Two Dimensional Navier-Stokes Equations For Channel Flow and Cavity Flow with an Obstacle

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Abstract

In this paper we consider an incompressible viscous fluid in a region $O \subset \mathbb{R}^2$. Our main problem is to consider the flow through a two dimensional channel/pipe. We use the physical description of the problem to formulate constraints and boundary conditions, allowing us to solve the problem analytically. We then solve the same problem numerically, and model the problem in Python. From here, we consider (in much less detail) fluid flow in a two dimensional cavity/box, which will feature flow around an obstacle.

1 Introduction

The velocity field of the fluid flow is given by

$$\vec{u} = \vec{u}(\vec{x}, t) = \big(u(\vec{x}, t),\ v(\vec{x}, t)\big) \tag{1}$$

$$\vec{x} = \big(x,\ y\big) \in O$$

We use the incompressible Navier-Stokes Equations (NSE)

$$\frac{\partial \vec{u}}{\partial t} + \big(\vec{u} \cdot \nabla\big)\vec{u} = -\frac{1}{\rho}\nabla p + \eta \Delta \vec{u} + \vec{F} \tag{2}$$

$$\nabla \cdot \vec{u} = 0 \tag{3}$$

where $\rho > 0$ is the constant density of the fluid, $\eta > 0$ is its kinematic viscosity, $p = p(\vec{x}, t) \in \mathbb{R}$ is the pressure, and $\vec{F} = (F_1, F_2) \in \mathbb{R}^2$ is a body force, as the PDE which governs our fluid flow in the region O. The top equation represents the momentum of the fluid, and is derived from Newton's Second Law ($F = ma$). The bottom equation represents the conservation of mass, and acts as a constraint satisfying the incompressibility of the fluid. It is also referred to as the divergence-free equation, pointing out that there will be no "compression" or "expansion" (i.e. divergence) of the fluid.

1.1 Momentum Equation (Equation of Motion)

Equation (2) is a form of Newton's Second Law in the following sense: Newton's Second Law states that the total resultant force $\mathbf{F} = (mass) \times (acceleration)$ in a system is given by summing all of the forces acting on the system, i.e. $\mathbf{F} = m\vec{a} = \Sigma F$. Starting with a velocity field $\vec{u}$ for the fluid, given by (1), we differentiate with respect to time to obtain an acceleration field. The result is

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}$$

just as in (2). It can be inferred that when Navier and Stokes derived these equations in the 1840s, they wondered what all of the forces acting on a fluid are. For an incompressible fluid (at a constant temperature), the influencing forces were found to be the gradient ∇p of the pressure scalar field $p = p(\vec{x}, t)$, the viscous stresses $\eta \Delta \vec{u}$, and any body forces $\vec{F}$ such as gravity.

1.2 Mass Conservation Equation (Incompressibility Constraint)

Equation (3) given by $\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is known as the incompressibility equation, mass conservation equation, continuity equation, and divergence-free equation. We will mostly refer to it as the incompressibility equation. Considering this equation as a constraint, we might wonder how it is enforced. In short, if you take the divergence of the momentum equation (2), the Laplacian Δp of the pressure funcion p arises, which can be solved for. The incompressibility equation $\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ arises, causing cancellation of terms, such as the time-related term $\frac{\partial \vec{u}}{\partial t}$ and the stress terms $\eta \Delta \vec{u}$. The resulting equation, known as the Pressure Poisson equation, is

$$-\frac{1}{\rho}\Delta p = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = (\frac{\partial u}{\partial x})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + (\frac{\partial v}{\partial y})^2 \tag{4}$$

We will see in our numerical solutions that the pressure p is computed using the (discretized) Pressure Poisson equation, and is used directly in the (discretized) momentum equations from NSE (2), which ensures incompressibility is satisfied.

2 Channel Flow

A two-dimensional channel is defined by a region $O = \mathbb{R} \times [y_\ell\ ,\ y_u] \subset \mathbb{R}^2$, where $y_\ell < y_u$, allowing us to define the height of the channel $h := y_u - y_\ell$. We can make the simplifying assumption that the flow will only be in the x direction. That is, $\vec{u} = \big(u(\vec{x}, t)\ ,\ 0\big)\ \ \forall \vec{x} \in O,\ \forall t \in \mathbb{R}$. Also, if we consider a fully developed flow (and keeping in mind we have an infinite channel), the velocity $\vec{u}$ will depend only on y, and not on x or t. To show this, we will solve the NSE (2) under the physical constraints identified from the channel flow problem.

