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# THE FOREST ROTATION PROBLEM WITH STOCHASTIC HARVEST AND AMENITY VALUE

LUIS H.R. ALVAREZ
Department of Economics
Quantitative Methods in Management
Turku School of Economics
FIN-20500 Turku
Finland
*E-mail:* luis.alvarez@tse.fi

ERKKI KOSKELA
Department of Economics
P.O. Box 17
University of Helsinki
FIN-00014 University of Helsinki
Finland, CESifo, München
and
IZA, Bonn
*E-mail:* erkki.koskela@helsinki.fi

ABSTRACT. We present a general approach to study optimal rotation policy with amenity valuation under stochastic forest stand value. We state a set of weak conditions under which a unique optimal harvesting threshold exists and derive the value of the optimal policy. We characterize the impact of forest stand value volatility on both the total and the marginal expected cumulative present value of the revenues accrued from amenities. We also illustrate our results numerically and find that depending on the precise characteristics of amenity valuation higher forest stand value volatility may accelerate the rotation policy by decreasing the optimal harvesting threshold.

KEY WORDS: Nature of amenity valuation, optimal Faustmannian and Wicksellian rotation policy, forest stand value volatility.

**1. Introduction.** In the usual Faustmann approach to the optimal forest rotation policy landowners are assumed to maximize the present value of net harvesting revenue, see e.g., Reed [1986] for a survey. It may be the case, however, that landowners are interested not only in the present value of net harvesting revenue, but also in the amenity services provided by forest stands. For instance, Binkley [1981], Kuuluvainen

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and Tahvonen [1999] and Pattanayak et al. [2002] provide some indirect empirical evidence in favor of this hypothesis, see also a survey of theoretical and empirical research about timber supply by Wear and Parks [1994]. Empirical findings raise among others the following question: How to explain the behavior of landowners? In this paper we re-examine the role of amenity services for the behavior of landowners with a certain focus. More precisely, we ask and provide new results for the following question: How does the nature of amenity valuation affect the optimal rotation policy under stochastic forest stand value?

The first analytical treatment to include amenity services of forest stands is the stand-level model by Hartman [1976]. In that paper it is shown that if amenity services are increasing with the forest stand age, the optimal rotation age will exceed the Faustmann rotation age thereby delaying harvesting, ceteris paribus. Clearly, if amenity services are decreasing with the forest stand age, the reverse happens, see also Strang [1983], Bowes and Krutilla [1985] and Snyder and Bhattacharyya [1990], where the maintenance of provision costs necessary to realize a quality flow of recreational services is explicitly included. Koskela and Ollikainen [2001, 2003a, 2003b], have studied both in the Faustmann and Hartman framework the behavioral impacts of various taxes associated with forestry in terms of how these affect the privately optimal rotation age and the optimal design of tax structure. The original Hartman model and these extensions, however, are deterministic so that future timber revenue and amenity values are known, which is a very heroic assumption to be relaxed.

To our knowledge, allowing for uncertainty in the Hartman framework has been theoretically analyzed in Reed [1993], in Reed and Ye [1994], in Conrad [2000] and Gong et al. [2005], respectively. In the seminal study Reed [1993] assumed that both the future timber value and future amenity services are stochastic and that there is a possibility of catastrophic risk, e.g., via forest fire. Future timber value and future amenity services are assumed to follow geometric (and possibly correlated) Brownian motion processes, and the catastrophic risk is modeled as a Poisson process. Reed shows under his assumptions that a risk-neutral landowner will harvest the forest stand only if the ratio of current timber value to current amenity valuation exceeds a threshold level, which among others depends on the option value associated both with timber value and amenity service volatilities. The Certainty-

Equivalence theorem does not hold even though the landowner is risk neutral due to the nonlinear effects of stochastic variables (for the Certainty-Equivalence theorem, see, e.g,. Laffont [1989, Chapter 3]). Reed shows in terms of comparative static properties of his model that the expected optimal rotation age will become longer due to a higher harvesting threshold when the expected growth rate of forest value and the volatilities of both forest value and amenity services will go up, while the expected optimal rotation age will become shorter as the expected growth rate of amenity, the discount rate, and the covariance between forest and amenity value stochasticities as well as the probability of catastrophic risk will increase.

