Ch 7.8 Positronium lifetime

composed by quark-antiquark

For example, $\pi^0 \rightarrow \gamma + \gamma$ is electromagnetic decay.
in this process $\underset{(-1)^{l+s}}{}$ $q + \bar{q} \rightarrow \gamma + \gamma$, $\underset{(-1)\times(-1)}{}$ $q\bar{q}$ annihilation.

$\Rightarrow \pi^0 \rightarrow \gamma + \gamma + \gamma$ is forbbiden by "C" conservation

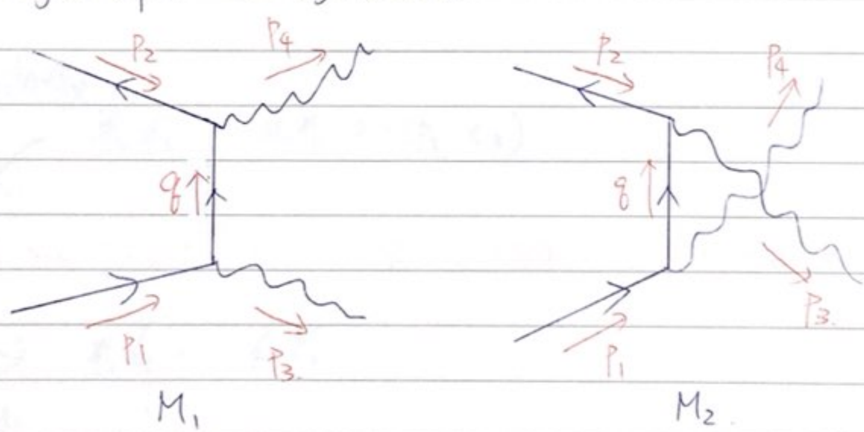
The cleaner example is positronium : $e^+ + e^- \rightarrow \gamma + \gamma$

e^+ e^- form bound state,
moving rather slowly (non-relativistic, $v \ll c$)
for calculating the amplitude we shall assume they are at rest

This is one of the cases we cannot average over initial spins, because the composite system is either in the singlet $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ $S=0$
or in triplet $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ $S=1$

charge conjugate
$C = (-1)^{l+s}$ for $q\bar{q}$, $C = -1$ for γ

EX 7.8. $\underset{p_2}{e^+} + \underset{p_1}{e^-} \rightarrow \underset{p_3}{\gamma} + \underset{p_4}{\gamma}$, assuming e^+ & e^- are at rest, and in the singlet spin configuration



$$M_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2} \bar{v}(2) \not{\epsilon}_4 (\not{p}_1 - \not{p}_3 + m) \not{\epsilon}_3 u(1)$$

$$M_2 = \frac{g_e^2}{(p_1 - p_4)^2 - m^2} \bar{v}(2) \not{\epsilon}_3 (\not{p}_1 - \not{p}_4 + m) \not{\epsilon}_4 u(1)$$

$$M_{tot} = M_1 + M_2$$

With the initial particles at rest, photons come out back-to-back, and we choose z axis to coincide of 1st photon:

$$p_1 = (m,0,0,0), \quad p_2 = (m,0,0,0)$$
$$p_3 = (m,0,0,m), \quad p_4 = (m,0,0,-m)$$

$$\Rightarrow (p_1-p_3)^2 - m^2 = (p_1-p_4)^2 - m^2 = -2m^2$$

$$\not{p}_1 \not{\epsilon}_3 = -\not{\epsilon}_3 \not{p}_1 + 2(p_1 \cdot \epsilon_3)$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

ϵ_3 has only spatial components (Coulomb gauge)

$$\Rightarrow p_1 \cdot \epsilon_3 = 0$$

$$\Rightarrow \not{p}_1 \not{\epsilon}_3 = -\not{\epsilon}_3 \not{p}_1$$

Similarly

$$\not{p}_3 \not{\epsilon}_3 = -\not{\epsilon}_3 \not{p}_3 + 2(p_3 \cdot \epsilon_3)$$

