7.4 photon

In classical electrodynamic, $\vec{E}$ and $\vec{B}$ produced by charge density ρ and current density $\vec{J}$ are determined by Maxwell eq.

divergence of a vector.

(i) $\nabla\cdot\vec{E} = 4\pi\rho$ (iii) $\nabla\cdot\vec{B} = 0$

(ii) $\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ (iv) $\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

curl.

(ii) and (iii) homogeneous Maxwell eq.
(i) and (iv) inhomogeneous Maxwell eq.

In relativistic notation, $\vec{E}$ and $\vec{B}$ form a anti-symmetric second-rank tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

ρ and $\vec{J}$ form a four-vector. $J^{\mu} = (c\rho, \vec{J})$

→ rewrite the Maxwell eq. in tensor notation

$$\partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}$$

Because $F^{\mu\nu} = -F^{\nu\mu}$ $\Rightarrow$ J^{μ} is divergenceless

$$\partial_{\mu} J^{\mu} = 0 \quad (\because \partial_{\mu}\partial_{\nu} F^{\mu\nu} = 0)$$

$$\Downarrow$$

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$ "continuity equation"

$\Rightarrow$ local charge conservation

In Maxwell eq. (iii), $\nabla \cdot \vec{B} = 0$

$\Rightarrow$ $\vec{B}$ can be written as curl of a vector potential $\vec{A}$

$\vec{B} = \nabla \times \vec{A}$ —— Eq. (7.75)

Then eq. (ii) $\Rightarrow$ $\nabla \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$

$\Rightarrow$ $\left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)$ can be written as the gradient of a scalar potential.

$\vec{E} = - \nabla V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$ —— Eq. (7.77)

In relativistic notation (ii) and (iii)

Eq. (7.78)

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where $A^\mu = (V, \vec{A})$ is a four-vector

Inhomogeneous Maxwell eq. (i) and (iv)

$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu$ —— Eq (7.80)

The potential V and $\vec{A}$ are not unique.

ex. $A'_\mu = A_\mu + \partial_\mu \lambda$

$\Rightarrow \partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$ $\Rightarrow$ In Eq. (7.78), $\vec{E}$ and $\vec{B}$ are unchanged

"Gauge transformation"

Base on this gauge freedom to put an extra constraint on.

$\partial_\mu A^\mu = 0$ "Lorentz condition."

then Eq.(7.80) simplify into.

inhomogeneous eq.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu \Longrightarrow \underline{\partial_\mu \partial^\mu} A^\nu = \frac{4\pi}{c} J^\nu$$ Eq(7.83)

$$\Box \equiv \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

is the relativistic extension of the Laplacian (∇^2).

Lorentz condition does not uniquely specify A^μ.
Further gauge transformations are possible Ex. $\Box\lambda = 0$. To formulate QED
In empty space ($J^\mu = 0$), we pick.

$A^0 = 0$ and Lorentz condition give $\nabla \cdot \vec{A} = 0$

call Coulomb gauge.

In QED, A^μ becomes the wave function of photon.
The free photon satisfies Eq. 7.83 and $J^\mu = 0$

$$\Box A^\mu = 0 \Rightarrow (\partial_\mu \partial^\mu) A^\nu = 0$$ Klein-Gordon Eq. for massless particle

eq.[7.88]

We look for plane-wave solutions with four-momentum $p = (E/c, \vec{p})$

$$\underset{\underset{\text{normalization.}}{\uparrow}}{A^{\mu}(x) = a}\, e^{-(i/\hbar)\, p \cdot x}\, \underbrace{\epsilon^{\mu}(p)}_{\text{polarization vector = spin of photon.}}$$

substitute to $\Box A^{\mu} = 0$

$\Rightarrow p^{\mu} p_{\mu} = 0$ or $E = |\vec{p}| c$ massless particle

Lorentz condition $\partial_{\mu} A^{\mu} = 0$

$\Rightarrow p^{\mu} \epsilon_{\mu} = 0$

In the Coulomb gauge

$\epsilon^{0} = 0$, $\vec{\epsilon} \cdot \vec{p} = 0$

polarization three-vector $\vec{\epsilon}$ is perpendicular to the direction of propagation : free photon is transversely polarized.