The model assumptions in Reed and Ye [1994] are identical to those in Reed [1993] with one exception: Instead of postulating a geometric Brownian motion process for both timber revenue and amenity valuation, they assume that discrete changes in the valuation of amenity services and in timber prices occur at random times characterized by geometric jump processes. Under this specification the ratio of the value of amenity services to timber values can be monotone (nondecreasing or nonincreasing), while still stochastic. This implies that there is no option value associated with the decision to harvest under stochastic future timber and amenity values. Hence, according to their findings the reason for the option values seems not to be so much stochasticity per se, but rather the possibility of reversals in the ratio between the value of timber revenues and amenity services. Conrad [2000] has applied option theory to the decision to extract resources and/or to develop a wilderness area and thereby amenity dividend. Like Reed [1993] he also assumes that both the value of timber revenues and amenity values follow different geometric Brownian motion processes. Gong et al. [2005] examine the joint effects of amenity benefits and timber price uncertainty on the optimal time to harvest by showing that incorporating nontimber benefits into the reservation price model increases the optimal reservation price of time and the land expectations value. As for empirical applications, Conrad [1997] has presented a model where a stand of old-growth forest has a known stumpage value and where future amenity value is uncertain and follows geometric Brownian motion (see also Bulte et al. [2002]). The main idea of these papers is to try empirically to outline the impact of stochasticity about forest conservation benefits on the incentive to harvest or conserve old-growth forests. To conclude, according to the current literature, volatilities will raise the optimal harvesting threshold, and thereby the expected rotation length, even under amenity valuation and risk-neutrality of landowners.

Our paper provides a new and more general approach compared with the existing literature to study the optimal rotation policy with amenity valuation under forest stand value uncertainty both in the case of the ongoing rotation and single rotation frameworks. In the small existing literature the authors have assumed different formulations for stochastic forest stand value and for amenity values without an explicit justification. First, we follow a more plausible approach by postulating stochastic forest stand value growth and assuming that the monetary value of amenities is a continuous and nonnegative function of the forest stand value, thus emphasizing the significance of the trade-off between timber revenues and amenity services. Second, our approach is based on the idea that the value of the optimal harvesting problem can be derived explicitly by solving an associated functional equation which can be motivated in terms of a running present value formulation familiar from cash flow management problems arising, among others, in financial economics. Instead of applying the ordinary Hamilton-Jacobi-Bellman approach or relying on quasi-variational inequalities (cf. Brekke and Øksendal [1998], Mundaca and Øksendal [1998], and Øksendal 1999) we take an alternative route and derive first the value accrued from following a harvesting strategy characterized by a single but otherwise arbitrary threshold at which the harvesting policy is exerted. Having derived this representation, we characterize both the value and the optimal policy in terms of exogenous parameters by relying on ordinary nonlinear-programming techniques. We state a set of typically satisfied conditions under which the optimal rotation threshold constitutes the unique solution of a standard first-order condition. The main advantage of this approach is that it simplifies the economic analysis of the optimal policy and its value by admitting the direct application of standard first order marginalistic interpretations of the optimal boundary and the resulting harvesting policy. Third, we characterize how both the expected cumulative present value and the expected marginal cumulative present value accrued from amenity services depend on the precise nature of the amenity valuation function and how forest stand value volatility affects these concepts in order to analyze the comparative static properties of the landowner's optimal policy. Finally, we illustrate our results numerically in a stochastic model based on logistic forest stand value growth. We determine the Faustmannian and Wicksellian harvesting thresholds and analyze their dependence on interest rate, forest stand value growth and volatility.