$p_3 \cdot \epsilon_3 = p_{3\mu} \epsilon_3^\mu = 0$ by virtue of Lorentz condition.

$$\not{p}_3 \not{\epsilon}_3 = -\not{\epsilon}_3 \not{p}_3$$

In M_1

$$(\not{p}_1 - \not{p}_3 + m)\not{\epsilon}_3 u(1) = \not{\epsilon}_3(-\not{p}_1 + \not{p}_3 + m)\, u(1) = \not{\epsilon}_3 \not{p}_3 u(1)$$

$(\not{p}_1 - m)\, u(1) = 0$
Dirac eq.

Similary

$$(\not{p}_1 - \not{p}_4 + m)\not{\epsilon}_4 u(1) = \not{\epsilon}_4 \not{p}_4 u(1)$$

Putting all together

$$\mathcal{M}_{tot} = -\frac{g_e^2}{2m^2}\,\bar{v}(2)\left[\underline{\not\epsilon_4 \not\epsilon_3 \not p_3 + \not\epsilon_3 \not\epsilon_4 \not p_4}\right] u(1).$$

Now $\not p_3 = m(\gamma^0 - \gamma^3)$, $\not p_4 = m(\gamma^0 + \gamma^3)$

$$m\left[(\not\epsilon_4 \not\epsilon_3 + \not\epsilon_3 \not\epsilon_4)\gamma^0 - (\not\epsilon_4 \not\epsilon_3 - \not\epsilon_3 \not\epsilon_4)\gamma^3\right]$$

$$\not\epsilon = -\vec{\epsilon}\cdot\vec{\gamma} = -\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\epsilon} \\ -\vec{\sigma}\cdot\vec{\epsilon} & 0 \end{pmatrix}.$$

$\uparrow$ ϵ is spatial

$$\Rightarrow \not\epsilon_3 \not\epsilon_4 = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\epsilon}_3 \\ -\vec{\sigma}\cdot\vec{\epsilon}_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\epsilon}_4 \\ -\vec{\sigma}\cdot\vec{\epsilon}_4 & 0 \end{pmatrix} = -\begin{pmatrix} (\vec{\sigma}\cdot\vec{\epsilon}_3)(\vec{\sigma}\cdot\vec{\epsilon}_4) & 0 \\ 0 & (\vec{\sigma}\cdot\vec{\epsilon}_3)(\vec{\sigma}\cdot\vec{\epsilon}_4) \end{pmatrix}$$

using $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$$\Rightarrow \not\epsilon_4 \not\epsilon_3 + \not\epsilon_3 \not\epsilon_4 = -2\,\vec{\epsilon}_3\cdot\vec{\epsilon}_4$$

$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$$\not\epsilon_4 \not\epsilon_3 - \not\epsilon_3 \not\epsilon_4 = 2i(\vec{\epsilon}_3\times\vec{\epsilon}_4)\cdot\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \equiv 2i(\vec{\epsilon}_3\times\vec{\epsilon}_4)\cdot\vec{\Sigma}$$

$$\mathcal{M}_{tot} = \frac{g_e^2}{m}\,\bar{v}(2)\left[(\vec{\epsilon}_3\cdot\vec{\epsilon}_4)\gamma^0 + i(\vec{\epsilon}_3\times\vec{\epsilon}_4)\cdot\vec{\Sigma}\,\gamma^3\right] u(1). \quad \text{— eq. (7.149)}$$

Then consider the spins of e^+ & e^- : singlet state

$$(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2} \quad \Rightarrow \quad \mathcal{M}_{singlet} = (\mathcal{M}_{\uparrow\downarrow} - \mathcal{M}_{\downarrow\uparrow})/\sqrt{2}$$

$\mathcal{M}_{\uparrow\downarrow}$ is obtained from eq. (7.149), spin up for e^- ($u^{(1)}$ in eq. 7.46) & spin-down for e^+ ($v^{-(2)}$ in eq. 7.47)

$$u(1) = \sqrt{2m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \bar{v}(2) = v^\dagger_{(2)}\gamma^0 = \sqrt{2m}\,(0, 0, 1, 0)$$