There are two linearly independent three-vector for $\vec{p}$

if $\vec{p}$ in $\hat{z}$ direction,

$$\vec{\epsilon}^{(1)} = (1, 0, 0), \quad \vec{\epsilon}^{(2)} = (0, 1, 0)$$

Two spin states for massless spin-1 particle.

Three spin states for massive spin-1 //
(2s+1)

Along its direction of motion, photon can only have $m_s = +1$ and $m_s = -1$ helicity.

They corresponds to right- and left- circular polarization

$$\vec{\epsilon}_{\pm} = \mp\left(\frac{\vec{\epsilon}^{(1)} \pm i\vec{\epsilon}^{(2)}}{\sqrt{2}}\right)$$

transverse modes.

$$\epsilon^{\mu}_{\pm} = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

logitudinal mode

We eliminate the **nonphysical** ($m_s = 0$) solution by using a particular gauge. (Coulomb gauge)

Without imposing the Coulomb gauge, logitudinal mode re-appear as "ghosts", but they 'decoupled' from everything else, so they do not affect the final results.

The Feynman Rules for QED

Electron
$$\psi(x) = a e^{-(i/\hbar)p\cdot x} u^{(s)}(p)$$

Positron
$$\psi(x) = a e^{(i/\hbar)p\cdot x} v^{(s)}(p)$$

satisfy momentum space Dirac eq.

$$\boxed{(\gamma^\mu p_\mu - mc)u = 0} \qquad \boxed{(\gamma^\mu p_\mu + mc)v = 0}$$

$$(\not{p} - mc)u = 0 \qquad (\not{p} + mc)v = 0$$

the adjoints, $\bar{u} \equiv u^\dagger \gamma^0$, $\bar{v} \equiv v^\dagger \gamma^0$

$$\boxed{\bar{u}(\gamma^\mu p_\mu - mc) = 0} \qquad \boxed{\bar{v}(\gamma^\mu p_\mu + mc) = 0}$$

They are orthogonal

$$\bar{u}^{(1)}u^{(2)} = 0 \qquad \bar{v}^{(1)}v^{(2)} = 0$$

normalized

$$\bar{u}u = 2mc \qquad \bar{v}v = -2mc$$

and complete (spin sum rule).

$$\boxed{\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc)} \qquad \boxed{\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)}$$

(average the particle spins)

Why we sum over the particle spins ?

Because in most of the experiment, they don't measure the spin of the particles.

photons

$$A_\mu(x) = a\, e^{-(i/\hbar)\, p\cdot x}\, \epsilon_\mu^{(s)} \quad , \quad s = 1, 2$$

Lorentz condition

$$\boxed{p^\mu \cdot \epsilon_\mu = 0}$$

They are orthogonal :

$$\epsilon_\mu^{(1)*}\, \epsilon^{(2)\mu} = 0$$

and normalized

$$\epsilon^{\mu *} \epsilon_\mu = -1 .$$

In Coulomb gauge

$$\epsilon^0 = 0 \quad , \quad \vec{\epsilon} \cdot \vec{p} = 0$$

and the polarization three-vectors obey completeness

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j$$ — Prob. 7.25.

spin-1.

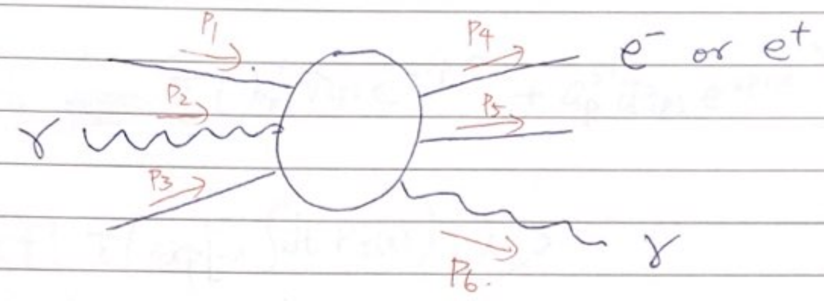
For massless : $\sum_s \epsilon_\mu^{*(s)} \epsilon_\nu^{(s)} \xrightarrow{\text{replace.}} -g_{\mu\nu}$. photon.