We demonstrate that, under nonnegative monetary value accrued from amenity services, higher forest stand value volatility is shown to increase the Wicksellian harvesting threshold, while it decreases the Faustmannian harvesting threshold. This new finding is essentially based on the fact that although higher volatility may increase the expected cumulative monetary present value accrued from amenities thus affecting the Wicksellian optimal policy, it may simultaneously decrease the marginal expected cumulative monetary present value which affects the Faustmannian optimal policy. Thus, in this case, higher forest stand value volatility increases the incentives to postpone the single rotation harvesting decision while it simultaneously decreases the incentives to postpone the ongoing harvesting decision.

We proceed as follows. In Section 2 a new and, in comparison to the current literature, more general, characterization of the optimal rotation problem with amenity valuation under forest value stochasticity is formulated. Section 3 provides the solution to the stochastic optimal rotation problem and characterizes its properties under a risk-neutral landowner. The Wicksellian single rotation problem and its relationship with the Faustmannian ongoing rotation problem is analyzed in Section 4. Our new numerical results are presented in Section 5 by using a model based on stochastic logistic growth of forest stand value. Finally, there is a brief concluding section.

**2. A characterization of the Faustmannian optimal rotation problem with amenity valuation.** In this section we characterize the optimal rotation problem when the forest owner is interested both in the expected cumulative harvest revenue and in the amenity services of forest stand under the circumstances where the forest stand value and thereby the amenity services as a continuous and nonnegative function of forest stand value are stochastic processes. More precisely, we proceed as follows. First, we specify the dynamics of the system and then define the optimal ongoing rotations problem, i.e., a stochastic impulse control problem.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered probability space, and assume that the dynamics of the stochastic forest stand value growth is described for all $t \geq 0$ by the generalized Itô-equation

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)\,ds + \int_0^t \sigma(X_s^\nu)\,dW_s - \sum_{\tau_k \leq t} \zeta_k, \tag{2.1}$$

where $\zeta_k$ denotes an admissible irreversible impulse (defined below), and $\mu : \mathbf{R}_+ \mapsto \mathbf{R}$ and $\sigma : \mathbf{R}_+ \mapsto \mathbf{R}$ are known to be sufficiently smooth, at least continuous, mappings guaranteeing the existence of a solution for the stochastic differential equation (2.1), cf. Borodin and Salminen [2002, pp. 47–48]. In line with standard models for the value dynamics of forest stand, we assume that the upper boundary $\infty$ of the state space is natural for the controlled diffusion in the absence of interventions, cf. Borodin and Salminen [2002, pp. 14–15]. For simplicity, we will also assume that the lower boundary 0 is unattainable for the controlled diffusion in the absence of harvesting. We would like to emphasize that although this assumption rules out the cases subject to a potentially finite time horizon, it is satisfied by most models applied in the literature on optimal forest rotation (logistic stochastic dynamics, geometric Brownian motion, etc.).

An *irreversible rotation policy*, i.e., an impulse control, for the system (2.1) is an infinite joint sequence $\nu = \{(\tau_k; \zeta_k)\}_{k\in\mathbf{N}}$, where $\{\tau_k\}_{k\in\mathbf{N}}$ is an increasing sequence of *rotation dates* for which $\tau_1 \geq 0$, and $\{\zeta_k\}_{k\in\mathbf{N}}$ denotes a sequence of irreversible harvests executed at the corresponding rotation dates. In line with standard models considering optimal rotation policies, we assume that the process is always instantaneously driven to the exogenously given generic initial state $x_0$ whenever the harvesting policy is carried out. This means that if the boundary at which the harvesting opportunity is exercised is given, then the size of the subsequent rotation policy is always a known constant. We assume that, for all $k \geq 1$, $X_{\tau_k+}^\nu = x_0 = X_{\tau_k-}^\nu - \zeta_k$ implying that the harvesting size is $\zeta_k = X_{\tau_k-}^\nu - x_0$. We denote as $\mathcal{V}$ the class of *admissible rotation policies* and assume that $\tau_k \to \infty$ almost surely for all $v \in \mathcal{V}$ and $x \in \mathbf{R}_+$. As usual, we denote the differential operator associated with the controlled process $X_t^\nu$ as

$$\mathcal{A} = \frac{1}{2}\,\sigma^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx}.$$

