac{0.25 \times u(3000)}{u(4000)} + \frac{0.75 \times u(0)}{u(4000)} \\
0.2 &> \frac{0.25 \times u(3000)}{u(4000)} \\
0.8 &> \frac{u(3000)}{u(4000)}.
\end{aligned} \tag{26}$$

The inequalities are reversed across the two choice situations. Expected utility predicts that people will either choose $A > B$ and $C > D$ or choose $B > A$ and $D > C$ if the independence axiom holds. It does not predict the empirically observed pattern. **pt** can be used to show the comparative theoretical predictions of all the decision making theories implemented in the package. Firstly show the EV.

```
compareEV(my_choices, digits=4)

##   cid gid   ev
## 1   1   1 3200
## 2   1   2 3000
## 3   2   1  800
## 4   2   2  750
```

Gambles A and C have the highest expected values. However, they are also the riskier choices. Compute the EU with a hypothetical concave utility function, and show the risk premium.

```
my_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=1))
eu_df <- compareEU(my_choices, utility=my_utility, digits=4)
eu_df

##   cid gid   ev   eu    ce                   rp
## 1   1   1 3200 1183  3104                95.91
## 2   1   2 3000 1148  3000 -0.000000000001819
## 3   2   1  800 295.7 642.4               157.6
## 4   2   2  750  287 620.8                129.2

ev <- as.numeric(eu_df$ev[1])
eu <- as.numeric(eu_df$eu[1])
ce <- as.numeric(eu_df$ce[1])
plotRP(my_title = "risk premium",
       my_title_colour="black", my_title_font_size=4,
       my_x_label = "objective consequence",
       my_y_label = "subjective value", xmin = 2500, xmax = 3500,
       my_color="violet",
       fun=power_uf,
       par=c(alpha=0.88, beta=0.88, lambda=1),
       ev=ev, eu=eu, ce=ce,
```
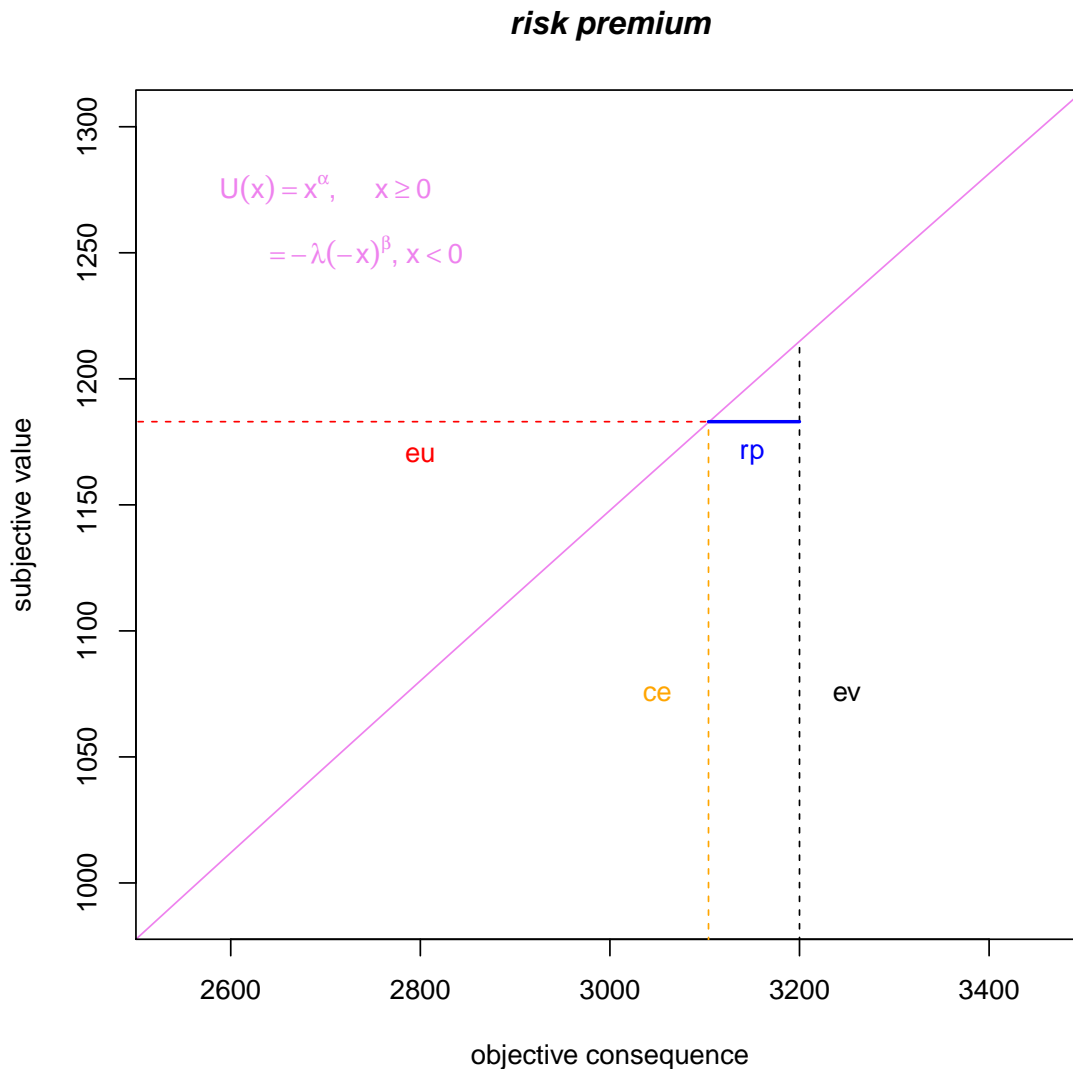
```
        my_labels=c(expression(paste(U(x)==x^alpha, ",      ", x>=0)),
                expression(paste(plain()==-lambda * (-x)^beta, ", ", x<0)),
                "ev","eu","ce","rp"),
        my_label_colors=c("violet","violet","black","red","orange","blue"),
        my_label_positions=list(c(2700,1275),c(2740,1250),c(3250,1075),
                c(2800,1170),c(3050,1075),c(3150,1170)),
        font_scaling=1)
```

risk premium

For this parameterisation $A > B$ and $C > D$. **pt** can be used to perform a sensitivity analysis to determine critical values for risk tolerances. The parameters for which the EU preferences are reversed so that $B > A$ and $D > C$ can be identified using the code below. Note that both these parameterisations use concave utility functions.

```
my_utility <- Utility(fun="power", par=c(alpha=0.78, beta=0.78, lambda=1))
compareEU(my_choices, utility=my_utility, digits=4)

##   cid gid   ev    eu    ce                  rp
## 1   1   1 3200 516.1  3005               195.2
## 2   1   2 3000 515.4  3000 0.000000000001364
## 3   2   1  800   129 508.1               291.9
## 4   2   2  750 128.9 507.3               242.7
```

```
my_utility <- Utility(fun="power", par=c(alpha=0.77, beta=0.78, lambda=1))
compareEU(my_choices, utility=my_utility, digits=4)

##   cid gid   ev    eu    ce    rp
## 1   1   1 3200   475  2994 206.3
## 2   1   2 3000 475.8  3000     0
## 3   2   1  800 118.7 494.7 305.3
## 4   2   2  750 118.9 495.7 254.3
```

Note the reversal in preferences. Using a linear utility function $u(x) = x$ gives the same result as EV.

```
my_utility <- Utility(fun="power", par=c(alpha=1, beta=1, lambda=1))
compareEU(my_choices, utility=my_utility, digits=4)

##   cid gid   ev   eu   ce rp
## 1   1   1 3200 3200 3200  0
## 2   1   2 3000 3000 3000  0
## 3   2   1  800  800  800  0
## 4   2   2  750  750  750  0
```

Compute the predictions for RDU.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.61))
compareRDU(my_choices, prob_weight=tk_1992_positive_prob_weight, utility=tk_1992_utility, digits=4)

##   cid gid   ev   rdu    ce                    rp
## 1   1   1 3200 898.1  2270                 929.9
## 2   1   2 3000  1148  3000 -0.000000000001819
## 3   2   1  800 385.5 868.4                -68.37
## 4   2   2  750 333.7   737                 12.99
```

In contrast, RDU predicts the preferences $B > A$ and $C > D$. PT reduces to RDU if all outcomes are strictly positive so its predictions are identical to RDU.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.61))
tk_1992_negative_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.69))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=tk_1992_positive_prob_weight,
        prob_weight_for_negative_outcomes=tk_1992_negative_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid   ev    pt    ce                    rp
## 1   1   1 3200 898.1  2270                 929.9
## 2   1   2 3000  1148  3000 -0.000000000001819
## 3   2   1  800 385.5 868.4                -68.37
## 4   2   2  750 333.7   737                 12.99
```

PT explains the common ratio paradox using a curved probability weighting function. This can be depicted using **pt**.

```
plotProbW(my_title=expression(paste("Tversky & Kahneman (1992), ",
        alpha==0.61)),
        my_title_colour="black", my_title_font_size=4,
        my_x_label = "p", my_y_label = "w(p)",
        pwf=kt_pwf, par=c(alpha=0.61),
        draw_reference_line_flag=TRUE, reference_line_colour="red",
        reference_line_style="dotted",
        my_labels=c(expression(paste(w(italic(p)) == frac(italic(p)^alpha,
                (italic(p)^alpha + (1-italic(p))^alpha)^(1/alpha))))),
        my_label_positions=list(c(0.4,0.8)),
        font_scaling=1.0)
```
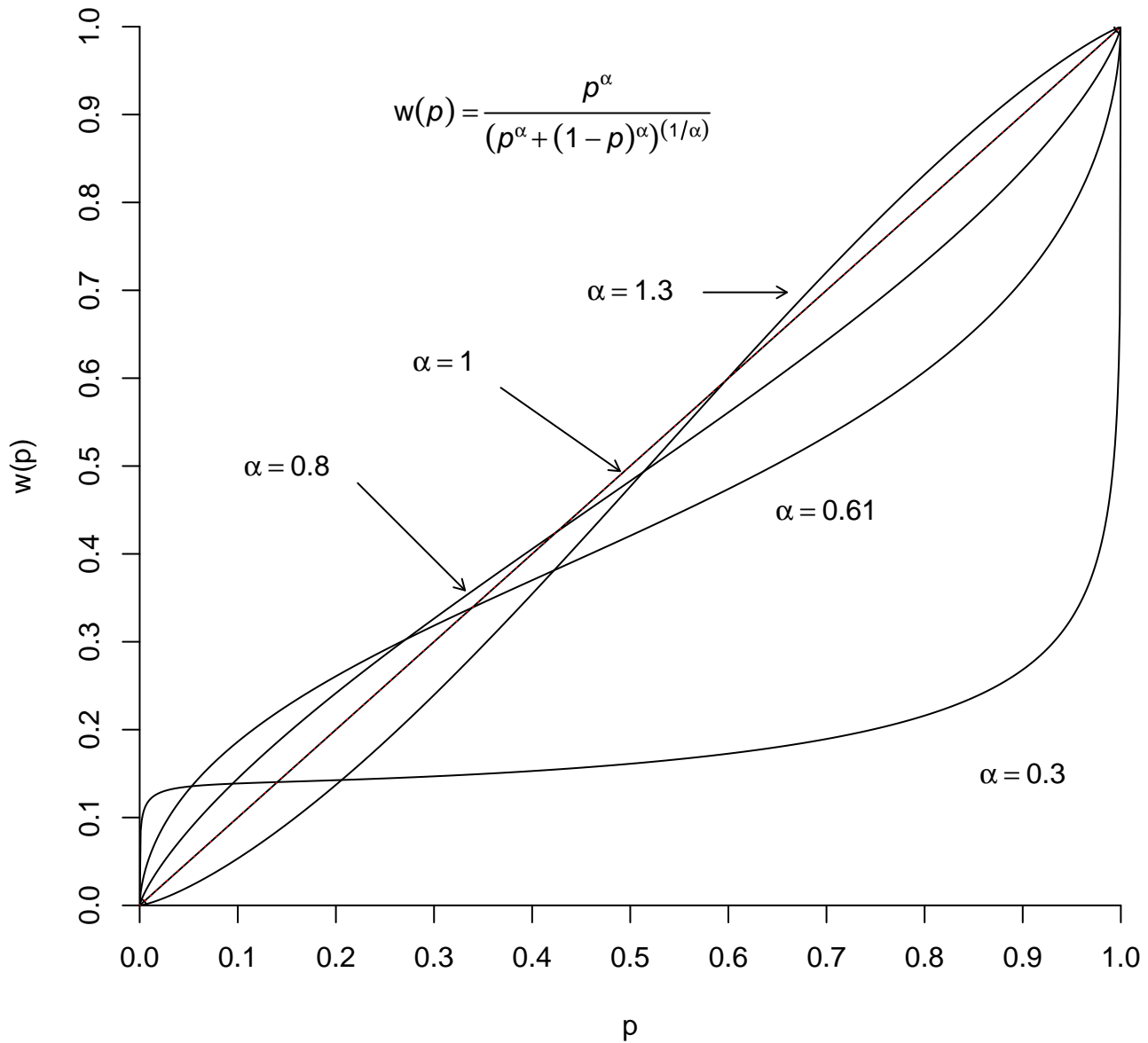
Tversky & Kahneman (1992), $\alpha = 0.61$