$\bar{V}(2)\,\gamma^0 U(1) = 0$, $\bar{V}(2)\,\vec{\Sigma}\,\gamma^3 U(1) = -2m\,\hat{z}$

So $M_{\uparrow\downarrow} = -2i g_e^2 \,(\vec{\epsilon}_3 \times \vec{\epsilon}_4)_z$

For $M_{\downarrow\uparrow}$, $U(1) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\bar{V}(2) = -\sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$\Rightarrow M_{\downarrow\uparrow} = 2i g_e^2 (\vec{\epsilon}_3 \times \vec{\epsilon}_4)_z = -M_{\uparrow\downarrow}$.

$\Rightarrow M_{singlet} = -2\sqrt{2}\, i g_e^2 (\vec{\epsilon}_3 \times \vec{\epsilon}_4)_z$

Since $M_{\uparrow\downarrow} = -M_{\downarrow\uparrow}$, triplet $(\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}$ gives zero, confirming two-photon decays is forbidden.

Finally, we put in appropriate photon polarization vectors.

spin-up, $m_s = +1$ spin-down, $m_s = -1$

$\vec{\epsilon}_+ = -\frac{1}{\sqrt{2}}(1, i, 0)$ $\vec{\epsilon}_- = \frac{1}{\sqrt{2}}(1, -i, 0)$

If photon is traveling in the $+\hat{z}$ direction, these correspond to right & left-circular polarization, respectively.

Since z component of total angular momentum must be zero, photon spins must be oppositely aligned: $\uparrow\downarrow$ or $\downarrow\uparrow$

In $\uparrow\downarrow$ case: $\vec{\epsilon}_3 = -\frac{1}{\sqrt{2}}(1, i, 0)$, $\vec{\epsilon}_4 = \frac{1}{\sqrt{2}}(1, -i, 0)$

$\Rightarrow \vec{\epsilon}_3 \times \vec{\epsilon}_4 = +i\,\hat{z}$

In $\downarrow\uparrow$ case

$\Rightarrow \vec{\epsilon}_3 \times \vec{\epsilon}_4 = -i\,\hat{z}$

So we need antisymmetric combination $(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$ for photon spins.

Finally, $M_{singlet} = -4g_e^2$ ✳

We calculate the total cross section for $e^+ e^-$ annihilation. In CM frame.

$$\frac{d\sigma}{d\Omega} = \frac{1}{\left(8\pi(E_1+E_2)\right)^2} \frac{|\vec{P_f}|}{|\vec{P_i}|} |M|^2.$$

$E_1 = E_2 = m$, $|\vec{P_f}| = m$, because of non-relativistic e^+ & e^-

$|\vec{P_i}| = m v$ ← incident e^+ (or e^-) speed

$$\frac{d\sigma}{d\Omega} = \frac{1}{v} \frac{\alpha^2}{m^2} \Rightarrow \sigma = \frac{4\pi}{v} \frac{\alpha^2}{m^2}$$

$\Rightarrow \sigma \propto \frac{1}{v}$: more slowly the e^+ & e^- approach one another, the more time is for them to interact, and the greater likelihood of annihilation.

Reminder: when we discussed the non-relativistic DM annihilation cross section, we expand $\sigma v = a + b v^2 + \cdots$

$$\frac{d\sigma}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega} \quad \text{or} \quad N = \mathcal{L}\sigma$$

$\mathcal{L}$ ← luminosity

N ← scattering event per unit time

$\mathcal{L} = \rho v$

ρ ← # of incident particles per unit volume

$$N = \rho v \sigma.$$

Positronium decay rate. Γ.