2.1 Physical Constraints

In this section, we use [(1)] and [(2)]. With our assumption that the fluid flow is fully developed, we have that $\vec{u}(\vec{x}) = \big(u(\vec{x})\ ,\ v(\vec{x})\big)$ does not depend on time. Also, since the flow is in an infinite channel (i.e.$-\infty < x < \infty$), we have that $\vec{u}$ has no vertical component, and thus

$$v(\vec{x}) = 0$$

It follows that

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} = 0 \tag{5}$$

Then since $\frac{\partial^2 v}{\partial y^2} = 0$, plugging this into the incompressibility equation (3) gives

$$\frac{\partial u}{\partial x} = 0 \Longrightarrow \frac{\partial^2 u}{\partial x^2} = 0 \tag{6}$$

In other words, the u component of the velocity $\vec{u}$ does not depend on x, only y (and we already know it does not depend on t). So our velocity field (1) reduces to

$$\vec{u}(\vec{x}, t) = \big(u(y)\ ,\ 0\big) \tag{7}$$

To use the fact that many of the partial derivatives vanish under the constraints of a channel flow, we first write out NSE (2) for its u and v components.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + F_1 \tag{8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + F_2 \tag{9}$$

Next we apply (5) and (6), and without loss of generality, assume $F_1 = F_2 = 0$. Also, recalling that we have a fully developed flow, we know $\vec{u}$ does not depend on t, and so the terms $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$ vanish, giving us

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \eta\frac{\partial^2 u}{\partial y^2} \tag{10}$$

$$\frac{\partial p}{\partial y} = 0 \tag{11}$$

Then since $\vec{u}$ is not a function of x, we have that the change in pressure in the x direction is constant, hence

$$\frac{\partial p}{\partial x} = \rho\eta\frac{\partial^2 u}{\partial y^2} = a \tag{12}$$

where a is a constant. We now have the differential equation

$$\frac{\partial^2 u}{\partial y^2} = a' \quad , \quad a' = \frac{a}{\rho\eta}$$

which we can integrate twice (and undo our substitutions $a' = \frac{a}{\rho\eta}$ and $a = \frac{\partial p}{\partial x}$) to obtain the following solution to (10)

$$u(y) = \frac{1}{2\rho\eta}\frac{\partial P}{\partial x}y^2 + Ay + B \tag{13}$$

So we have that the velocity $\vec{u}$ in the 2D channel has a parabolic profile. We will see that the analysis we have done to reach this solution agrees with the experimental results that we reach in the next section. Before that, we consider the boundary conditions for the problem, so that we can fully solve it.

2.2 Boundary Conditions

Recall $O = \mathbb{R} \times [y_\ell\ ,\ y_u]$ and make the simplifying assumption that $y_\ell = 0$ and $y_u = h$. We apply the "no-slip" boundary condition on the walls of the pipe $y = 0$ and $y = h$. That is,

$$\vec{u}(x, 0, t) = \vec{u}(x, h, t) = \vec{0} \tag{14}$$

Since we obtained the solution (13), this boundary condition can be written

$$u(0) = u(h) = 0$$

This is not a vector equation, it is the u (or "x") component of the velocity $\vec{u}$, which is the only component that is not identically zero everywhere in O in the case of channel flow, as we found in Section 2.1 (see (7)).

Our other boundary condition is periodic, and is applied to both ends of the channel. For simplicity, say these ends are $x = 0$ and $x = L$. The periodic boundary condition acts to fulfill the "infinite" length of the channel, where the idea is that the fluid "leaving" through the end $x = L$ gets sent back to, or reappears at, $x = 0$. You can visualize a 2D video game, where walking your character off the screen causes it to reappear on the opposite end.

The periodic boundary condition is formulated as follows

$$\vec{u}(x, y, t) = \vec{u}(x + L, y, t) \tag{15}$$

2.3 The Full Problem

Now that we have addressed the governing PDE of fluid flow through a 2D channel, its physical constraints, and its boundary conditions, we can set up the full problem:

$$\begin{gathered}
\frac{\partial p}{\partial x} = \rho\eta\frac{\partial^2 u}{\partial y^2} \quad \text{in } O \\
\frac{\partial p}{\partial y} = 0 \quad \text{in } O \\
u(x, 0, t) = u(x, h, t) = 0 \\
v(x, 0, t) = v(x, h, t) = 0 \\
u(x, y, t) = u(x + L, y, t) \\
v(x, y, t) = v(x + L, y, t)
\end{gathered}$$

We can apply the boundary condition (14) to the solution (13) to identify the constants of integration A and B. Applying the BC $u(x, 0, t) = 0$ gives

$$B = 0$$

and applying the BC $u(x, h, t) = 0$ gives

$$A = -\frac{h}{2\rho\eta}\frac{\partial p}{\partial x}$$

where we remind the reader that $\partial p/\partial x = a$ is a constant, as from (12). Thus our solution (13) is

$$u(y) = -\frac{1}{2\rho\eta}\frac{\partial p}{\partial x}(y^2 - hy) \tag{16}$$

Equating to zero gives us $y = 0$ and $y = h$, i.e. the fluid has zero velocity on the walls of the channel, as expected. Also, maximizing as in differential calculus gives us that the velocity of the fluid is greatest at $y = h/2$, the center of the channel.