It's not actual equality. The replacement is valid as long as both sides are dotted into the rest QED amplitude $\mathcal{M}$

For massive spin-1 : $\sum_s \epsilon_\mu^{*(s)} \epsilon_\nu^{(s)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$ for W, Z bosons.

Feynman Rules QED.

1.



external lines $p_1, p_2 - - -, p_n$
draw arrows forward in time.

internal lines $q_1, q_2 \cdots q_m$
draw an arrows (arbitrary direction).

2. External lines:

$$\text{Electrons}: e^- \begin{cases} \text{Incoming} (\rightarrow\!\bullet) : u. \\ \text{Outgoing} (\bullet\!\rightarrow) : \bar{u} \end{cases}$$

$$\text{Positrons}: e^+ \begin{cases} \text{Incoming} (\leftarrow\!\bullet) : \bar{v} \\ \text{Outgoing} (\bullet\!\leftarrow) : v \end{cases}$$

$$\text{Photons}: \gamma \begin{cases} \text{Incoming} (\sim\!\bullet) : \epsilon_\mu \\ \text{Outgoing} (\bullet\!\sim) : \epsilon_\mu^* \end{cases}$$

3. Vertex factor:

related to the charge of electron

$$i\, g_e \gamma^\mu$$

$$g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$$

4. Propagator :

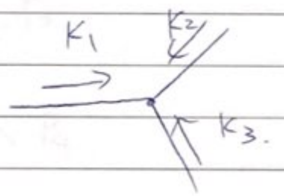
e⁻ and e⁺ : $\frac{i(\gamma^{\mu} q_{\mu} + mc)}{q^2 - m^2 c^2}$ or $\frac{i(\not{q} + mc)}{q^2 - m^2 c^2}$

photon : $\frac{-i g_{\mu\nu}}{q^2}$

scalar : $\frac{i}{q^2 - m^2 c^2}$

5. For each vertex, write a delta function

$(2\pi)^4 \delta^4 (k_1 + k_2 + k_3)$



6. Integrate over internal momenta:

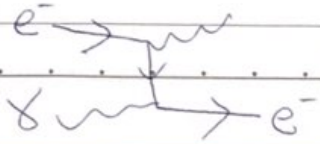
$\frac{d^4 q}{(2\pi)^4}$

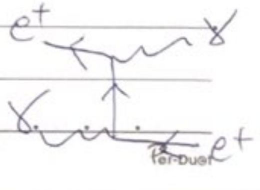
7. Cancel the delta function.

$(2\pi)^4 \delta^4 (P_1 + P_2 + \cdots - P_n) \xrightarrow{\text{replace}} \text{by } i$

The safest procedure is to track each fermion line backward through the diagram.

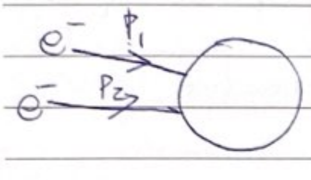
Fermion line would not break.

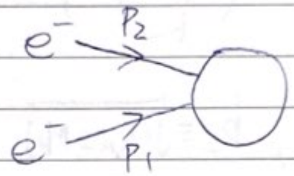




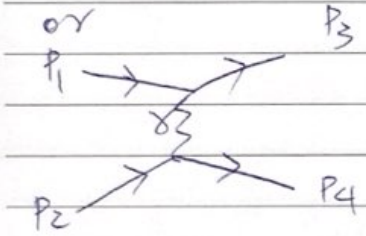
due to antisymmetrization of fermion wave functions

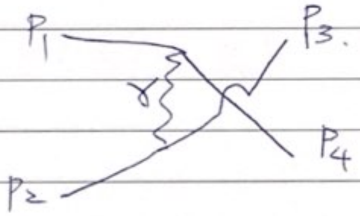
Antisymmetrization: Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) identical particles electrons (or identical anti-particles positrons), or of an incoming electron with an outgoing positron.
particles ... anti-particle.