$$w(p) = \frac{p^\alpha}{(p^\alpha + (1-p)^\alpha)^{(1/\alpha)}}$$
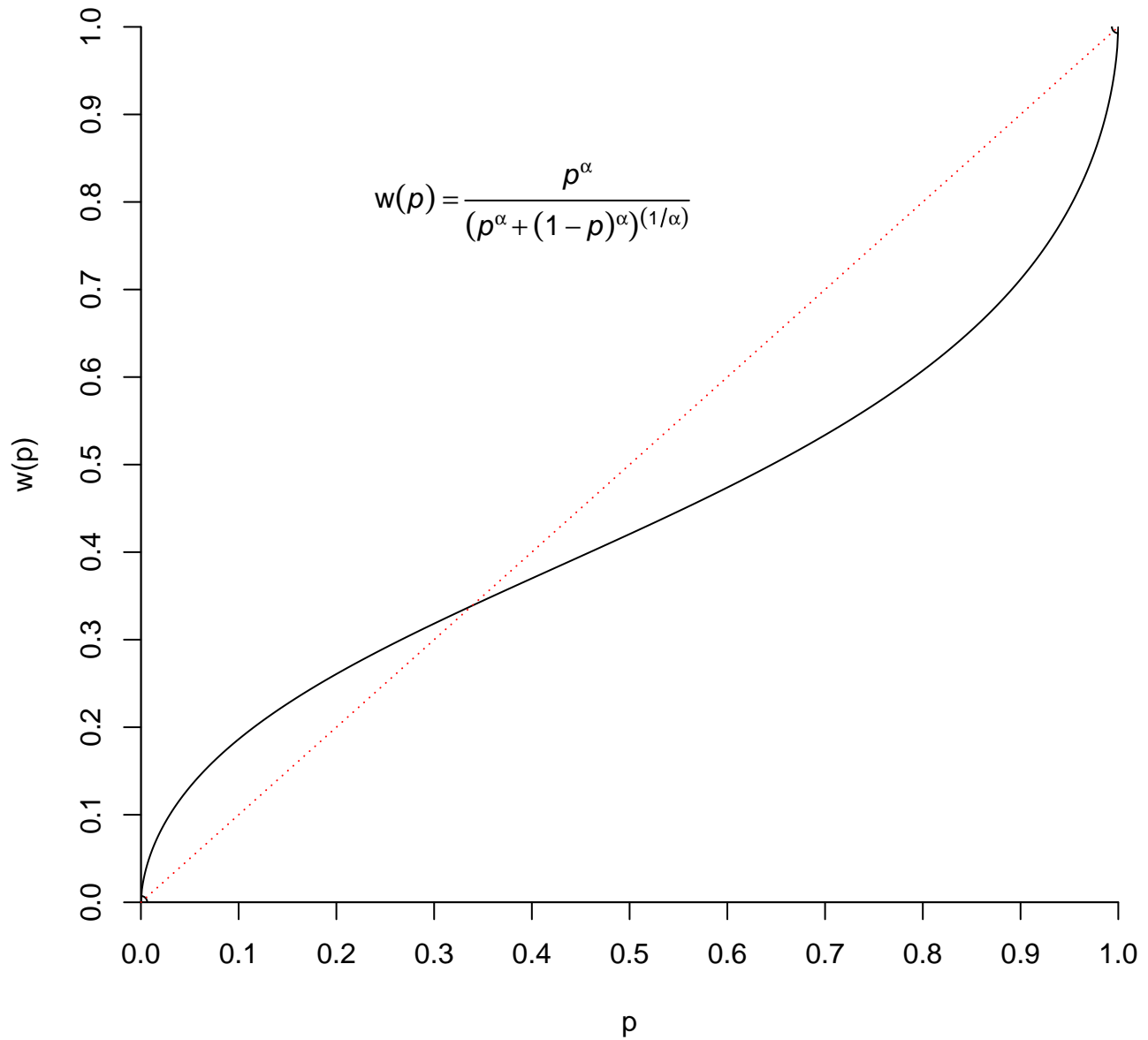
This function is only strictly increasing for $\alpha \geq 0.28$. A family of α values can be plotted (see e.g. (Wakker, 2010, p. 207 Figure 7.2.1)). The lower the α parameter, the more the decision maker exhibits likelihood insensitivity.

```
plotOneParProbWFam(my_title="Tversky & Kahneman (1992) family",
        my_title_colour="black", my_title_font_size=4,
        my_x_label = "p", my_y_label = "w(p)", pwf=kt_pwf,
        par=c(0.3, 0.61, 0.8, 1.0, 1.3),
        draw_reference_line_flag=TRUE, reference_line_colour="red",
        reference_line_style="dotted",
        my_labels=c(expression(paste(alpha == 0.3)),
        expression(paste(alpha == 0.61)),
        expression(paste(alpha == 0.8)),
        expression(paste(alpha == 1.0)),
        expression(paste(alpha == 1.3)),
        expression(paste(w(italic(p)) == frac(italic(p)^alpha,
        (italic(p)^alpha + (1-italic(p))^alpha)^(1/alpha))))),
        my_label_positions=list(c(0.9,0.15),c(0.7,0.45),c(0.15,0.5),
        c(0.31,0.62),c(0.5,0.7),c(0.42, 0.9)),
```

```
        font_scaling=1.0,
        arrow_positions = list(c(0.3,0.5,0.39,0.41),c(0.42,0.58,0.52,0.51),
                c(0.59,0.66,0.66,0.66)))
```

Tversky & Kahneman (1992) family

$$w(p) = \frac{p^\alpha}{(p^\alpha + (1-p)^\alpha)^{(1/\alpha)}}$$

$\alpha = 1.3$

$\alpha = 1$

$\alpha = 0.8$

$\alpha = 0.61$

$\alpha = 0.3$

Rather than use the single parameter probability weighting functional form of Tversky and Kahneman (1992), Gonzalez and Wu (1999) have proposed the use of a two-parameter "linear in log odds" probability weighting function to model probability discriminability and gamble attractiveness (Gonzalez and Wu, 1999, Figure 4 p. 140). The parameter γ controls curvature and δ controls elevation. **pt** can be used to show the effects of varying δ on the elevation. Lower values of δ decrease the elevation.
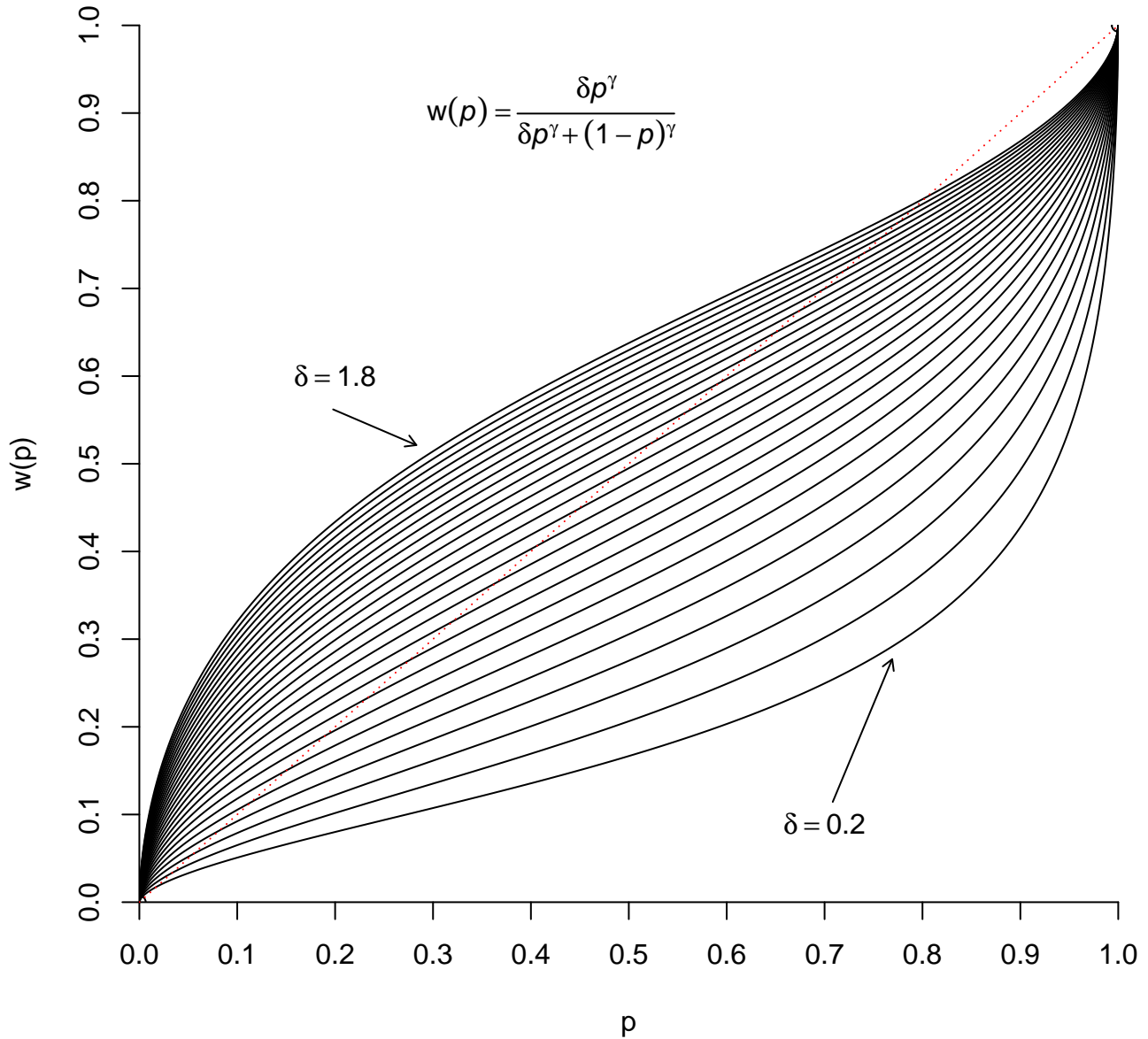
```
plotTwoParProbWFam(my_title=expression(paste("linear in log odds,  ",
        gamma == 0.6)),
        my_title_colour="black", my_title_font_size=4,
        my_x_label = "p", my_y_label = "w(p)", pwf=linear_in_log_odds_pwf,
        par=list(a_list=c(0.6), b_list=seq(from=0.2, to=1.8, by=0.06)),
        draw_reference_line_flag=TRUE, reference_line_colour="red",
        reference_line_style="dotted",
        my_labels=c(expression(paste(delta == 0.2)),
```

```
            expression(paste(delta == 1.8)),
            expression(paste(w(italic(p)) == frac(delta * italic(p)^gamma,
                  delta * italic(p)^gamma + (1-italic(p))^gamma)))),
     my_label_positions=list(c(0.7,0.09),c(0.2,0.6),c(0.42, 0.9)),
     font_scaling=1.0,
     arrow_positions = list(c(0.28,0.56,0.35,0.53),c(0.7,0.23,0.75,0.35)))
```