3 Computational Solution

In this section, we use [(3)]. We wish to write the NSE momentum equation (2) and the Pressure Poisson equation (4) as discretized numerical differential equations. Let subscripts i and j indicate the coordinate $\vec{x}_{ij} = (x_i, y_j)$. Use superscript n to denote the time step, so that we can consider the velocity $\vec{u}_{ij}^n$ at point $\vec{x}_{ij}$ and at time n, with $i, j, n \in \mathbb{Z}^+$.

3.1 Finite Difference Method

To carry out this process, we use the u and v components of the NSE momentum equation (2), given by equations (8) and (9). We will use the finite difference method, computing a forward-step difference for the time derivatives, backward-step differences for the spacial derivatives of the fluid velocity, and centered differences for the spatial derivatives of the pressure. Upon doing so, we obtain (for the u component):

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} + u_{ij}^n \frac{u_{ij}^n - u_{i-1,j}^n}{\Delta x} + v_{ij}^n \frac{u_{ij}^n - u_{i,j-1}^n}{\Delta y} = -\frac{1}{\rho}\frac{p_{i+1,j}^n - p_{i-1,j}^n}{2\Delta x} + \eta\left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{(\Delta x)^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{(\Delta y)^2}\right)$$

where $\Delta t, \Delta x, \Delta y > 0$ are the respective mesh sizes. From here, we can solve for the first term, u_{ij}^{n+1}, using only algebra. The result is

$$u_{ij}^{n+1} = u_{ij}^n - \frac{\Delta t}{\Delta x} u_{ij}^n (u_{ij}^n - u_{i-1,j}^n) - \frac{\Delta t}{\Delta y} v_{ij}^n (u_{ij}^n - u_{i,j-1}^n) - \frac{\Delta t}{\rho}\frac{p_{i+1,j}^n - p_{i-1,j}^n}{2\Delta x} + \eta\Delta t\left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{(\Delta x)^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{(\Delta y)^2}\right)$$

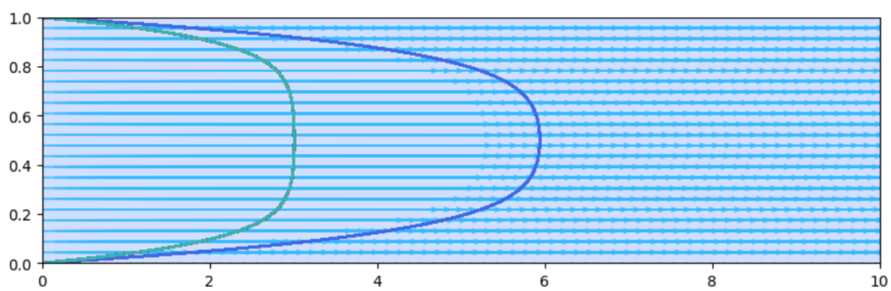
for the u component, and

$$v_{ij}^{n+1} = v_{ij}^n - \frac{\Delta t}{\Delta x} u_{ij}^n (v_{ij}^n - v_{i-1,j}^n) - \frac{\Delta t}{\Delta y} v_{ij}^n (v_{ij}^n - v_{i-1,j}^n) - \frac{\Delta t}{\rho}\frac{p_{i,j+1}^n - p_{i,j-1}^n}{2\Delta y} + \eta\Delta t\left(\frac{v_{i+1,j}^n - 2v_{ij}^n + v_{i-1,j}^n}{(\Delta x)^2} + \frac{v_{i,j+1}^n - 2v_{ij}^n + v_{i,j-1}^n}{(\Delta y)^2}\right)$$

for the v component of the fluid velocity $\vec{u}$. We use the same exact approach on the Pressure Poisson equation (4), only we omit it from the paper, as the result is very long (but the reader can trust that it is the same process, and only requires more time and patience to compute, not any additional techniques). It is from this computation that we obtain p_{ij}^n, which is used directly in the discretized u and v velocities that we just found. This is how incompressibility is satisfied numerically.

