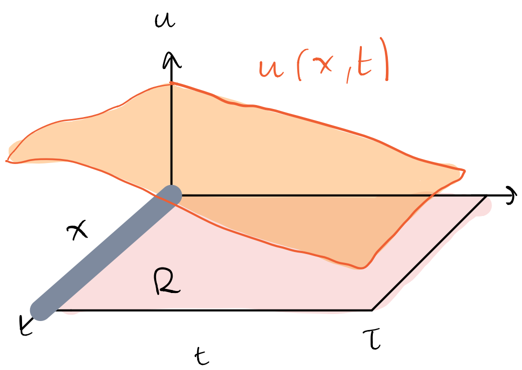
Proof of maximum principle

Assume $u(x,t)$ solves $u_t = k u_{xx}$ on $R = [0,\ell] \times [0,T)$



$\left(u(x,t) \text{ solves a diff eq} \right) \Rightarrow \left[u \text{ is differentiable on } R \right]$

$\Rightarrow \left[u \text{ is continuous on } R \right] \underset{\text{EVT}}{\Rightarrow} \left[u \text{ attains a max on } R \right]$

On the edges where $t=0$, $x=0$, $x=\ell$

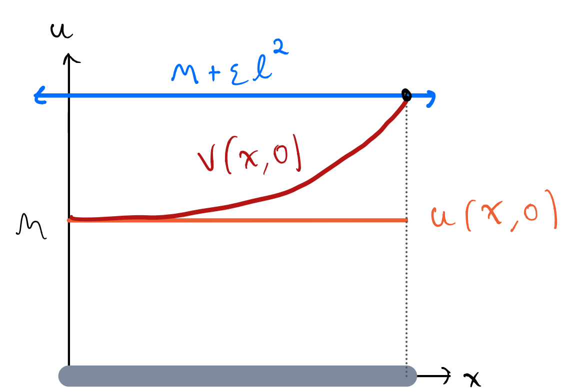
$\left(u(x,0),\ u(0,t)\ \ u(\ell,t) \text{ continuous on } [0,\ell], [0,T], [0,T] \right.$

$\Rightarrow$ All 〃 〃 attain a max on 〃 〃 〃

Call these m_1, m_2, m_3 $\quad M = \max\{m_1, m_2, m_3\}$

Let $\varepsilon > 0$ be any constant

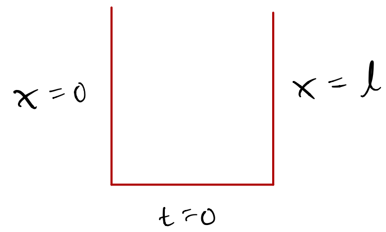
$$\boxed{v(x,t) := u(x,t) + \varepsilon x^2}$$



We want $u(x,t) \le M \quad \forall (x,t) \in R$

First,

$\boxed{\text{WTS}}$ $\quad V(x,t) \le M + \varepsilon l^2 \qquad \forall (x,t) \in R \qquad (1)$



Recall (M is max edge temperature)

$\implies$ [$u(x,t) \le M$ on the three edges $t=0, \; x=0, \; x=l$

Also, $\quad 0 \le x \le l \implies x^2 \le l^2 \implies \varepsilon x^2 \le \varepsilon l^2$

Thus $\quad V(x,t) \le M + \varepsilon l^2$ holds on all 3 edges

Now, consider

$V(x,t) := u(x,t) + \varepsilon x^2$

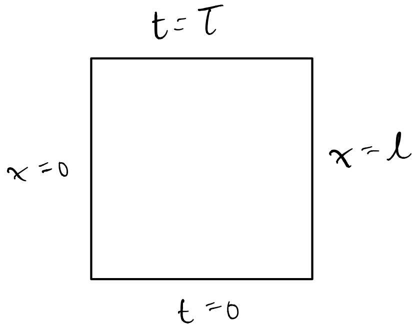
$$V_t - kV_{xx} = u_t - k(u_{xx} + 2\varepsilon)$$

$(u_t = ku_{xx})$

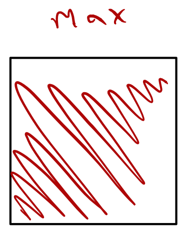
$$= u_t - ku_{xx} - 2k\varepsilon = -2k\varepsilon < 0$$

$$\boxed{\therefore V_t - kV_{xx} < 0} \qquad (2)$$

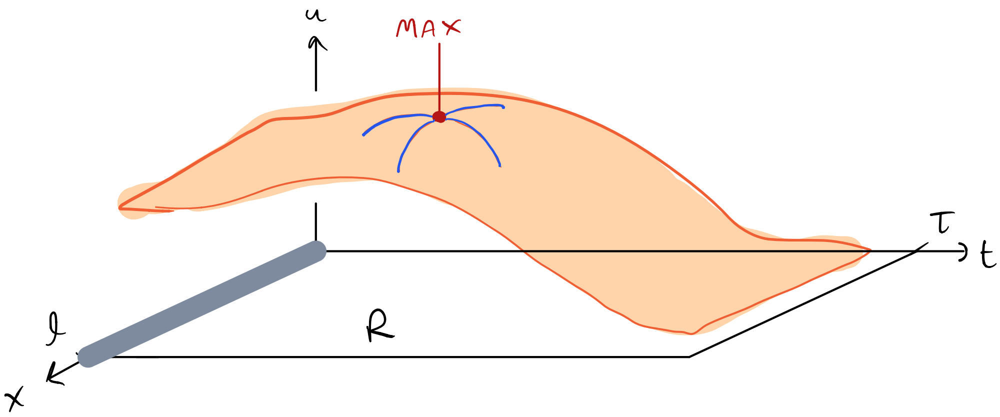
"Diffusion Inequality"



... So where is the max attained?



