APPENDIX

Full Conditional Posterior Distributions

Let $\mathbf{Y} = \{\mathbf{N}_1, \ldots, \mathbf{N}_n; t_1^*, \ldots, t_n^*; \mathbf{X}_1, \ldots, \mathbf{X}_n\}$ denote the observed data, where $\mathbf{N}_i = (N_{i1}, \ldots, N_{iM_i})'$ and $\mathbf{X}_i = (\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iM_i})'$. Also, let $\mathbf{\Phi} = \{\phi_i, \phi_{ij}, i = 1, \ldots, n; j = 1 \ldots, M_i\}$, $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_M)'$, and $\mathbf{\Gamma} = \{\gamma_{hk}, h = 1 \ldots, M; k = 1, \ldots, p\}$. Given the above notation, the likelihood is proportional to

$$L(\mathbf{\Phi}, \boldsymbol{\nu}, \mathbf{\Gamma}|\mathbf{Y}) = \prod_{i=1}^{n}\prod_{j=1}^{M_i}\left(\phi_i\nu_0\widehat{\lambda}_{0j}\prod_{h=1}^{j}\phi_{ih}\nu_h\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}}\right)^{N_{ij}}\exp\left\{-r_{ij}\phi_i\nu_0\widehat{\lambda}_{0j}\prod_{h=1}^{j}\phi_{ih}\nu_h\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}}\right\}. \tag{A-1}$$

Under the dynamic gamma model, the full conditional posterior distributions of the ϕ_i and ϕ_{ij}'s are also gamma due to the Poisson form of (A-1):

$$\pi(\phi_i|\cdot) = G_i = \mathcal{G}\left(\psi_1 + N_i^*, \psi_1 + m_i^*\right) \tag{A-2}$$

$$\pi(\phi_{is}|\cdot) = G_{is} = \mathcal{G}\left(\psi_2 + N_{is}^*, \psi_2 + m_{is}^*\right), \tag{A-3}$$

where the notation $a|\cdot$ denotes a given all other variables and $N_i^* = \sum_{j=1}^{M_i} N_{ij}$, $N_{is}^* = \sum_{j=s}^{M_i} N_{ij}$

$$\begin{aligned} m_i^* &= \nu_0 \textstyle\sum_{j=1}^{M_i} r_{ij}\widehat{\lambda}_{0j}\prod_{h=1}^{j}\phi_{ih}\nu_h\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}} \\ m_{is}^* &= \phi_i\nu_0\textstyle\sum_{j=s}^{M_i} r_{ij}\widehat{\lambda}_{0j}(\prod_{h=1}^{j}\nu_h\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}})\prod_{f\neq s}^{j}\phi_{if}, \end{aligned}$$

for $s = 1, \ldots, M_i$. The full conditional posterior densities of the elements of $\boldsymbol{\nu}$ and Γ are

$$\pi(\nu_0|\cdot) = \mathcal{G}\left(\kappa + \sum_{i=1}^{n}N_i^*, \kappa + \sum_{i=1}^{n}\phi_i\sum_{j=1}^{M_i}r_{ij}\widehat{\lambda}_{0j}\prod_{h=1}^{j}\phi_{ih}\nu_h\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}}\right) \tag{A-4}$$

$$\pi(\nu_s|\cdot) = \mathcal{G}\left(\psi_3 + \sum_{i\in R_s}N_{is}^*, \psi_3 + \nu_0\sum_{i\in R_s}\phi_i\sum_{j=s}^{M_i}r_{ij}\widehat{\lambda}_{0j}\left(\prod_{h=1}^{j}\phi_{ih}\prod_{k=1}^{p}\gamma_{hk}^{x_{ijk}}\right)\left(\prod_{f\neq s}^{j}\nu_f\right)\right) \tag{A-5}$$

$$\begin{aligned} \pi(\gamma_{sk}|\cdot) \propto\ & \exp\left\{\left(\psi_4 + \sum_{i\in R_s}\sum_{j=s}^{M}N_{ij}x_{ijk}\right)\log(\gamma_{sk}) - \gamma_{sk}\psi_4\right. \\ & \left.-\nu_0\sum_{i\in R_s}\sum_{j=s}^{M}r_{ij}\phi_i\widehat{\lambda}_{0j}\prod_{h=1}^{j}\phi_{ih}\nu_h\prod_{l=1}^{p}\gamma_{hl}^{x_{ijl}}\right\}, \end{aligned} \tag{A-6}$$

where $R_s = \{i : M_i \geq s\}$, for $s = 1, \ldots, M$.