linear in log odds, $\gamma = 0.6$

$$w(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1-p)^{\gamma}}$$

$\delta = 1.8$

$\delta = 0.2$

Likewise, **pt** can be used to show the effects of varying γ on the curvature. Lower values of γ increase the curvature.

```
plotTwoParProbWFam(my_title=expression(paste("linear in log odds,  ",
     delta == 0.6)),
     my_title_colour="black", my_title_font_size=4,
     my_x_label = "p", my_y_label = "w(p)", pwf=linear_in_log_odds_pwf,
     par=list(a_list=seq(from=0.2, to=1.8, by=0.06), b_list=c(0.6)),
     draw_reference_line_flag=TRUE, reference_line_colour="red",
     reference_line_style="dotted",
     my_labels=c(expression(paste(gamma == 0.2)),
```
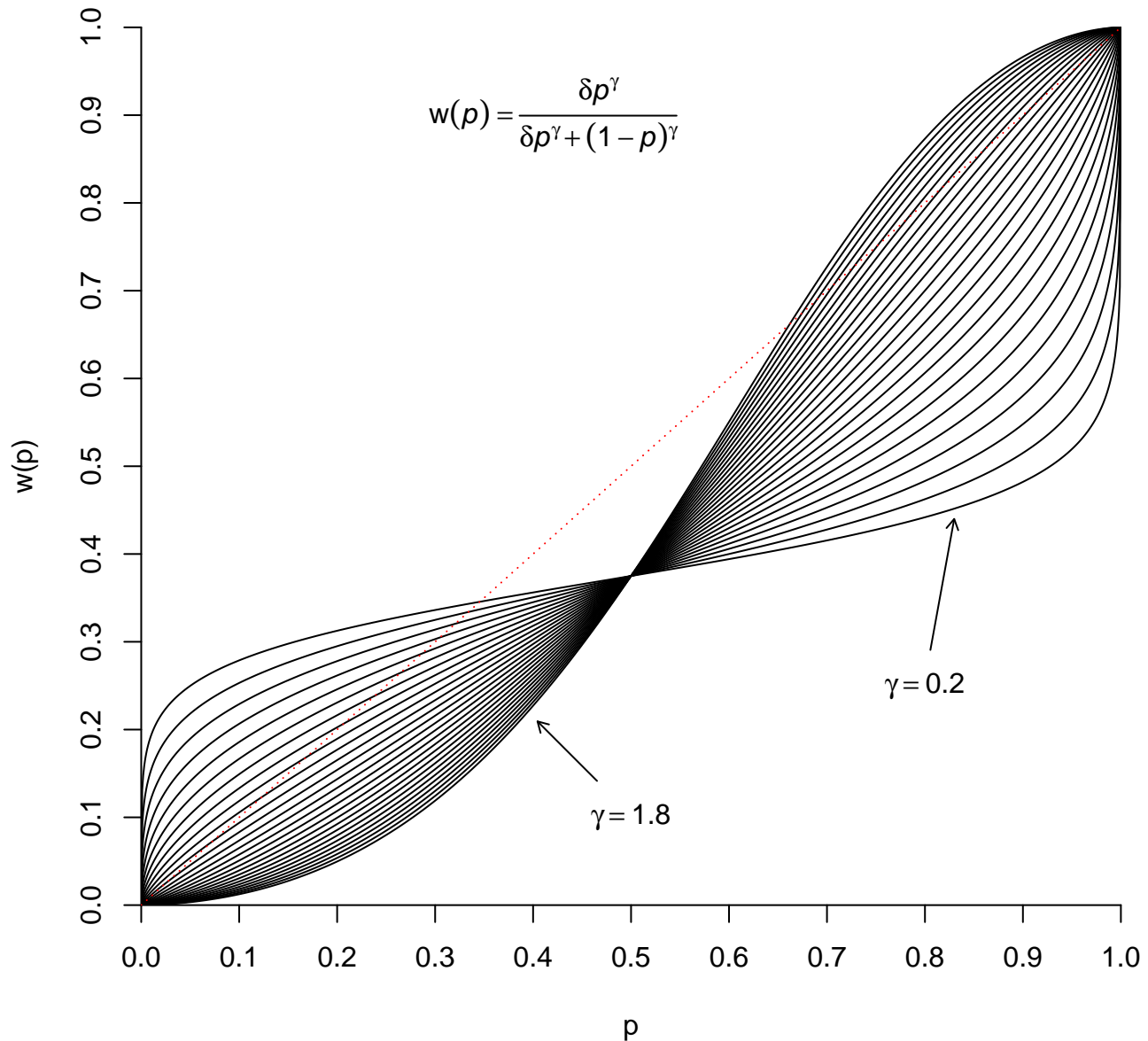
```
            expression(paste(gamma == 1.8)),
            expression(paste(w(italic(p)) == frac(delta * italic(p)^gamma,
                  delta * italic(p)^gamma + (1-italic(p))^gamma)))),
      my_label_positions=list(c(0.8,0.25),c(0.5,0.1),c(0.42, 0.9)),
      font_scaling=1.0, arrow_positions = list(c(0.5,0.25,0.45,0.3),
            c(0.78,0.36,0.8,0.47))))
```

linear in log odds, $\delta = 0.6$

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma}$$

$\gamma = 0.2$

$\gamma = 1.8$

Repeating the PT calculations using the Gonzalez and Wu (1999) probability weighting function gives a similar prediction to before.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
linear_in_log_odds_prob_weight <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.61, beta=0.724))
comparePT(my_choices,
      prob_weight_for_positive_outcomes=linear_in_log_odds_prob_weight,
      prob_weight_for_negative_outcomes=linear_in_log_odds_prob_weight,
      utility=tk_1992_utility, digits=4)

## cid gid  ev    pt    ce                  rp
```

```
## 1    1    1 3200 928.1   2357                  843.4
## 2    1    2 3000   1148   3000 -0.000000000001819
## 3    2    1  800 350.6 779.4                  20.58
## 4    2    2  750 310.2 678.4                   71.6
```

Next, SWU predicts the preferences $B > A$ and $C > D$.

```
my_utility <- Utility(fun="power", par=c(alpha=0.4, beta=0.4, lambda=1))
my_pwf <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.4, beta=0.4))
compareSWU(my_choices, prob_weight=my_pwf, utility=my_utility, digits=4)

##   cid gid   ev   swu    ce                   rp
## 1   1   1 3200 11.33 431.9                 2768
## 2   1   2 3000  24.6  3000 -0.000000000001819
## 3   2   1  800 5.155 60.34                739.7
## 4   2   2  750  5.04 57.04                  693
```

SWAU predicts the preferences $B > A$ and $C > D$.

```
my_utility <- Utility(fun="power", par=c(alpha=0.4, beta=0.4, lambda=1))
my_pwf <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.4, beta=0.4))
compareSWAU(my_choices, prob_weight=my_pwf, utility=my_utility, digits=4)

##   cid gid   ev  swau    ce                   rp
## 1   1   1 3200 18.96  1566                 1634
## 2   1   2 3000  24.6  3000 -0.000000000001819
## 3   2   1  800  8.63 218.8                581.2
## 4   2   2  750 8.573 215.2                534.8
```

RAM predicts the preferences $B > A$ and $C > D$.

```
branch_weight_list <- list(c(1), c(0.3738, 0.6262))
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareRAM(my_choices, branch_weight_list=branch_weight_list,
        prob_weight=power_prob_weight, utility=my_utility, digits=4)

##   cid gid   ev  ramu    ce    rp
## 1   1   1 3200  2447  2447 753.2
## 2   1   2 3000  3000  3000     0
## 3   2   1  800 737.9 737.9 62.12
## 4   2   2  750 650.1 650.1 99.89
```

GDU predicts the preferences $B > A$ and $C > D$.

```
my_pwf <- ProbWeight(fun="compound_invariance", par=c(alpha=0.542, beta=1.382))
my_utility <- Utility(fun="power", par=c(alpha=1, beta=1, lambda=1))
compareGDU(my_choices, prob_weight=my_pwf, utility=my_utility, digits=4)

##   cid gid   ev   gdu    ce    rp
## 1   1   1 3200  2167  2167  1033
## 2   1   2 3000  3000  3000     0
## 3   2   1  800 668.7 668.7 131.3
## 4   2   2  750 576.3 576.3 173.7
```

PRT predicts the preferences $B > A$ and $C > D$.

```
my_utility <- Utility(fun="power", par=c(alpha=0.631, beta=0.631, lambda=1))
comparePRT(my_choices, utility=my_utility, gamma=0.676, digits=4)

##   cid gid   ev  prtu    ce                   rp
```

```
## 1    1    1 3200 131.7   2287                    912.7
## 2    1    2 3000 156.3   3000 -0.0000000000004547
## 3    2    1  800 55.71  584.7                    215.3
## 4    2    2  750 51.75  520.2                    229.8
```

TAX predicts the preferences $B > A$ and $C > D$.