Given the stochastic dynamics of forest stand value growth described in (2.1) and our technical assumptions, we now define the *total expected cumulative present value of the net revenues* (*including harvesting and monetary value of amenity services*) *from the present up to a potentially infinite future* as

$$J^{\nu}(x) = \mathbf{E}_x \left[ \sum_{k=1}^{\infty} e^{-r\tau_k} (X^{\nu}_{\tau_k-} - c) + \int_0^{\infty} e^{-r_s} \pi(X^{\nu}_s)\, ds \right], \tag{2.2}$$

where $c > 0$ is the regeneration cost associated with harvesting and replanting and $\pi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a given continuous and nonnegative mapping measuring the *monetary value accrued from amenity services.* This is because we have not separated timber prices and forest volume. In line with the rotation literature, we assume that the revenues from harvesting are negative at the generic initial state, so that $x_0 < c$. Given the definition of the total expected cumulative present value of the net revenues $J^{\nu}(x)$ we consider the optimal rotation problem which is a stochastic impulse control problem

$$V(x) = \sup_{\nu \in V} J^{\nu}(x), \quad x \in \mathbf{R}_+ \tag{2.3}$$

and characterize the admissible rotation policy $\nu^* \in V$ for which this value is attained.

**3. Solution of the stochastic optimal rotation problem with amenity valuation.** In this section we solve the optimal ongoing rotation problem with amenity valuation in our general framework and characterize its properties. In order to accomplish that, we follow the approach applied in Alvarez [2001, 2004a, 2004b]. The key idea is to derive an associated functional from which the optimal rotation threshold and its value can be explicitly solved in terms of exogenous variables by relying solely on standard nonlinear programming techniques. More precisely, we proceed as follows: First we define the auxiliary Markovian functional, i.e., the value accrued from exercising the irreversible rotation policy at a given predetermined threshold, and show how it can be expressed in terms of the fundamental solutions of an associated ordinary differential equation. Using this auxiliary functional, we show how the optimal rotation threshold at which the irreversible harvesting opportunity should be exercised can be determined from an ordinary first order condition. Second, we present a set of sufficient conditions under which a unique optimal threshold exists and under which the associated functional constitutes the value of optimal rotation problem. Third, we present a set of plausible sufficient conditions under which the cumulative present value from amenity services is concave in terms of current state and negatively related to increasing volatility of forest stand value. We also state sufficient conditions for the negative relationship between volatility and the value of the landowner's optimal rotation policy. Finally, we characterize how under a certain set of conditions the properties of both the expected cumulative present value of the revenues accrued form amenity services and its marginal value depends on the precise nature of the amenity valuation function and, very importantly, what is the impact of the volatility of forest stand value on these concepts. We emphasize that answering the last question results in new findings which are in contrast to the existing literature.

Denote now for simplicity as $X_t$ the underlying process in the absence of interventions and define the mapping $F : \mathbf{R}_+ \mapsto \mathbf{R}$ for any constant threshold $y \in (x_0, \infty)$ recursively as (a running present value formulation)

$$F(x) = \mathbf{E}_x\left[e^{-r\tau_y}(F(x_0) + X_{\tau_y} - c) + \int_0^{\tau_y} e^{-rs}\pi(X_s)\,ds\right], \tag{3.1}$$