When the Dirichlet process is used to model the frailty terms, the full conditionals of ϕ_i and ϕ_{is} are not G_i and G_{is}. Using the Pólya urn representation of the Dirichlet process (Blackwell and MacQueen, 1973; MacEachern, 1994; West, 1990), one can show that the prior distribution of ϕ_i given $\boldsymbol{\phi}^{(i)} = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n)'$ is the mixture

$$\left(\frac{\alpha_{01}}{\alpha_{01}+n-1}\right)G_{01} + \left(\frac{1}{\alpha_{01}+n-1}\right)\sum_{l=1}^{h^{(i)}} n_l^{(i)}\delta_{\theta_l^{(i)}}, \tag{A-7}$$

where δ_θ denotes the degenerate distribution with all its mass at θ, and the prior for ϕ_{is} given $\boldsymbol{\phi}_s^{(i)} = \{\phi_{i's} : i' \in R_s, i' \neq i\}$ and $n_{(s)}$ total subjects in R_s is

$$\left(\frac{\alpha_{02}}{\alpha_{02}+n_{(s)}-1}\right)G_{02} + \left(\frac{1}{\alpha_{02}+n_{(s)}-1}\right)\sum_{l=1}^{h_s^{(i)}} n_{sl}^{(i)}\delta_{\theta_{sl}^{(i)}}, \tag{A-8}$$

where $\boldsymbol{\theta}^{(i)}$ and $\boldsymbol{\theta}_s^{(i)}$ denote the $h^{(i)}$ and $h_s^{(i)}$ unique values of $\boldsymbol{\phi}^{(i)}$ and $\boldsymbol{\phi}_s^{(i)}$, respectively, $n_l^{(i)}$ elements of $\boldsymbol{\phi}^{(i)}$ have value $\theta_l^{(i)}$, and $n_{sl}^{(i)}$ elements of $\boldsymbol{\phi}_s^{(i)}$ have value $\theta_{sl}^{(i)}$. After factoring in the likelihood, the full conditional posterior of ϕ_i is

$$q_{i0}G_i + \sum_{l=1}^{h^{(i)}} q_{il}\delta_{\theta_l^{(i)}} \tag{A-9}$$

where

$$q_{il} = \begin{cases} \frac{c_1\alpha_{01}C(\psi_1,\psi_1)}{C(\psi_1+N_i^*,\psi_1+m_i^*)} & l = 0 \\ c_1 n_l^{(i)}(\theta_l^{(i)})^{N_i^*}\exp\{-\theta_l^{(i)}m_i^*\} & l > 0, \end{cases}$$

$C(a,b) = b^a/\Gamma(a)$, and c_1 is a normalizing constant. Similarly, the full conditional posterior of ϕ_{is} $(s = 1, \ldots, M_i)$ is

$$q_{is0}G_{is} + \sum_{l=1}^{h_s^{(i)}} q_{isl}\delta_{\theta_{sl}^{(i)}} \tag{A-10}$$

where

$$q_{isl} = \begin{cases} \frac{c_{s+1}\alpha_{02}C(\psi_2,\psi_2)}{C(\psi_2+N_{is}^*,\psi_2+m_{is}^*)} & l = 0 \\ c_{s+1} n_{sl}^{(i)}(\theta_{sl}^{(i)})^{N_{is}^*}\exp\{-\theta_{sl}^{(i)}m_{is}^*\} & l > 0. \end{cases}$$

Thus, the full conditional distributions of ϕ_i and $(\phi_{i1}, \ldots, \phi_{iM_i})'$ are mixtures of the gamma posteriors obtained under the dynamic gamma model and multinomial distributions with

support on the unique values of each frailty component.

Updating Algorithm

In order to sample efficiently under the Dirichlet process mixture, we invoke a sampling scheme similar to that provided by MacEachern (1994) and West et al. (1994). Let there be h unique values in $(\phi_1, \ldots, \phi_n)'$ and h_s unique values in $\{\phi_{is} : i \in R_s\}$, which we denote by $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_h)'$ and $\boldsymbol{\theta}_s = (\theta_{s1}, \ldots, \theta_{sh_s})'$, respectively, for $s = 1 \ldots, M$. We also define the discrete random variables $\mathcal{S}_i$ and $\mathcal{S}_{is}$ such that $\mathcal{S}_i = k$ if $\phi_i = \theta_k$ and $\mathcal{S}_{is} = l$ if $\phi_{is} = \theta_{sl}$, for $i \in R_s$. Our MCMC algorithm proceeds as follows:

Step 1. Sample ν_0 from (A-4), given the current values of $\boldsymbol{\Phi}$, $\nu_1, \ldots, \nu_M$, and $\boldsymbol{\Gamma}$.