```
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid   ev   tax    ce    rp
## 1   1   1 3200  1934  1934  1266
## 2   1   2 3000  3000  3000     0
## 3   2   1  800 732.8 732.8  67.2
## 4   2   2  750 633.4 633.4 116.6
```

Note that this TAX example uses a linear probability weighting function. TAX explains the Allais paradoxes through the increased branch weighting given to the lowest branches of a choice. Given there are three objective consequences $\{0, 3000, 4000\}$, we can show the gambles in the probability simplex. The iso-expected value lines are shown in black, the expected utility indifference lines in purple and the prospect theory indifference curves in red.

```
my_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=1))

drawSimplex(x1=0, x2=3000, x3=4000,
        line_dot_density=1000,
        draw_ev_flag=TRUE, ev_colour="black",
        draw_pt_flag=TRUE, alpha=0.61, beta=0.724, pt_colour="red",
        draw_utility_flag=TRUE, utility=my_utility, eu_colour="purple",
        start_points=list(c(0.1,0.9),c(0.2,0.8),c(0.3,0.7),c(0.4,0.6),
                c(0.5,0.5),c(0.6,0.4),c(0.7,0.3),c(0.8,0.2),c(0.9,0.1)),
        labels=c("A","B","C","D","increasing preference"),
        label_positions=list(c(0.03,0.04),c(0.21,0.75),
                c(0.79,0.04),c(0.85,0.18),c(0.7,0.7)),
        label_colours=c("orange","purple","orange","purple","red"),
        label_font_sizes=c(12,12,12,12,16),
        label_font_faces=c("plain","plain","plain","plain","bold"),
        label_rotations=c(0,0,0,0,-45),
        circle_radii=c(0.01,0.01,0.01,0.01),
        circle_outline_colours=c("black","black","black","black"),
        circle_fill_colours=c("orange","purple","orange","purple"),
        circle_positions=list(c(0,0),c(0.2,0.8),c(0.75,0),c(0.8,0.2)),
        lines=list(c(0,0,0.2,0.8),c(0.2,0.8,0.8,0.2),
                c(0.8,0.2,0.75,0),c(0.75,0,0,0)),
        line_widths=c(1, 1, 1, 1),
        line_styles=c("dashed", "dashed", "dashed", "dashed"),
        line_colours=c("blue","blue","blue","blue"),
        arrows=list(c(0.8,0.5,0.5,0.8)),
        arrow_widths=c(2), arrow_styles=c("solid"), arrow_colours=c("red"))
```
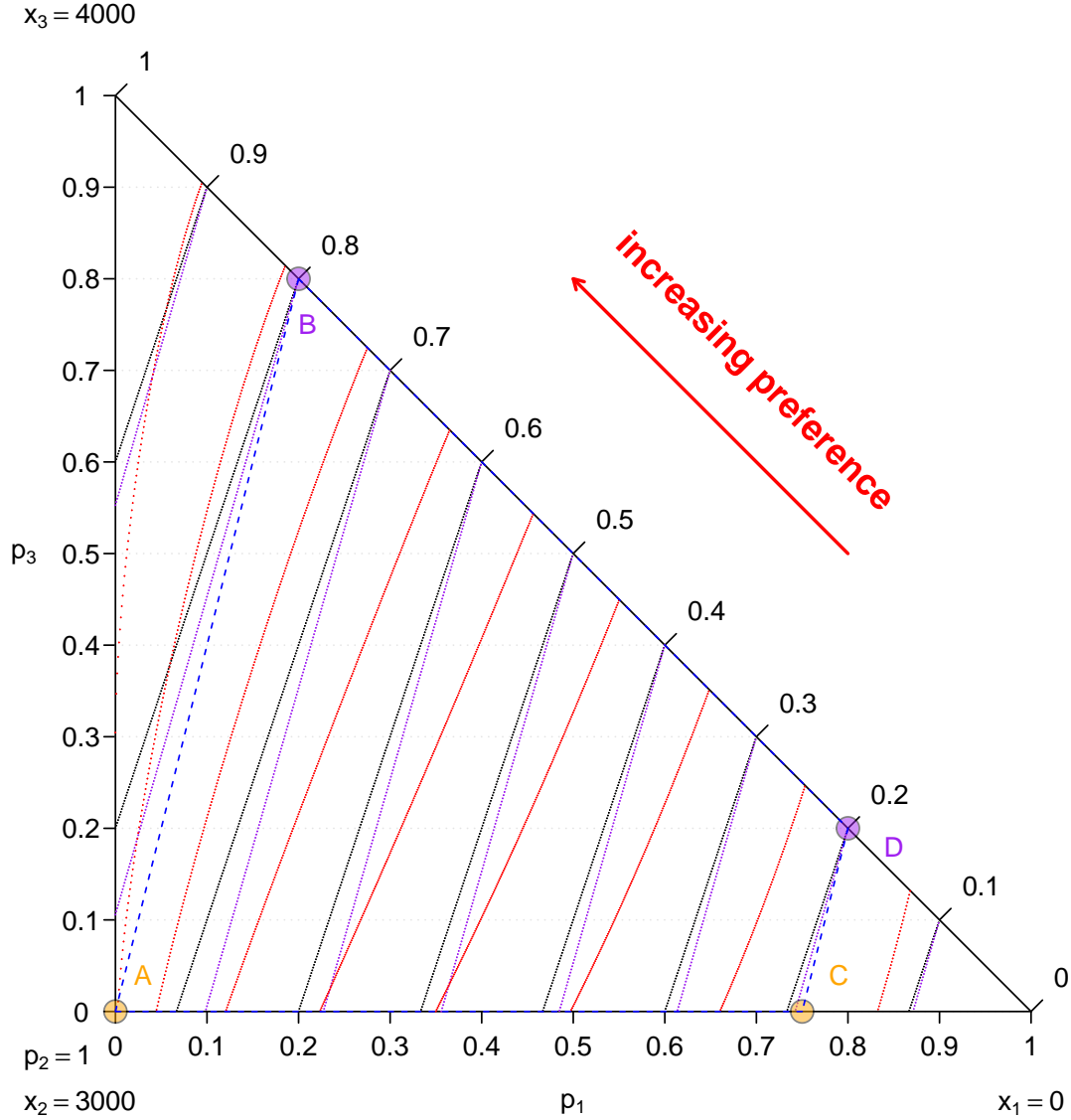
The introduction of a reference point allowed PT to generated novel predictions such as the four-fold pattern of risk attitudes and loss aversion. These successes will be described in the following sections.

6 Four-fold pattern of risk attitudes

One of the most distinctive predictions of prospect theory has been the four-fold pattern of risk attitudes within individual decision makers. Although EU could predict both risk aversion and risk seeking using different utility curves, it could not do this simultaneously in the same decision maker. The combination of reference points, curvature of the probability weighting function and kinked utility function allows PT to model this pattern. A key concept in PT is outcomes are regarded as losses or gains relative to a reference point defined via a framing event. The reference point is generally taken to be the "status quo". Put very roughly, people don't mind winning, but are averse to losing. To capture this effect, losses are weighted more than gains. Small probability outcomes are overestimated relative to the actual probability of the outcome happening. Middle and high probability outcomes are underweighted relative to the actual probabilities of these outcomes. Negative outcomes are given more weight (loss aversion). The weight of rare events is increased. The predictions are given in Table 1.

Table 1: Four-fold pattern of risk attitudes.

	medium to high probability event	low probability event
loss	risk seeking	risk averse
gain	risk averse	risk seeking

Consider problems 11, 12. 14 and 14' from Kahneman and Tversky (1979, p. 273, 281) which illustrate the four-fold pattern.

```r
choice_ids <- c(1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4)
gamble_ids <- c(1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2)
outcome_ids <- c(1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1)
objective_consequences <- c(1000, 0, 500, -0, -1000, -500,
        5000, 0, 5, -5000, -0, -5)
probability_strings <- c("1/2", "1/2", "1", "1/2", "1/2",
        "1", "0.001", "0.999", "1", "0.001", "0.999", "1")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##    cid gid oid    pr    oc
## 1    1   1   1   1/2  1000
## 2    1   1   2   1/2     0
## 3    1   2   1     1   500
## 4    2   1   1   1/2     0
## 5    2   1   2   1/2 -1000
## 6    2   2   1     1  -500
## 7    3   1   1 0.001  5000
## 8    3   1   2 0.999     0
## 9    3   2   1     1     5
## 10   4   1   1 0.001 -5000
## 11   4   1   2 0.999     0
## 12   4   2   1     1    -5

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.03,
        circle_radius=0.02, y_split_gap=0.07, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=10,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=10,
        label=c("A","B","C","D","E","F","G","H"),
        label_font_colour=c("orange","magenta","green","blue",
                "purple","pink","grey","violet"),
        label_font_size=c(11,11,11,11,11,11,11,11),
        label_positions=list(c(0.26,0.94),c(0.26,0.78),c(0.26,0.69),
                c(0.26,0.53),c(0.26,0.44),c(0.26,0.28),c(0.26,0.2),
                c(0.26,0.03)))
```
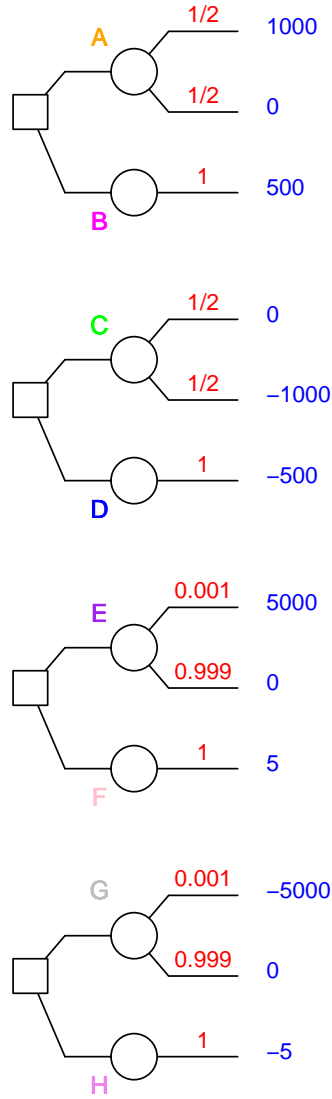
The PT predictions can be calculated as follows.