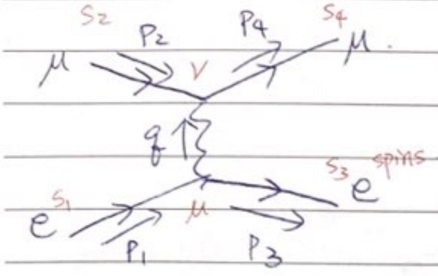


or





Ex 7.1 Electron - muon scattering $e^- + \mu^- \to e^- + \mu^-$



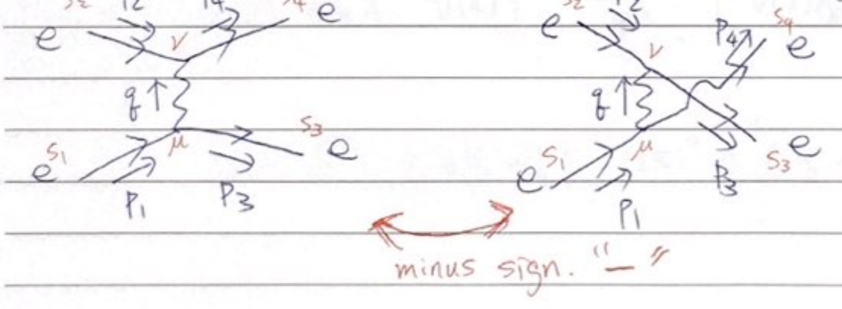
Walking backward.

$$\int \left[\bar{u}^{(s_3)}(p_3)\ (ig_e \gamma^{\mu})\ u^{(s_1)}(p_1) \right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}^{(s_4)}(p_4)\ (ig_e\ \gamma^{\nu})\ u^{(s_2)}(p_2) \right]$$

$$\times\ \delta^4(p_1 - p_3 - q)\ \ \delta^4(p_2 + q - p_4)\ \ d^4q$$

$$\Rightarrow\ \mathcal{M} = -\frac{g_e^2}{(p_1-p_3)^2} \left[\bar{u}^{(s_3)}(p_3)\ \gamma^{\mu}\ u^{(s_1)}(p_1) \right] \left[\bar{u}^{(s_4)}(p_4)\ \gamma_{\mu}\ u^{(s_2)}(p_2) \right]$$

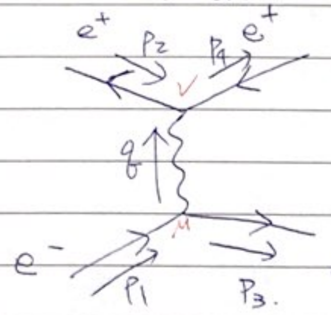
Ex 7.2 Electron - electron scattering $e^- + e^- \to e^- + e^-$

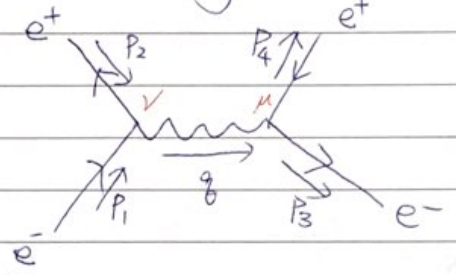


$$\mathcal{M} = \boxed{-} \frac{g_e^2}{(p_1-p_3)^2} \left[\bar{u}(3)\ \gamma^{\mu}\ u(1) \right] \left[\bar{u}(4)\ \gamma_{\mu}\ u(2) \right]$$

$$\boxed{+} \frac{g_e^2}{(p_1-p_4)^2} \left[\bar{u}(4)\ \gamma^{\mu}\ u(1) \right] \left[\bar{u}(3)\ \gamma_{\mu}\ u(2) \right]$$