Case 1 – Interior



Assume by contr. $\left[V \text{ attains its max on } \operatorname{int} R = (0,l)\times(0,T) \right]$

$$\Longrightarrow \left[\exists\, (x_0,t_0) \in \operatorname{int} R \;:\; V(x_0,t_0) \geq V(x,t) \quad \forall\, (x,t) \in R \right]$$

From Calc III, $V_x = 0$, $V_t = 0$

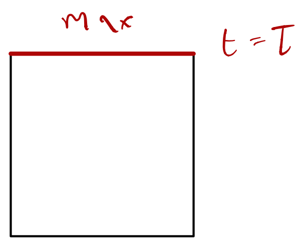
At (x_0,t_0): $V_{xx} \leq 0$ $\quad V_{tt} \leq 0$

From (2): $0 > V_t - kV_{xx} \Rightarrow 0 > -kV_{xx}$

$kV_{xx} > 0 \Rightarrow V_{xx} > 0$

This contradicts our assumption that the max is attained in int R

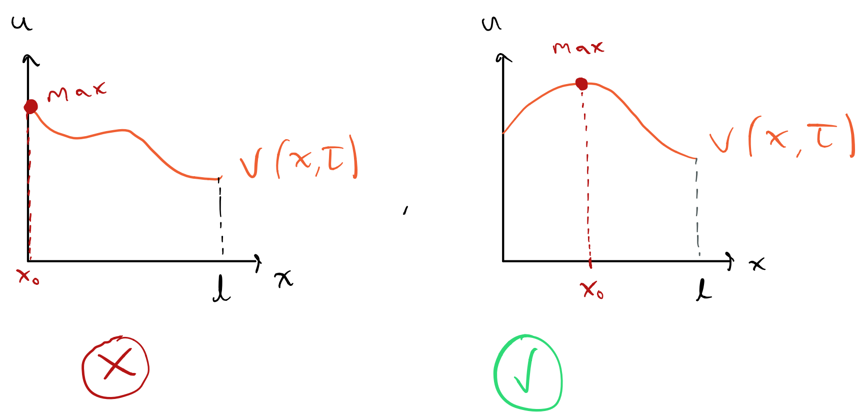
Case 2 - Top edge $(t = \tau)$ ✗



Assume by contr.

$$\left[\, \exists x_0 \in (0, \ell) : \; v(x_0, \tau) \geq v(x,t) \quad \forall (x,t) \in R \,\right] \qquad (3)$$

$t = \tau$:



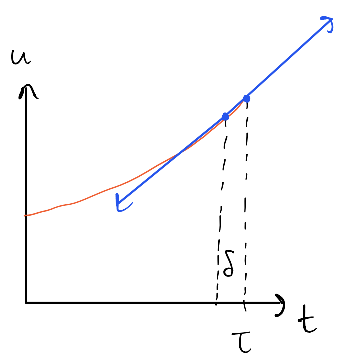
So then $\quad v_x = 0 \,, \quad v_{xx} \leq 0$

Furthermore, $\quad (3) \Longrightarrow \left[\, \underbrace{v(x_0, \tau) \geq v(x_0, \tau - \delta)} \,\right] \quad \forall\, \delta > 0$

Then

$$v_t(x_0, \tau) = \lim_{\delta \to 0} \frac{v(x_0, \tau) - v(x_0, \tau - \delta)}{\delta} \geq 0$$

at (x_0, τ) : $\quad \therefore \; v_t \geq 0$

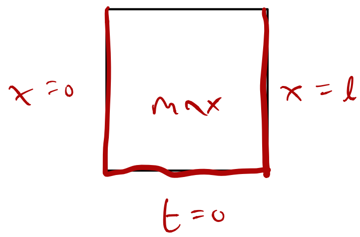


Plug into (2)

$$0 > \underbrace{v_t}_{\geq 0} - k \underbrace{v_{xx}}_{\leq 0} \geq 0 \qquad \cancel{\ast}$$

This contradicts our assumption that the max is attained at $t=\tau$

Case 3 - Edge $t=0$, $x=0$, $x=\ell$



So: $\forall (x,t) \in R$,

$$v(x,t) \le M + \varepsilon \ell^2 \implies u(x,t) + \varepsilon x^2 \le M + \varepsilon \ell^2$$

$$\implies u(x,t) \le M + \varepsilon(\ell^2 - x^2)$$

$\varepsilon \to 0$ gives

$$\boxed{u(x,t) \le M} \qquad \forall (x,t) \in R$$

$\square$