where $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$ denotes the next rotation date. Thus, $F(x)$ can be interpreted as the value accrued from exercising the irreversible rotation policy at a given but potentially suboptimal threshold $y \in (x_0, \infty)$. After the rotation opportunity has been utilized, the forest stand value process is instantaneously driven to the state $x_0$ at which it is restarted through replantation. In terms of the considered class of policies, we study the admissible *Faustmannian* rotation policy $\nu_y \in \mathcal{V}$ defined as

$$\nu_y = \{\tau_1^y, \ldots, \tau_k^y, \ldots ; \zeta_1^y, \ldots, \zeta_k^y, \ldots\}_{k \in \mathbf{N}} \tag{3.2}$$

where the cutting dates of the forest are defined as $\tau_{k+1}^y = \inf\{t \geq \tau_k^y : X_t^{\nu_y} \geq y\}$ and the constant harvests are defined as $\zeta_k^y = y - x_0$. The admissibility of the considered harvesting policy naturally implies that $F(x) \le V(x)$ for all $x \in \mathbf{R}_+$ and $y \in (x_0, \infty)$, where the value of the optimal harvesting policy $V(x)$ has been defined in (2.3). Thus, it is sufficient to establish conditions under which the direction of this inequality is reversed and, therefore, under which the proposed auxiliary functional actually constitutes the value of the optimal policy. It is also worth pointing out that although the controlled forest stand value process is not a diffusion because of the jumps created by the harvesting strategy, the controlled process behaves like a diffusion between consecutive harvesting dates so that the standard techniques apply within a rotation cycle.

Applying Dynkin's theorem, cf. Øksendal [2003, p. 124], to the expected cumulative present value

$$(R_r\pi)(x) = \mathbf{E}_x \int_0^\infty e^{-rs}\pi(X_s)\, ds$$

implies that (3.1) can be rewritten as

$$F(x) = (R_r\pi)(x) + \mathbf{E}_x\left[e^{-r\tau_y}(F(x_0) + X_{\tau_y} - c - (R_r\pi)(X_{\tau_y}))\right].$$

Given this expression, it is now a standard exercise in ordinary differential equations to establish that $F(x)$ can be rewritten as

(3.3)

$$F(x) = \begin{cases} F(x_0) + x - c & x \ge y \\ (R_r\pi)(x) + \big(F(x_0) + y - c - (R_r\pi)(y)\big)\dfrac{\psi(x)}{\psi(y)} & x < y, \end{cases}$$

where $\psi(x)$ denotes the increasing fundamental solution of the ordinary second order differential equation $(\mathcal{A}u)(x) = ru(x)$, cf. Borodin and Salminen [2002, p. 19]. Noticing that $F(x)$ is continuous for any possible choice of the harvesting threshold $y \in (x_0, \infty)$ implies that $F(x)$ satisfies the value-matching condition $F(y) = F(x_0) + y - c$ for any admissible harvesting threshold $y \in (x_0, \infty)$. Since this condition can be rewritten as $F(y) + c = F(x_0) + y$, we find that *independently of the optimality of the considered harvesting threshold* $y \in (x_0, \infty)$ the value of the considered cutting strategy has to satisfy the familiar balance equation stating that *the net harvesting returns* $y + F(x_0)$ (*current project value + future harvesting potential*) *have to be equal to the full costs* $c + F(y)$, *where* $c$ *measures the direct regeneration cost associated* *with harvesting and replanting and $F(y)$ measures the opportunity costs of harvesting.*

It is now clear from (3.3) that letting $x$ tend to $x_0$ yields after a simple algebraic manipulation

$$F(x_0) = \frac{\psi(x_0)(y - c - (R_r\pi)(y)) + \psi(y)(R_r\pi)(x_0)}{\psi(y) - \psi(x_0)}. \tag{3.4}$$