For a single atom, electron density is $|\psi(0)|^2$

$$N = \rho v \sigma$$

$$\Gamma = |\psi(0)|^2 v \sigma = 4\pi \frac{\alpha^2}{m^2} |\psi(0)|^2.$$

From prob. 5.23, $|\psi(0)|^2 = \frac{1}{\pi}\left(\frac{\alpha m}{2}\right)^3.$

So the lifetime of positronium is.

$$\tau = \frac{1}{\Gamma} = \frac{2}{\alpha^5 m} = 1.25 \times 10^{-10} \text{ sec.}$$

p^+ e^- hydrogen atom.

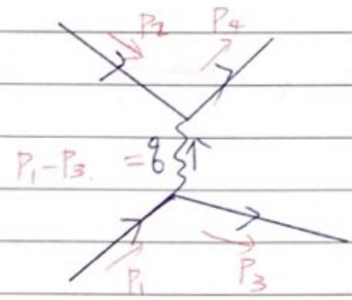
$p^+ + e^- \rightarrow$.

p^+ ← baryon #

e^- ← le #.

Ch 7.9 renormalization

Let's consider "electron-muon" scattering again:

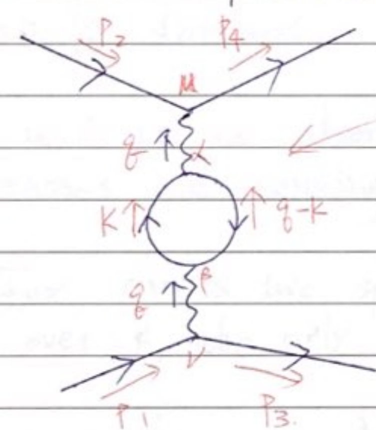


leading-order.

amplitude

$$M_{LO} = -g_e^2 \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \frac{g_{\mu\nu}}{q^2} \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

There are a number of fourth-order corrections, of which is "vacuum polarization":



here the virtual photon splits into an electron-positron pair, leading to a modification in the effective charge of the electron.

(Feynman rule: for a closed fermion loop include a factor -1 and take the trace)

The amplitude for this higher-order diagram is

$$M_{HO} = \frac{-i g_e^4}{q^4} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]$$

$$\times \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu(\not{k}+m_e)\gamma_\nu(\not{k}-\not{q}+m_e)\right]}{(k^2-m_e^2)\left[(k-q)^2-m_e^2\right]} \right\} \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

$M_{LO} + M_{HO} \Rightarrow$ lead to a modification of photon propagator:

$$\frac{g_{\mu\nu}}{q^2} \longrightarrow \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu} \quad , \quad I_{\mu\nu} \equiv -g_e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{k}+m_e)\gamma_\nu(\not{k}-\not{q}+m_e)]}{(k^2-m_e^2)[(k-q)^2-m_e^2]}$$

This integral is divergent. Rugh estimate, it should go like

$$\int |k|^3 d|k| \frac{|k|^2}{|k|^4} = \int |k| dk = |k|^2 \quad , \text{as } |k| \to \infty$$

quadratically divergent.

But because of cancellations, it goes like $\ln|k|$, so it is logarithmically divergent.

The strategy will be to absorb the infinities into "renormalized" masses and coupling constants.

The integral $I_{\mu\nu}$ carries two space-time indices; once we have integrated over k, the only four-vector left is q^μ.

$I_{\mu\nu}$ must have the form $g_{\mu\nu}(\;) + q_\mu q_\nu(\;)$

$$\Rightarrow I_{\mu\nu} = -i g_{\mu\nu} q^2 I(q) + \underline{\underline{q_\mu q_\nu J(q^2)}}$$

The 2nd term contribute zero in M_{HO}, since q^μ contracts with γ^μ

$$[\bar{u}(p_3)\gamma^\mu u(p_1)] q_\mu = [\bar{u}(p_3)(\not{p}_1 - \not{p}_3)u(p_1)] = 0$$

eq. of motion

$\not{p}_1 u(p_1) = m\, u(p_1) \quad , \quad \bar{u}(p_3)\not{p}_3 = \bar{u}(p_3) m$

So we ignore the 2nd term.