Step 2.1. Sample $\mathcal{S}_i$, for $i = 1, \ldots, n$ from a multinomial distribution with $\Pr(\mathcal{S}_i = l) = q_{il}$, for $l = 0, 1, \ldots, h^{(i)}$, with a new ϕ_i drawn from G_i if $\mathcal{S}_i = 0$.

Step 2.2. Given the updated values of h and $\mathcal{S}_1, \ldots, \mathcal{S}_n$, generate a new $\boldsymbol{\theta}$ by sampling each θ_k from its full conditional posterior distribution, $\mathcal{G}\big(\psi_1 + \sum_{i:\mathcal{S}_i=k} N_i^*, \psi_1 + \sum_{i:\mathcal{S}_i=k} m_i^*\big)$, for $k = 1, \ldots, h$. Assign the appropriate value of $\boldsymbol{\theta}^{(i)}$ to ϕ_i as indicated by $\mathcal{S}_i$.

For $s = 1, \ldots, M$, perform the following steps:

Step 3.s(a). For $i \in R_s$, sample $\mathcal{S}_{is}$ from the multinomial distribution with $\Pr(\mathcal{S}_{is} = l) = q_{isl}$, for $l = 0, 1, \ldots, h_s^{(i)}$, with a new ϕ_{is} drawn from G_{is} if $\mathcal{S}_{is} = 0$.

Step 3.s(b). Update $\boldsymbol{\theta}_s$ by sampling each θ_{sl} from the full conditional posterior, $\mathcal{G}\big(\psi_2 + \sum_{i:\mathcal{S}_{is}=l} N_{is}^*, \psi_2 + \sum_{i:\mathcal{S}_{is}=l} m_{is}^*\big)$, for $l = 1, \ldots, h_s$. Assign the appropriate value of $\boldsymbol{\theta}_s^{(i)}$ to ϕ_{is}.

Step 3.s(c-d). Sample ν_s from (A-5) and γ_{sk} from (A-6) for $k = 1, \ldots, p$.

Step 4.1-4.2. Update ψ_1 and ψ_2. Under the DP model, the full conditional posterior densities of ψ_1 and ψ_2 depend only on $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_M$,

$$\pi(\psi_1|\cdot) \propto \left(\frac{\psi_1^{\psi_1}}{\boldsymbol{\Gamma}(\psi_1)}\right)^h \psi_1^{a_1-1} \exp\left\{-\psi_1\left(b_1 + \sum_{k=1}^{h}(\theta_k - \log\theta_k)\right)\right\} \quad \text{(A-11)}$$

$$\pi(\psi_2|\cdot) \propto \left(\frac{\psi_2^{\psi_2}}{\boldsymbol{\Gamma}(\psi_2)}\right)^{\sum_{s=1}^{M} h_s} \psi_2^{a_2-1} \exp\left\{-\psi_2\left(b_2 + \sum_{s=1}^{M}\sum_{k=1}^{h_s}(\theta_{sk} - \log\theta_{sk})\right)\right\}, \quad \text{(A-12)}$$

while under the dynamic gamma frailty model, the posteriors depend on each ϕ_i and ϕ_{ij}, respectively. Since (A-11) and (A-12) do not have closed forms, we recommend updating ψ_1 and ψ_2 using a Metropolis-Hastings random walk.

In our analysis of the chemoprevention data (Section 4), we modified the algorithm slightly to speed up computation. In steps 2.1 and 3.s(a), we sampled each $\mathcal{S}_i$ and $\mathcal{S}_{is}$ using the cluster configuration at the previous iteration. This allowed us vectorize the code used to sample these latent variables and remove computationally intensive loops. Although this modification may have slowed convergence down slightly, it is unlikely that it affected our results due to the length of our chain.

It is also fairly straightforward to sample from predictive distributions at each iteration of the Gibbs sampler. Given $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_M$, the frailty of a future subject, $\boldsymbol{\xi}_{n+1}$, may be predicted by sampling from the distributions

$$\pi(\phi_{n+1}|\boldsymbol{\phi}) = \left(\frac{\alpha_{01}}{\alpha_{01}+n}\right)G_{01} + \left(\frac{1}{\alpha_{01}+n}\right)\sum_{l=1}^{h} n_l \delta_{\theta_l} \quad \text{(A-13)}$$

$$\pi(\phi_{(n+1)s}|\boldsymbol{\phi}_s) = \left(\frac{\alpha_{02}}{\alpha_{02}+n_s}\right)G_{02} + \left(\frac{1}{\alpha_{02}+n_s}\right)\sum_{l=1}^{h_s} n_{sl} \delta_{\theta_{sl}} \quad \text{(A-14)}$$

for $s = 1, \ldots, M$. Count data for a future subject may then be simulated using the non-homogeneous Poisson process in equation (4), Section 2.1.

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