```r
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
linear_in_log_odds_prob_weight <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.61, beta=0.724))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=linear_in_log_odds_prob_weight,
        prob_weight_for_negative_outcomes=linear_in_log_odds_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid   ev     pt     ce                       rp
## 1   1   1  500  183.3  373.1                    126.9
## 2   1   2  500  237.2    500   -0.0000000000002842
## 3   2   1 -500 -412.5 -373.1                   -126.9
## 4   2   2 -500 -533.7   -500    0.0000000000002842
## 5   3   1    5  19.08  28.52                   -23.52
## 6   3   2    5  4.122      5 -0.00000000000000008882
## 7   4   1   -5 -42.92 -28.52                    23.52
## 8   4   2   -5 -9.274     -5  0.00000000000000008882
```

Kahneman and Tversky (1979) found empirical support for the PT predictions. These results are illustrated in Table 2.

The first two choices are exactly the same choices but are framed differently. A sample of 70 subjects were given 1000 in choice 1 and 68 different subjects were given 2000 in choice 2. They were then asked to make two choices. In choice 1 84% preferred $B > A$. In choice 2 69% preferred $C > D$. However, the end states are the same in both choices: 1500 for certain for options B and D and an equal chance gamble to gain 1000 or 2000 for options A and C. In the first choice

Table 2: Reflection effect patterns (from Kahneman and Tversky, 1979).

Problem 11. (p. 273)			
Risk averse for potential large gains under medium to high probability	(B > A)	84%	$N = 70$
Problem 12. (p. 273)			
Risk seeking to avoid potential large losses	(C > D)	69%	$N = 68$
Problem 14. (p. 281)			
Risk seeking for highly unlikely large gains (Gambling)	(E > F)	72%	$N = 72$
Problem 14'. (p. 281)			
Risk averse for highly unlikely large losses (Insurance)	(H > G)	83%	$N = 72$

(a gains frame), people exhibited risk aversion while in the second choice (a loss frame) they exhibited risk seeking. These patterns have been empirically supported by other researchers, however the patterns may be sensitive to different elicitation procedures. Harbaugh et al. (2010) found more complex choice elicitation procedures led to greater support for the four-fold pattern compared to simpler procedures, and suggest increased cognitive load may be responsible for the four-fold pattern. Prospect theory can also predict the (M4) Markowitz (1952) four-fold pattern of risk preferences for outcome magnitudes using decreasingly elastic utility functions (Scholten and Read, 2014) (see Table 3).

Table 3: (M4) Markowitz four-fold pattern for outcome magnitudes.

	large outcome	small outcome
loss	risk seeking	risk averse
gain	risk averse	risk seeking

To illustrate, consider the following choices from Hershey and Schoemaker (1980).

```
choice_ids <- c(1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4)
gamble_ids <- c(1, 2, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2)
outcome_ids <- c(1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2)
objective_consequences <- c(1, 0, 100, 10000, 0, 1000000,
        -1, 0, -100, -10000, 0, -1000000)
probability_strings <- c("1", "0.99", "0.01", "1", "0.99", "0.01",
        "1", "0.99", "0.01", "1", "0.99", "0.01")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##    cid gid oid   pr        oc
## 1    1   1   1    1         1
## 2    1   2   1 0.99         0
## 3    1   2   2 0.01       100
## 4    2   1   1    1     10000
## 5    2   2   1 0.99         0
## 6    2   2   2 0.01   1000000
## 7    3   1   1    1        -1
## 8    3   2   1 0.99         0
## 9    3   2   2 0.01      -100
## 10   4   1   1    1    -10000
## 11   4   2   1 0.99         0
## 12   4   2   2 0.01  -1000000

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.03,
        circle_radius=0.02, y_split_gap=0.07, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=10,
```
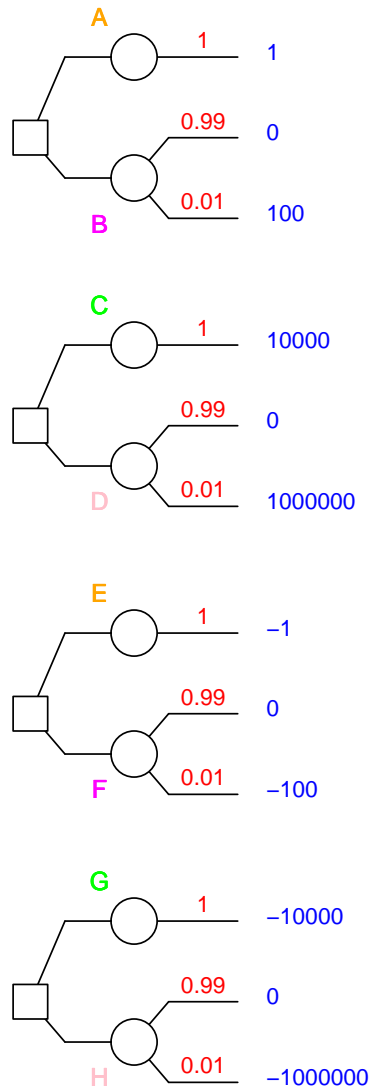
```
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=10, label=c("A","B","C","D","E","F","G","H"),
        label_font_colour=c("orange","magenta","green","pink","orange","magenta","green","pink"),
        label_font_size=c(11,11,11,11,11,11,11,11),
        label_positions=list(c(0.26,0.98),c(0.26,0.8),c(0.26,0.73),c(0.26,0.56),
                c(0.26,0.48),c(0.26,0.31),c(0.26,0.23),c(0.26,0.06)))
```


A sample of 36 people were asked to consider the first choice between gambles A and B, with 81% choosing the risky option B (see (Hershey and Schoemaker, 1980, p. 406 Table 3, Experiment 1, Question 13)). For small outcome gains, people were risk seeking. In the second choice situation between gambles C and D, 88% of 37 people chose C, the safe option. For large outcome gains, people were risk averse. In the third choice situation between gambles E and F, 63% of 36 people chose E, the safe option. For small outcome losses, people were risk averse. Finally, in the fourth choice situation between gambles G and H, 68% of 37 people chose H, the risky option. For large outcome losses, people were risk seeking. The PT predictions using parameterisations from Scholten and Read (2014) can be calculated as follows.

```
norm_log_utility <- Utility(fun="normalized_logarithmic", par=c(alpha=0.032, beta=0.0031, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.4496))
tk_1992_negative_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.6704))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=tk_1992_positive_prob_weight,
        prob_weight_for_negative_outcomes=tk_1992_negative_prob_weight,
        utility=norm_log_utility, digits=4)

##   cid gid     ev     pt     ce     rp
```

```
## 1    1    1       1 0.9843 11.86 -10.86
## 2    1    2       1  4.382 13.23 -12.23
## 3    2    1   10000  180.4  3690    6310
## 4    2    2   10000  31.67 31.68    9968
## 5    3    1      -1 -2.247     1      -2
## 6    3    2      -1 -8.447 3.776  -4.776
## 7    4    1  -10000  -2515 10000  -20000
## 8    4    2  -10000 -251.5 133.6  -10134
```

The predicted preferences are $B > A$, $C > D$, $E > F$ and $H > G$, in line with the empirical data.
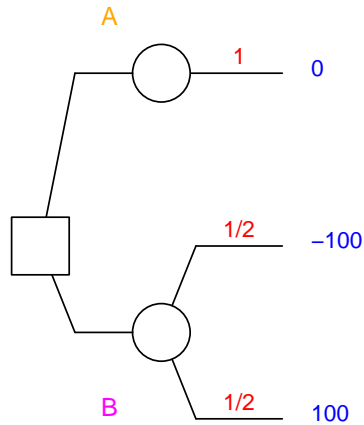
7   Loss aversion

Birnbaum (2008, p. 466) defines loss aversion as "the behavioral finding that people show risk aversion for mixed gambles". Prospect theory explains loss aversion through a kinked utility function centred around a reference point (which usually but not always represents the status quo). This construct underpins the famous quote from Kahneman and Tversky (1979, p. 279) that "losses loom larger than gains". It extends the EU utility function into the domain of losses and embodies the conceptual advance that decision makers consider changes in wealth states relative to the reference point of their current wealth (rather than only final wealth states), developing a line of thought from Markowitz (1952). Kahneman (2011, p. 269-277) offers an account of how this idea came about.

Consider the choice from Erev (2013, Table 5 p. 221) consisting of the option of maintaining the status quo or the option of taking a symmetrical gamble where the absolute value of the loss is the same as the gain. Assume the units of the gambles are of high nominal value and that real money is involved. **pt** can model this choice.

```
choice_ids <- c(1, 1, 1)
gamble_ids <- c(1, 2, 2)
outcome_ids <- c(1, 1, 2)
objective_consequences <- c(0, -100, 100)
probability_strings <- c("1", "1/2", "1/2")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##   cid gid oid  pr   oc
## 1   1   1   1   1    0
## 2   1   2   1 1/2 -100
## 3   1   2   2 1/2  100

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.05,
        circle_radius=0.025, y_split_gap=0.15, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=10,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=10, label=c("A","B"),
        label_font_colour=c("orange","magenta"), label_font_size=c(11,11),
        label_positions=list(c(0.26,0.7),c(0.26,0.36)))
```

In the Erev (2013) study, 88% of the 46 undergraduate students respondents in the study were risk averse and chose the status quo, supporting prospect theory's absolute loss aversion hypothesis. There is further empirical support for this phenomenon (see, e.g. Brooks and Zank, 2005; Birnbaum and Bahra, 2007; Ert and Erev, 2008; McGraw et al., 2010; Redelmeier and Tversky, 1992). However, a number of experimental manipulations such as introducing the presence of status quo framing effects seem to enhance its presence (Erev, 2013). Rabin (2000) provided a proof that with the assumption of a concave utility function, an expected utility maximizer would reject an extremely favourable $50-50$ gamble with large nominal value (e.g. $(-100, 0.5; 2.5 \times 10^9, 0.5)$) if they rejected a $50-50$ gamble with small nominal value (e.g. $(-100, 0.5; 110, 0.5)$). Rabin and Thaler (2001) suggests a combination of loss aversion and mental accounting explains risk averse behaviour for small nominal value gambles rather than expected utility. From an evolutionary psychology perspective, de Martino et al. (2010) suggest the amygdala regulates loss aversion.

Erev (2013) tabled the loss aversion coefficient λ across a number of studies such as Köbberling and Wakker (2005); Abdellaoui et al. (2007), and found it fell within in the range $[0.78, 2.61]$, with a strong positive correlation between the magnitude of λ and the absolute value of the objective consequences. Expressed very roughly, decision makers require gains to be at least twice the absolute value of the losses in an equal chance binary gamble before they will consider taking on the gamble (see Tversky and Kahneman, 1991, p. 1053-1054). Calculations for PT can be performed via the following code.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
linear_in_log_odds_prob_weight <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.61, beta=0.724))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=linear_in_log_odds_prob_weight,
        prob_weight_for_negative_outcomes=linear_in_log_odds_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid ev    pt    ce    rp
```

```
## 1   1   1 0      0      0      0
## 2   1   2 0 -30.21 -19.13 19.13
```

Prospect theory gives a negative value of -30.2 for gamble B, less than the 0 of the certain gamble A. Thus the riskless choice is preferred. Note that loss aversion can be explained by other theories such as TAX, without the use of a kinked utility function (Birnbaum, 2006). In the following example a linear utility function is used in the TAX calculation. In the internal calculations for TAX, branch weights of $2/3$ and $1/3$ are given to the -100 and 100 outcomes, respectively.

```
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid ev    tax     ce     rp
## 1   1   1 0      0      0      0
## 2   1   2 0 -33.33 -33.33 33.33
```

The value is -33.3 for gamble B, less than the 0 of gamble A. Thus the riskless choice is preferred as well.

8   Critical tests of prospect theory

PT assumes transitivity, coalescing, first-order stochastic dominance, restricted branch independence, upper tail independence, lower cumulative independence, upper cumulative independence and gain-loss separability (Birnbaum, 2008). However, empirical research has detected violations of some of these assumptions to various extents. Violations of gain-loss separability and coalescing will first be discussed as these seem fundamental issues. Violations of first-order stochastic dominance will also be discussed to demonstrate the implementation of the Birnbaum (1997, p. 74) stochastic dominance violation recipe in **pt**.