After long computation, the 1st term gives

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{ \underbrace{\int_{m_e^2}^{\infty} \frac{dz}{z}}_{\text{isolates the logarithmic divergence}} - 6\int_0^1 z(1-z)\ln\left[1-\frac{q^2}{m_e^2}z(1-z)\right]dz \right\}$$

To handle it, we impose a cutoff M, at the end taking $M\to\infty$.

$$\int_{m^2}^{\infty}\frac{dz}{z} \to \int_{m^2}^{M}\frac{dz}{z} = \ln\frac{M^2}{m}$$

The 2nd integral is finite:

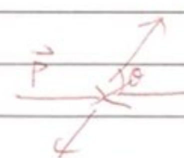
$$f(x) = 6\int_0^1 z(1-z)\ln\left[1+xz(1-z)\right]dz$$

$$= -\frac{5}{3} + \frac{4}{x} + \frac{2(x-2)}{x}\sqrt{\frac{x+4}{x}}\tanh^{-1}\sqrt{\frac{x}{x+4}}$$

$$f(x) \cong \begin{cases} x/5, & x \ll 1 \\ \ln x, & x \gg 1 \end{cases}, \quad f(x\to 0) = 0$$

Thus: $\boxed{I(q^2) = \frac{g_e^2}{12\pi^2}\left\{\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2}\right)\right\}}$.

here q^2 is negative : $q^2 = -4|\vec{p}|^2 \sin\frac{\theta}{2}$



The amplitude for $e-\mu$ scattering, including vacuum polarization is

$$M_{LO}+M_{HO} = -g_e^2\,[\bar{u}(p_3)\gamma^\mu u(p_1)]\,\frac{g_{\mu\nu}}{q^2}\left\{1-\frac{g_e^2}{12\pi^2}\left[\ln\left(\frac{M^2}{m^2}\right)-f\left(\frac{-q^2}{m^2}\right)\right]\right\}$$

$$\times\,[\bar{u}(p_4)\gamma^\nu u(p_2)]$$ valide to order g_e^4.

Here comes critical step, we absorb the infinity by introducing the "renormalized" coupling constant:

$$g_R \equiv g_e\sqrt{1-\frac{g_e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)}$$

$$\Rightarrow M_{LO}+M_{HO} = -g_R^2\,[\bar{u}(p_3)\gamma^\mu u(p_1)]\,\frac{g_{\mu\nu}}{q^2}\left\{1+\frac{g_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2}\right)\right\}[\bar{u}(p_4)\gamma^\nu u(p_2)]$$

Two important things:

1、The infinities are gone.
Everything is now written in terms of g_R, instead of g_e.
g_R is what we actually measure in the laboratory.

2、There remains the finite correction term, and it depends on q^2. We can absorb this into g_R, then g_R depends on q^2; "running" coupling constant

$$g_R(q^2) = g_R(0)\sqrt{1+\frac{g_R(0)^2}{12\pi^2}f\left(\frac{-q^2}{m^2}\right)}$$

fine structure constant $\alpha \equiv \frac{g_R^2}{4\pi}$

$$\Rightarrow \alpha(q^2) = \alpha(0)\left\{1+\frac{\alpha(0)}{3\pi}f\left(\frac{-q^2}{m^2}\right)\right\}$$

$$\alpha(q^2=0) = \frac{1}{137.035999084}$$

The variation is extremely slight, in non-relativistic situations:

Ex. $e^+ \xrightarrow[v=\frac{c}{10}]{} \longleftarrow e^-$

$$\frac{\Delta\alpha}{\alpha} \simeq 6\times10^{-6}$$

$$\alpha(q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-q^2}{M^2}\right)\right\}$$

The effective charge of the electron (and muon), depends on the momentum transferred in the collision.

Higher momentum transfer means closer approach.
$\Rightarrow$ the effective charge of each particle depends on how far apart they are.

This is a consequence of vacuum polarization, which "screens" each charge.