Ex 7.3 Electron − positron scattering $e^- + e^+ \to e^- + e^+$





M_1 ? $(\pm) M_2$

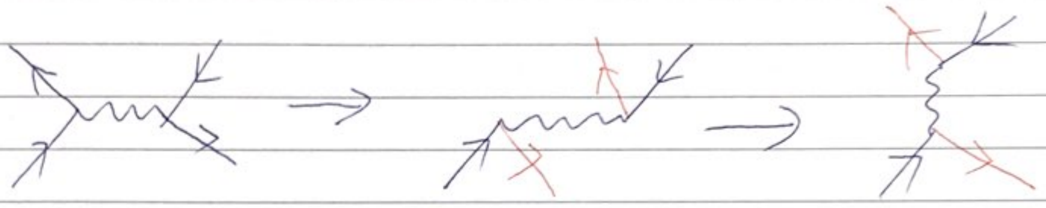
$$M_1 = \int \left[\bar{u}(3)\ i g_e \gamma^\mu u(1)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{v}(2)\ i g_e \gamma_\nu v(4)\right] (2\pi)^4 \delta^4(P_1 - P_3 - q) \cancel{(2\pi)^4} \delta^4(P_2 + q - P_4) \frac{d^4 q}{\cancel{(2\pi)^4}}$$

$$= -\frac{g_e^2}{(P_1 - P_3)^2} \left[\bar{u}(3)\ \gamma^\mu u(1)\right] \left[\bar{v}(2)\ \gamma_\mu v(4)\right]$$

$$M_2 = \int \left[\bar{u}(3)\ i g_e \gamma^\mu v(4)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{v}(2) i g_e \gamma^\nu u(1)\right] (2\pi)^4 \delta^4(P_1 + P_2 - q)\ (2\pi)^4 \delta^4(q - P_3 - P_4) \frac{d^4 q}{(2\pi)^4}.$$

$$= -\frac{g_e^2}{(P_1 + P_2)^2} \left[\bar{u}(3)\ \gamma^\mu v(4)\right] \left[\bar{v}(2)\ \gamma_\mu u(1)\right]$$

"$\pm$" ? Interchange the incoming positron and the outgoing electron in second diagram.



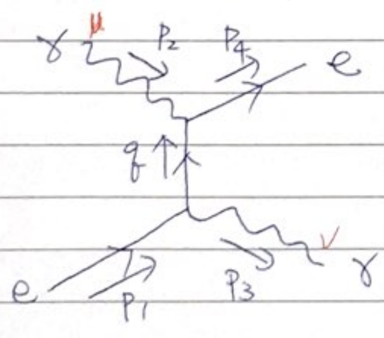
⇒ there we need "$-$"

$$M = M_1 - M_2$$

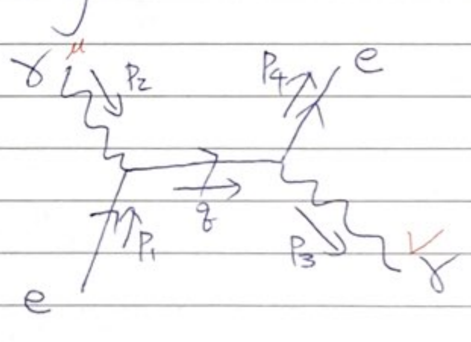
$$= -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{v}(2)\gamma_\mu v(4)\right]$$

$$+ \frac{g_e^2}{(p_1+p_2)^2}\left[\bar{u}(3)\gamma^\mu v(4)\right]\left[\bar{v}(2)\gamma_\mu u(1)\right]$$

Ex 7.6 Compton Scattering $e^- + \gamma \to e^- + \gamma$



M_1



M_2