8.1   Violations of gain-loss separability

A PT assumption is the separability of gains and losses, implicit in the kinked shape of the utility function around a reference point. The PT utility function of Eq. (9) can be plotted as follows.

```
plotUtility(my_x_label = "objective consequence",
        my_y_label = "subjective value",
        xmin = -10, xmax = 10, fun=power_uf,
        par=c(alpha = 0.88, beta = 0.88, lambda = 2.25),
        fun_colour = "purple",
        draw_reference_line_flag = TRUE,
        reference_line_colour = "red",
        reference_line_style = 1)
```
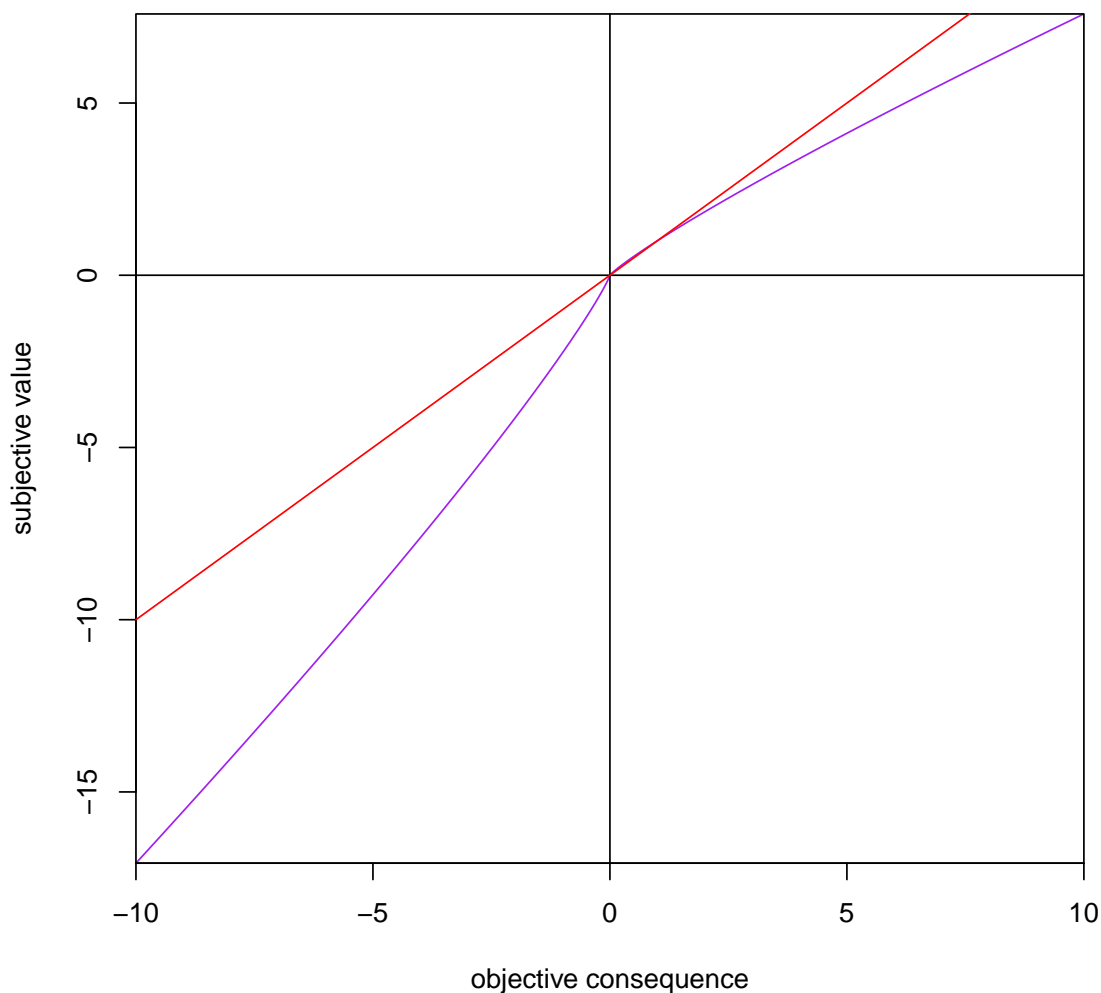
At present very little empirical research has been conducted on gain-loss separability. The limited research available has generated evidence where gain-loss separability is violated (Birnbaum and Bahra, 2007; Wu and Markle, 2008; Por and Budescu, 2013). Consider tests 15, 13 and 11 from Birnbaum and Bahra (2007, Table 4 p. 1021).

```
choice_ids <- c(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3)
gamble_ids <- c(1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2)
outcome_ids <- c(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3)
objective_consequences <- c(100, 0, 0, 50, 50, 0,
        -0, -50, -50, -0, -0, -100,
        100, 0, -50, 50, -0, -100)
probability_strings <- c("0.25", "0.25", "0.5", "0.25", "0.25", "0.5",
        "0.5", "0.25", "0.25", "0.5", "0.25", "0.25",
        "0.25", "0.25", "0.5", "0.5", "0.25", "0.25")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##    cid gid oid   pr   oc
## 1    1   1   1 0.25  100
## 2    1   1   2 0.25    0
## 3    1   1   3 0.5     0
```
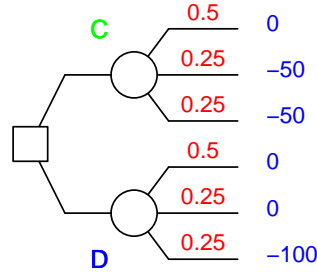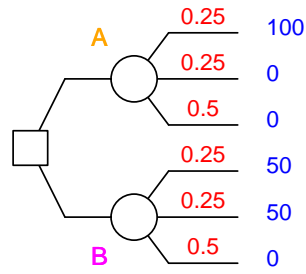
```
## 4    1   2   1 0.25   50
## 5    1   2   2 0.25   50
## 6    1   2   3  0.5    0
## 7    2   1   1  0.5    0
## 8    2   1   2 0.25  -50
## 9    2   1   3 0.25  -50
## 10   2   2   1  0.5    0
## 11   2   2   2 0.25    0
## 12   2   2   3 0.25 -100
## 13   3   1   1 0.25  100
## 14   3   1   2 0.25    0
## 15   3   1   3  0.5  -50
## 16   3   2   1  0.5   50
## 17   3   2   2 0.25    0
## 18   3   2   3 0.25 -100

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.03,
        circle_radius=0.02, y_split_gap=0.04, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=10,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=10,
        label=c("A","B","C", "D","E","F"),
        label_font_colour=c("orange","magenta","green",
                "blue","purple","pink"),
        label_font_size=c(11,11,11,11,11,11),
        label_positions=list(c(0.26,0.93),c(0.26,0.74),
                c(0.26,0.6),c(0.26,0.4),c(0.26,0.28),c(0.26,0.05)))
```

In a sample of 178 participants 71% preferred $B > A$, 65% preferred $D > C$ and 76% preferred $E > F$. The calculations for PT can be run as follows.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.61))
tk_1992_negative_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.69))
comparePT(my_choices,
          prob_weight_for_positive_outcomes=tk_1992_positive_prob_weight,
          prob_weight_for_negative_outcomes=tk_1992_negative_prob_weight,
          utility=tk_1992_utility, digits=4)

##   cid gid  ev     pt      ce      rp
## 1   1   1   25  16.73   24.57   0.433
## 2   1   2   25  13.15   18.69   6.311
## 3   2   1  -25 -31.94  -20.38  -4.618
## 4   2   2  -25    -38  -24.83  -0.1664
## 5   3   1    0 -15.21  -8.771   8.771
## 6   3   2    0 -24.85  -15.33   15.33
```

PT incorrectly predicts the preferences $A > B$ and $C > D$ but correctly predicts $E > F$. Por and Budescu (2013) suggest violations of gain-loss separability could limit the ability of PT to generalise from purely positive or negative domain gambles to mixed gambles.

In contrast, TAX explains loss aversion through the increased weighting provided to negative choice branches rather than through using a kinked utility function. Note the linear utility $u(x) = x$ and loss aversion coefficient $\lambda = 1$ used in the following TAX calculation.

```
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid  ev     tax      ce     rp
## 1   1   1  25   13.79   13.79  11.21
## 2   1   2  25   20.69   20.69  4.308
## 3   2   1 -25  -20.69  -20.69 -4.308
## 4   2   2 -25  -13.79  -13.79 -11.21
## 5   3   1   0  -15.51  -15.51  15.51
## 6   3   2   0  -34.49  -34.49  34.49
```

In this case TAX successfully predicts $B > A$, $D > C$ and $E > F$, in line with the empirical results. However, this is not true for other cases. Consider test 7 from Wu and Markle (2008, Table 1 p. 1326). These choices can be modelled as follows.
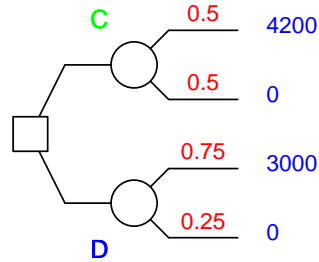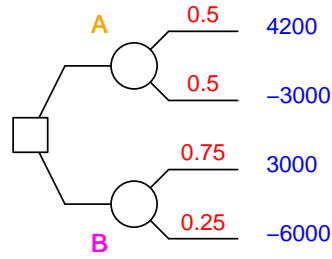
```
choice_ids <- c(1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3)
gamble_ids <- c(1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2)
outcome_ids <- c(1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2)
objective_consequences <- c(4200, -3000, 3000, -6000,
        4200, 0, 3000, 0,
        0, -3000, 0, -6000)
probability_strings <- c("0.5", "0.5", "0.75", "0.25",
        "0.5", "0.5", "0.75", "0.25",
        "0.5", "0.5", "0.75", "0.25")
my_choices <- Choices(choice_ids=choice_ids,
        gamble_ids=gamble_ids,
        outcome_ids=outcome_ids,
        objective_consequences=objective_consequences,
        probability_strings=probability_strings)
my_choices

##    cid gid oid   pr    oc
## 1    1   1   1  0.5  4200
## 2    1   1   2  0.5 -3000
## 3    1   2   1 0.75  3000
## 4    1   2   2 0.25 -6000
## 5    2   1   1  0.5  4200
## 6    2   1   2  0.5     0
## 7    2   2   1 0.75  3000
## 8    2   2   2 0.25     0
## 9    3   1   1  0.5     0
## 10   3   1   2  0.5 -3000
## 11   3   2   1 0.75     0
## 12   3   2   2 0.25 -6000

drawChoices(my_choices,
        decision_square_x=0.2, decision_square_edge_length=0.03,
        circle_radius=0.02, y_split_gap=0.06, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=10,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=10,
        label=c("A","B","C", "D","E","F"),
        label_font_colour=c("orange","magenta","green",
                "blue","purple","pink"),
        label_font_size=c(11,11,11,11,11,11),
        label_positions=list(c(0.26,0.93),c(0.26,0.74),
                c(0.26,0.6),c(0.26,0.4),c(0.26,0.28),c(0.26,0.05)))
```

A — 0.5 → 4200, 0.5 → −3000
B — 0.75 → 3000, 0.25 → −6000

C — 0.5 → 4200, 0.5 → 0
D — 0.75 → 3000, 0.25 → 0

E — 0.5 → 0, 0.5 → −3000
F — 0.75 → 0, 0.25 → −6000

With 81 participants, 51.9% preferred $A > B$, 85.2% preferred $D > C$ and 63% preferred $F > E$. The PT predictions can be generated via the following code.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.61))
tk_1992_negative_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.69))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=tk_1992_positive_prob_weight,
        prob_weight_for_negative_outcomes=tk_1992_negative_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid    ev     pt      ce      rp
## 1   1   1    600 -523.3 -488.9    1089
## 2   1   2    750 -742.8   -728    1478
## 3   2   1   2100  649.2   1570   530.1
## 4   2   2   2250  652.3   1578   671.6
## 5   3   1  -1500  -1172  -1223  -277.1
## 6   3   2  -1500  -1395  -1490 -9.982
```