3.2 Channel Flow in Python

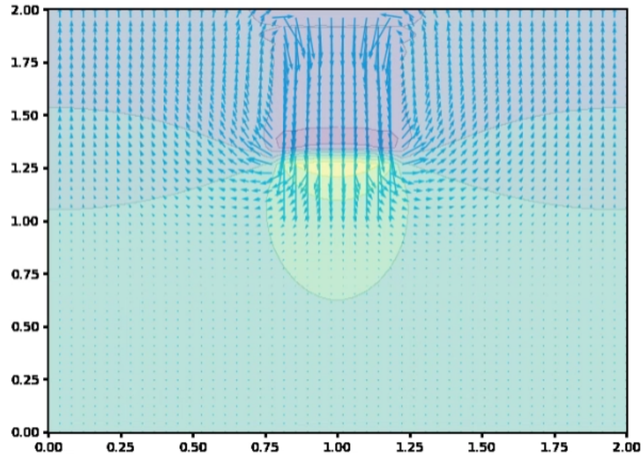
We will use the formulas for u_{ij}^{n+1}, v_{ij}^{n+1}, and (the omitted) p_{ij}^n, along with the boundary conditions we identified in Section 2.3, to simulate fluid flow through an infinite 2D channel. We also provide the initial conditions that the fluid velocity $\vec{u}$ is 0 everywhere at time $t = 0$, and the pressure is 1 everywhere at time $t = 0$. That is, $\vec{u}_{ij}^0 = \vec{0}$ and $p_{ij}^0 = 1$ for all $i \in [0, L]$ and for all $j \in [0, h]$. We also apply a constant force F_1 to the u component of the velocity.



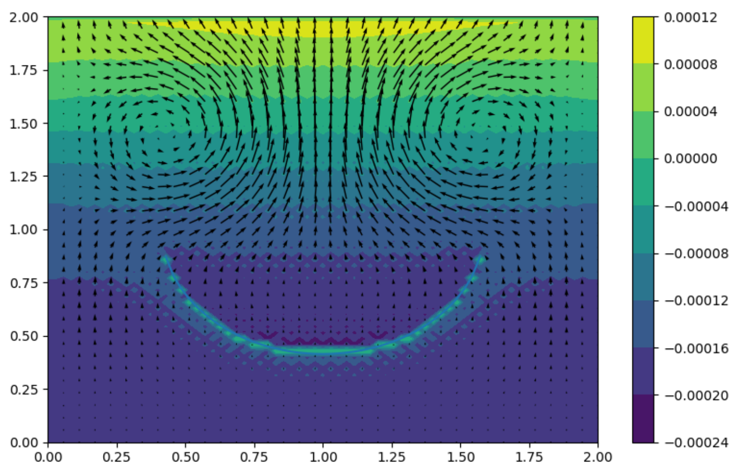
After 100 iterations (representing only a couple seconds of flow), we examine the velocity field along with its velocity profile. Notice that the dark blue velocity profile resembles a parabola, agreeing with our solution (16) from Section 2.3. As for the teal velocity profile, which is the average velocity profile over the iterations, it is flatter along the center of the channel. This is because the flow is not fully developed, and the initial velocity profile at time $t = 0$ is constant, and is given by the body force F_1.

3.3 Cavity Flow in Python

The only differences between the cavity flow problem and the channel flow problem are the boundary conditions. In a 2D cavity, we have the no-slip constraint on each wall $x = 0$, $x = L$, $y = 0$, and $y = h$, and we have no periodic boundaries. We also introduce a moving obstacle (a small line segment), which traverses through the channel, causing the fluid to flow around it. This is carried out using another boundary condition: one such that there is the no-slip constraint on the line segment, and the line segment traverses at a constant velocity with only a v component that is different from zero. In this model, we show not only the fluid velocity vector field $\vec{u}$, but the pressure scalar field as well, as a contour plot/heat map.



We also show the flow around a half circle, where the initial flow is pressure-driven in the vertical direction. Note that we are using the finite difference method, which is not very efficient here since a circle is not accurately partitioned into rectangles.



References

[1] Tian J, Zhang B. *On the Solutions of Navier-Stokes Equations for Turbulent Channel Flows in a Particular Function Class*. Journal of Mathematical Physics. 2018, October 18.

[2] Polyanin, A. *Exact Solutions of the Navier Stokes Equations with Generalized Separation of Variables*. Doklady Physics. 2001, January.

[3] Barba, L. *12 Steps to Navier-Stokes*. CFD Python. 2014, January. https://github.com/barbagroup/CFDPython.