This describes the total expected cumulative present value of future harvests and amenity services accrued over a rotation cycle terminated whenever the underlying forest stand value process becomes greater than $y$. Inserting (3.4) into (3.3) then finally yields

$$F(x) = \begin{cases} (x - c) + (R_r\pi)(x_0) + u(y)\psi(x_0) & x \geq y \\ (R_r\pi)(x) + u(y)\psi(x) & x < y, \end{cases} \tag{3.5}$$

where

$$u(y) = \frac{y - c + (R_r\pi)(x_0) - (R_r\pi)(y)}{\psi(y) - \psi(x_0)}. \tag{3.6}$$

It is now clear from (3.5) that the arbitrary harvesting boundary $y$ affects the value $F(x)$ of the considered class of harvesting strategies only through the mapping $u(y)$. As intuitively is clear, this mapping plays a key role in the verification of the optimality of the proposed harvesting strategy. In order to justify this argument, we first establish the following lemma.

**Lemma 3.1.** *Assume that the mapping $u(y)$ attains a unique global maximum at $x^* \in (x_0, \infty)$, that $u(y)$ is nonincreasing on $(x^*, \infty)$, and define the mapping $F^*(x)$ as*

$$F^*(x) = \begin{cases} (x - c) + (R_r\pi)(x_0) + u(x^*)\psi(x_0) & x \geq x^* \\ (R_r\pi)(x) + u(x^*)\psi(x) & x < x^*. \end{cases} \tag{3.7}$$

*Then, $F(x)$ is continuously differentiable on $\mathbf{R}_+$, twice continuously differentiable on $\mathbf{R}_+ \setminus \{x^*\}$, and $F^*(x) \geq F(x)$ for all $x \in \mathbf{R}_+$. Moreover, the optimal threshold $x^*$ satisfies the ordinary first order condition*

$$\frac{1 - (R_r\pi)'(x^*)}{\psi'(x^*)} = u(x^*). \tag{3.8}$$

*Proof.* The assumed differentiability of the mapping $u(y)$ implies that an interior maximum $x^*$ necessarily satisfies the condition $u'(x^*) = 0$ which is equivalent with (3.8). Proving the alleged results is now straightforward. □

Lemma 3.1 establishes that whenever the auxiliary mapping $u(x)$ attains a unique maximum at $x^* \in (x_0, \infty)$ and is nonincreasing on $(x^*, \infty)$, the potentially suboptimal value $F^*(x)$ constitutes the maximal cumulative harvesting yield which can be attained by applying an admissible threshold harvesting policy $\nu_y$ described in (3.2). Lemma 3.1 also demonstrates that if a maximizing threshold exists, then the cumulative harvesting yield satisfies both the *value-matching condition* $F^*(x) = x^* - c + F^*(x_0)$ and the *smooth fit principle* $F^{*\prime}(x^*) = 1$, cf. Dixit and Pindyck [1994, p. 109]. It is worth observing that (3.5) clearly indicates that this argument can actually be reversed. More precisely, the expected cumulative present value of the net revenues, including harvesting and monetary value of amenity services, from the present up to a potentially infinite future $F(x)$ is continuously differentiable on its domain only if the harvesting threshold $y$ satisfies the ordinary first order condition (3.8).

Unfortunately, the conditions of Lemma 3.1 are not sufficient for establishing the optimality of the proposed time invariant rotation policy characterized by the sequence of harvesting dates $\tau_{k+1}^{x^*}$ and the constant impulses $\zeta_k^{x^*} = x^* - x_0$. Therefore, further analysis is required for the verification of the optimality of these quantities. A set of general sufficiency conditions under which the auxiliary functional coincides with the value of the considered optimal rotation problem are now summarized in

**Theorem 3.2.** *Assume that the mapping $u(y)$ attains a unique global maximum at $x^* \in (x_0, \infty)$ and that $(1 - (R_r\pi)'(x))/\psi'(x)$ is nonincreasing on $\mathbf{R}_+$. Then, the expected cumulative present value of the net revenues, including harvesting and monetary value of amenity services, from the present up to a potentially infinite future reads as $V(x) = F^*(x)$ and $\nu_{x^*}$ is the optimal rotation policy.*