With the parameterisations from Tversky and Kahneman (1992), PT correctly predicts the preferences $A > B$ and $D > C$, but incorrectly predicts $E > F$. The predictions for TAX are generated via the following code.

```
my_utility <- Utility(fun="linear", par=c(lambda=1))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
```

```
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid   ev   tax    ce    rp
## 1   1   1   600  -600  -600   1200
## 2   1   2   750 -1900 -1900   2650
## 3   2   1  2100  1400  1400    700
## 4   2   2  2250  1367  1367  883.4
## 5   3   1 -1500 -1000 -1000   -500
## 6   3   2 -1500 -1267 -1267 -233.2
```

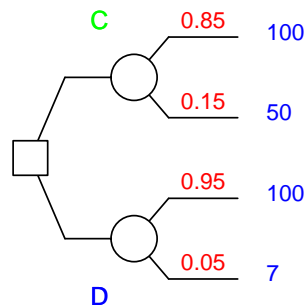TAX correctly predicts $A > B$ and $E > F$, but incorrectly predicts $C > D$.

8.2 Violations of coalescing

PT assumes gamble outcomes with the same objective consequences can be coalesced without affecting choice preferences. Birnbaum (2004), however, identified situations where this was not true for the majority of decision makers. Consider problems 14 and 8 from Birnbaum (2004, Table 3 p. 95) which are tests of coalescing.

```
choice_ids <- c(1, 1, 1, 1, 1, 1, 2, 2, 2, 2)
gamble_ids <- c(1, 1, 1, 2, 2, 2, 1, 1, 2, 2)
outcome_ids <- c(1, 2, 3, 1, 2, 3, 1, 2, 1, 2)
objective_consequences <- c(100, 50, 50,
       100, 100, 7,
       100, 50, 100, 7)
probability_strings <- c("0.85", "0.1", "0.05",
       "0.85", "0.1", "0.05",
       "0.85", "0.15", "0.95", "0.05")
my_choices <- Choices(choice_ids=choice_ids,
       gamble_ids=gamble_ids,
       outcome_ids=outcome_ids,
       objective_consequences=objective_consequences,
       probability_strings=probability_strings)
my_choices

##    cid gid oid   pr   oc
## 1    1   1   1 0.85 100
## 2    1   1   2  0.1  50
## 3    1   1   3 0.05  50
## 4    1   2   1 0.85 100
## 5    1   2   2  0.1 100
## 6    1   2   3 0.05   7
## 7    2   1   1 0.85 100
## 8    2   1   2 0.15  50
## 9    2   2   1 0.95 100
## 10   2   2   2 0.05   7

drawChoices(my_choices,
       decision_square_x=0.2, decision_square_edge_length=0.03,
       circle_radius=0.02, y_split_gap=0.07, x_split_offset=0.03,
       probability_text_digits=4, y_probability_text_offset=0.015,
       y_value_text_offset=0.005, x_value_text_offset=0.025,
       probability_text_font_colour="red", probability_text_font_size=10,
       objective_consequence_text_font_colour="blue",
       objective_consequence_text_font_size=10,
       label=c("A","B","C","D"),
       label_font_colour=c("orange","magenta","green","blue"),
       label_font_size=c(11,11,11,11),
       label_positions=list(c(0.26,0.9),c(0.26,0.6),
               c(0.26,0.37),c(0.26,0.13)))
```
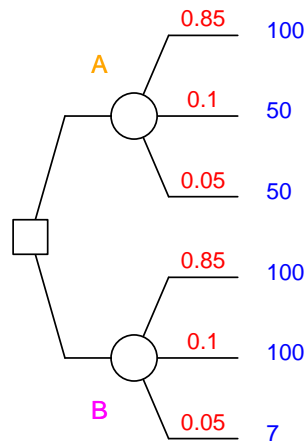
In a sample of 100 participants, 63% preferred $B > A$ and 75% preferred $C > D$ (Birnbaum, 2004, Table 3 p. 95 (unframed condition)). However, PT incorrectly predicts the preferences $A > B$ and $C > D$.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
tk_1992_positive_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.61))
tk_1992_negative_prob_weight <- ProbWeight(fun="Tversky_Kahneman_1992", par=c(alpha=0.69))
comparePT(my_choices,
        prob_weight_for_positive_outcomes=tk_1992_positive_prob_weight,
        prob_weight_for_negative_outcomes=tk_1992_negative_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid    ev    pt    ce    rp
## 1   1   1  92.5 48.44 82.23 10.27
## 2   1   2 95.35 46.79 79.05  16.3
## 3   2   1  92.5 48.44 82.23 10.27
## 4   2   2 95.35 46.79 79.05  16.3
```

For these choices TAX correctly predicts the preferences $B > A$ and $C > D$.

```
my_utility <- Utility(fun="linear", par=c(lambda=2.25))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_choices, prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid    ev   tax    ce    rp
```

```
## 1    1    1  92.5 68.37 68.37 24.13
## 2    1    2 95.35  69.7  69.7 25.65
## 3    2    1  92.5  75.7  75.7  16.8
## 4    2    2 95.35    62    62 33.35
```

8.3   Violations of first-order stochastic dominance

Birnbaum (2005a, p. 263-264) defines first-order stochastic dominance for two different gambles A and B with A stochastically dominating B as "the probability of winning prize x or greater given gamble A is greater than or equal to the probability to win x or more in gamble B, for all x, and if this probability is strictly higher for at least one value of x". Yates (1990, p. 221) provides an intuitive definition of dominance as follows: "One alternative is said to dominate another if it is just as good on all the pertinent aspect dimensions and better on at least one". Prospect theory assumes decision makers satisfy first-order stochastic dominance at all times, if it is detected. Birnbaum (1997, p. 74) created a recipe for gambles where first order stochastic dominance is systematically violated in choices between the gambles. These were empirically investigated in Birnbaum and Navarrete (1998). **pt** can be used to create choices using this recipe.

```
my_list <- vsdChoices(x=12, y=96, p="0.1", q="0.9", x_plus=14, y_minus=90, r="0.05")
my_list

## $g0
##   cid gid oid  pr oc
## 1   1   1   1 0.1 12
## 2   1   1   2 0.9 96
##
## $gplusminus
##   cid gid oid   pr oc
## 1   1   1   1 0.05 12
## 2   1   1   2 0.05 14
## 3   1   1   3  0.9 96
## 4   1   2   1  0.1 12
## 5   1   2   2 0.05 90
## 6   1   2   3 0.85 96
##
## $gsplusminus
##   cid gid oid   pr oc
## 1   1   1   1 0.05 12
## 2   1   1   2 0.05 14
## 3   1   1   3 0.05 96
## 4   1   1   4 0.85 96
## 5   1   2   1 0.05 12
## 6   1   2   2 0.05 12
## 7   1   2   3 0.05 90
## 8   1   2   4 0.85 96
```

```
drawChoices(my_list[[2]],
        decision_square_x=0.2, decision_square_edge_length=0.05,
        circle_radius=0.025, y_split_gap=0.1, x_split_offset=0.03,
        probability_text_digits=4, y_probability_text_offset=0.015,
        y_value_text_offset=0.005, x_value_text_offset=0.025,
        probability_text_font_colour="red", probability_text_font_size=11,
        objective_consequence_text_font_colour="blue",
        objective_consequence_text_font_size=11, label=c("A","B"),
        label_font_colour=c("orange","magenta"), label_font_size=c(11,11),
        label_positions=list(c(0.26,0.7),c(0.26,0.3)))
```
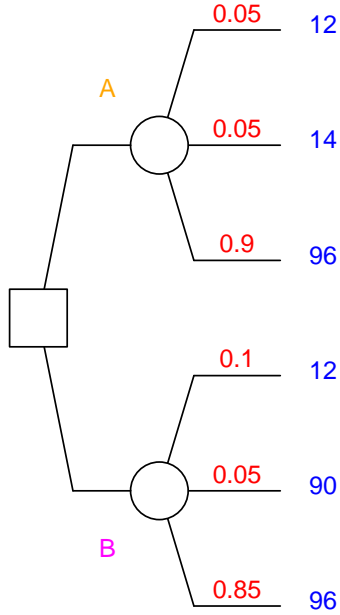
In this case gamble A stochastically dominates gamble B. This can be seen by calculating the probability of receiving an amount greater than or equal to each objective consequence of A and B, depicted in Table 4. For the amounts 96 and 14, A dominates B. For the amounts 90 and 12, A has probabilities at least the same as B.

Table 4: Gamble A stochastically dominates gamble B.

A			B	
oc	pr		oc	pr
96	0.9	>	96	0.85
90	0.9	same	90	0.9
14	0.95	>	14	0.9
12	1.0	same	12	1.0

However, Birnbaum and Navarrete (1998, Table 1 p. 61) found in 100 participants, 73% preferred $B > A$. For any set of functional forms and parameterisations, PT predicts $A > B$.

```
tk_1992_utility <- Utility(fun="power", par=c(alpha=0.88, beta=0.88, lambda=2.25))
linear_in_log_odds_prob_weight <- ProbWeight(fun="linear_in_log_odds", par=c(alpha=0.61, beta=0.724))
comparePT(my_list[[2]],
        prob_weight_for_positive_outcomes=linear_in_log_odds_prob_weight,
        prob_weight_for_negative_outcomes=linear_in_log_odds_prob_weight,
        utility=tk_1992_utility, digits=4)

##   cid gid   ev    pt    ce    rp
```

```
## 1   1   1 87.7 43.24 72.27 15.43
## 2   1   2 87.3 42.96 71.73 15.57
```

However TAX predicts $B > A$.

```
my_utility <- Utility(fun="linear", par=c(lambda=1.0))
power_prob_weight <- ProbWeight(fun="power", par=c(alpha=0.7, beta=1))
compareTAX(my_list[[2]], prob_weight=power_prob_weight, utility=my_utility, delta=-1, digits=4)

##   cid gid   ev    tax    ce    rp
## 1   1   1 87.7 45.77 45.77 41.93
## 2   1   2 87.3  63.1  63.1  24.2
```

Whereas under PT stochastic dominance is always obeyed, other theories such as TAX can predict violations of stochastic dominance under certain conditions with stochastic dominance being obeyed most of the time. Levy (2008) explains these violations as effects of bounded rationality. In his experiments the more transparent he made the first-order stochastic dominance, the fewer violations he detected.

9   Summary

As new risky decision making theories are developed, they can be incorporated into the R package **pt** for comparison with the currently implemented theories. This will allow a computational library to be developed to support decision making research. The package can be used to facilitate Gedanken experiments for developing new critical tests of decision making. The intention is to provide computational insights to develop theories of risky decision making that can more accurately account for empirical data than current theories, and to generate ideas for novel predictions.

Acknowledgments

References

Abdellaoui, M., Bleichrodt, H., and Paraschiv, C. (2007). Loss aversion under prospect theory: A parameter-free measurement. *Management Science*, 53(10):1659–1674.

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école Américaine. *Econometrica*, 21(4):503–546.

Allais, M. (1979). The foundations of a positive theory of choice involving risk and a criticism of the postulates and axioms of the American School. In Allais, M. and Hagen, O., editors, *Expected utility hypothesis and the Allais paradox*, pages 27–145. Reidel, Dordrecht: The Netherlands.

Becker, J. L. and Sarin, R. K. (1987). Lottery dependent utility. *Management Science*, 33(11):1367–1382.

Bernoulli, D. (1738/1954). Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36. Translation of Bernoulli, D. Specimen theoriae novae de mensura sortis. Commentarii Academiae Scientiarum Imperialis Petropolitanae, 5, 175-192 (1738).

Birnbaum, M. H. (1997). Violations of monotonicity in judgment and decision making. In Marley, A. A. J., editor, *Choice, decision, and measurement: Essays in honor of R. Duncan Luce*, pages 73–100. Erlbaum, Mahwah, NJ.

Birnbaum, M. H. (1999). Paradoxes of allais, stochastic dominance, and decision weights. In Shanteau, J., Mellers, B. A., and Schum, D. A., editors, *Decision science and technology: Reflections on the contributions of Ward Edwards*, pages 27–52. Kluwer Academic Publishers, Norwell, MA.

Birnbaum, M. H. (2004). Causes of Allais common consequence paradoxes: An experimental dissection. *Journal of Mathematical Psychology*, 48(2):87–106.

Birnbaum, M. H. (2005a). A comparison of five models that predict violations of first-order stochastic dominance in risky decision making. *Journal of Risk and Uncertainty*, 31(3):263–287.

Birnbaum, M. H. (2005b). Three new tests of independence that differentiate models of risky decision making. *Management Science*, 51(9):1346–1358.

Birnbaum, M. H. (2006). Evidence against prospect theories in gambles with positive, negative, and mixed consequences. *Journal of Economic Psychology*, 27(6):737–761.

Birnbaum, M. H. (2007). Tests of branch splitting and branch-splitting independence in Allais paradoxes with positive and mixed consequences. *Organizational Behavior and Human Decision Processes*, 102(2):154–173.

Birnbaum, M. H. (2008). New paradoxes of risky decision making. *Psychological Review*, 115(2):463–501.

Birnbaum, M. H. (2010). Testing lexicographic semiorders as models of decision making: Priority dominance, integration, interaction, and transitivity. *Journal of Mathematical Psychology*, 54(4):363–386.

Birnbaum, M. H. (2011). Testing theories of risky decision making via critical tests. *Frontiers in Psychology*, 2(315):1–3.

Birnbaum, M. H. and Bahra, J. P. (2007). Gain-loss separability and coalescing in risky decision making. *Management Science*, 53(6):1016–1028.

Birnbaum, M. H. and Navarrete, J. B. (1998). Testing descriptive utility theories: Violations of stochastic dominance and cumulative independence. *Journal of Risk and Uncertainty*, 17(1):49–79.

Birnbaum, M. H., Patton, J. N., and Lott, M. K. (1999). Evidence against rank-dependent utility theories: Tests of cumulative independence, interval independence, stochastic dominance, and transitivity. *Organizational Behavior and Human Decision Processes*, 77(1):44–83.

Birnbaum, M. H. and Schmidt, U. (2008). An experimental investigation of violations of transitivity in choice under uncertainty. *Journal of Risk and Uncertainty*, 37(1):77–91.

Brooks, P. and Zank, H. (2005). Loss averse behavior. *Journal of Risk and Uncertainty*, 31(3):301–325.

Colbert, G., Murray, D., and Nieschwietz, R. (2009). The use of expected value in pricing judgment. *Journal of Risk Research*, 12(2):199–208.

de Martino, B., Camerer, C. F., Adolphs, R., and Purves, D. (2010). Amygdala damage eliminates monetary loss aversion. *Proceedings of the National Academy of Sciences of the United States of America*, 107(8):3788–3792.

Edwards, W. (1954). The theory of decision making. *Psychological Bulletin*, 51(4):380–417.

Edwards, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review*, 69:109–135.

Erev, I. (2013). On the descriptive value of loss aversion in decisions under risk: Six clarifications. *Judgment and Decision Making*, 8(3):214–235.

Ert, E. and Erev, I. (2008). The rejection of attractive gambles, loss aversion, and the lemon avoidance heuristic. *Journal of Economic Psychology*, 29:715–723.

Gilboa, I. (1987). Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics*, 16(1):65–88.

Gonzalez, R. and Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology*, 38:129–166.

Harbaugh, W. T., Krause, K., and Vesterlund, L. (2010). The fourfold pattern of risk attitudes in choice and pricing tasks. *Economic Journal*, 120:595–611.

Hastie, R. (2001). Problems for judgment and decision making. *Annual Review of Psychology*, 52:653–683.

Hershey, J. C. and Schoemaker, P. J. H. (1980). Prospect theory's reflection hypothesis: A critical examination. *Organizational Behavior and Human Performance*, 25(3):395–418.

Kahneman, D. (2011). *Thinking, fast and slow*. Farrar, Straus and Giroux, New York, NY.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291.

Karmarkar, U. S. (1978). Subjectively weighted utility: A descriptive extension of the expected utility model. *Organizational Behavior and Human Performance*, 21(1):61–72.

Karmarkar, U. S. (1979). Subjectively weighted utility and the Allais Paradox. *Organizational Behavior and Human Performance*, 24(1):67–72.

Köbberling, V. and Wakker, P. P. (2005). An index of loss aversion. *Journal of Economic Theory*, 122:119–131.

Lattimore, P. K., Baker, J. R., and Witte, A. D. (1992). The influence of probability on risky choice: A parametric examination. *Journal of Economic Behavior and Organization*, 17(3):377–400.

Levy, H. (2008). First degree stochastic dominance violations: Decision weights and bounded rationality. *Economic Journal*, 118:759–774.

Li, S. (2003). The role of expected value illustrated in decision-making under risk: Single-play vs multiple-play. *Journal of Risk Research*, 6(2):113–124.

Lichtenstein, S., Slovic, P., and Zink, D. (1969). Effect of instruction in expected value on optimality of gambling decisions. *Journal of Experimental Psychology*, 79(2, Pt.1):236–240.

Luce, R. D. (2000). *Utility of gains and losses: Measurement-theoretical and experimental approaches*. Lawrence Erlbaum Associates, Mahwah, NJ.

Luce, R. D. and Marley, A. A. J. (2005). Ranked additive utility representations of gambles: Old and new axiomatizations. *Journal of Risk and Uncertainty*, 30(1):21–62.

MacCrimmon, K. R. and Larsson, S. (1979). Utility theory: Axioms versus "paradoxes". In Allais, M. and Hagen, O., editors, *Expected Utility Hypotheses and the Allais Paradox*, pages 333–409. Reidel, Dordrecht, The Netherlands.

Machina, M. J. (1987). Choice under uncertainty: Problems solved and unsolved. *Journal of Economic Perspectives*, 1(1):121–154.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy*, 60:151–158.

Marley, A. A. J. and Luce, R. D. (2001). Ranked-weighted utilities and qualitative convolution. *Journal of Risk and Uncertainty*, 23(2):135–163.

Marley, A. A. J. and Luce, R. D. (2005). Independence properties vis-à-vis several utility representations. *Theory and Decision*, 58(1):77–143.

Marschak, J. (1950). Rational behavior, uncertain prospects, and measurable utility. *Econometrica*, 18(2):111–141.

McGraw, A. P., Larsen, J. T., Kahneman, D., and Schkade, D. (2010). Comparing gains and losses. *Psychological Science*, 21(10):1438–1445.

Montgomery, H. and Adelbratt, T. (1982). Gambling decisions and information about expected value. *Organizational Behavior and Human Performance*, 29(1):39–57.

Por, H.-H. and Budescu, D. V. (2013). Revisiting the gain-loss separability assumption in prospect theory. *Journal of Behavioral Decision Making*, 26(4):385–396.

Post, T., van den Assem, M. J., Baltussen, G., and Thaler, R. H. (2008). Deal or no deal? decision making under risk in a large-payoff game show. *American Economic Review*, 98(1):38–71.

Preston, M. G. and Baratta, P. (1948). An experimental study of the auction-value of an uncertain outcome. *American Journal of Psychology*, 61:183–193.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3(4):324–343.

Quiggin, J. (1985). Subjective utility, anticipated utility, and the Allais paradox. *Organizational Behavior and Human Decision Processes*, 35(1):94–101.

Quiggin, J. (1993). *Generalized expected utility theory: The rank dependent model*. Kluwer, Boston, MA.

R Core Team (2013). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rabin, M. (2000). Risk aversion and expected-utility theory: A calibration theorem. *Econometrica*, 68(5):1281–1292.

Rabin, M. and Thaler, R. H. (2001). Anomalies: Risk aversion. *Journal of Economic Perspectives*, 15(1):219–232.

Redelmeier, D. A. and Tversky (1992). On the framing of multiple prospects. *Psychological Science*, 3(3):191–193.

Savage, L. J. (1954). *The foundations of statistics*. Wiley, New York, NY.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–587.

Scholten, M. and Read, D. (2014). Prospect theory and the "forgotten" fourfold pattern of risk preferences. *Journal of Risk and Uncertainty*, DOI 10.1007/s11166-014-9183-2.

Tversky, A. and Kahneman, D. (1991). Loss aversion in riskless choice: A reference-dependent model. *Quarterly Journal of Economics*, 106(4):1039–1061.

Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323.

Viscusi, W. K. (1989). Prospective reference theory: Toward an explanation of the paradoxes. *Journal of Risk and Uncertainty*, 2(3):235–263.

von Neumann, J. and Morgenstern, O. (1947). *Theory of games and economic behavior*. Princeton University Press, Princeton, NJ, second edition.

Wakker, P. P. (2003). The data of Levy and Levy (2002) "Prospect theory: Much ado about nothing?" actually support prospect theory. *Management Science*, 49(7):979–981.

Wakker, P. P. (2008). Explaining the characteristics of the power (CRRA) utility family. *Health Economics*, 17(12):1329–1344.

Wakker, P. P. (2010). *Prospect theory: For risk and ambiguity*. Cambridge University Press, Cambridge, UK.

Wu, G. and Markle, A. B. (2008). An empirical test of gain-loss separability in prospect theory. *Management Science*, 54(7):1322–1335.

Yates, J. F. (1990). *Judgment and decision making*. Prentice Hall, Englewood Cliffs, NJ.