*Proof.* The alleged result essentially follows from the fact that the proposed value function is continuously differentiable on $\mathbf{R}_+$, twice continuously differentiable on $\mathbf{R}_+\setminus\{x^*\}$, and satisfies the quasi-variational inequalities $\min\{rF^*(x)-\pi(x)-(AF^*)(x),\ F^*(x)-F^*(x_0)-(x-c)\}=0$. The complete proof is available upon request. $\square$

Theorem 3.2 states a set of typically satisfied conditions under which the auxiliary mapping $F^*(x)$ constitutes the maximal attainable harvesting yield and under which $\nu_{x^*}$ is the optimal harvesting policy. An interesting implication of the findings of Theorem 3.2, which relates the marginal value of the optimal rotation policy to *Tobin's marginal* $q$, is now presented in our next corollary.

**Corollary 3.3.** *Assume that the conditions of Theorem* 3.2 *are satisfied. Then, the marginal value of the optimal rotation policy can be expressed as*

$$V'(x) = 1-\psi'(x)\left(\frac{1-(R_r\pi)'(x)}{\psi'(x)}-\frac{1-(R_r\pi)'(x^*)}{\psi'(x^*)}\right)^+, \tag{3.9}$$

*where the marginal value of the optimal rotation policy satisfies the inequality* $V'(x)\le 1$ *for all* $x\in\mathbf{R}_+$.

*Proof.* Standard differentiation of (3.7) yields that

$$\begin{aligned} V'(x)-1 &= F^{*\prime}(x)-1 \\ &= \begin{cases} 0 & x\ge x^* \\ (R_r\pi)'(x)-1+(1-(R_r\pi)'(x^*))\,\dfrac{\psi'(x)}{\psi'(x^*)} & x<x^*, \end{cases} \end{aligned}$$

from which the alleged result follows by the monotonicity of $(1-(R_r\pi)'(x))/\psi'(x)$. $\square$

Corollary 3.3 demonstrates that rotation is suboptimal as long as the marginal value falls short of the marginal harvesting returns which, in the present example, are equal to one. This observation is of interest since it indicates how the considered optimal rotation problem is connected to traditional capital theoretic models considering rational investment behavior (for excellent surveys of this classical $q$-theory of investment, cf. Caballero [1999]).

Although the conditions of Theorem 3.2 are considerably weak, establishing the required monotonicity of the function $(1-(R_r\pi)'(x))/\psi'(x)$ is not always an easy task. Fortunately, there is a class of easily verifiable conditions, which can be expressed in terms of the mapping $\theta(x)=\pi(x)+\mu(x)-r(x-c)$ measuring the expected growth rate of the total monetary value of a forest stand, under which the requirements of Theorem 3.2 are always fulfilled. These conditions are summarized in the following.

**Theorem 3.4.** *Assume that the expected growth rate of the total monetary value of a forest stand $\theta(x)$ is nonincreasing on $\mathbf{R}_+$ and satisfies the inequalities $\lim_{x\downarrow 0}\theta(x) > 0 > \lim_{x\to\infty}\theta(x)$. Then, a unique optimal harvesting threshold $x^* = \arg\max_{y\geq x_0}\{u(y)\}$ exists, the expected cumulative present value of the net revenues, including harvesting and monetary value of amenity services, from the present up to a potentially infinite future reads as $F^*(x) = V(x)$, and $\nu_{x^*}$ is the optimal rotation policy.*

*Proof.* See Appendix A. □

Theorem 3.4 states in terms of the expected growth rate of the total monetary value of a forest stand a set of sufficient conditions under which the auxiliary functional $F^*(x)$ constitutes the value of the optimal rotation problem (2.3) and under which the optimal rotation threshold is $x^*$. It is worth noticing that one of the main implications of Theorem 3.4 is that the monotonicity of the expected growth rate of the total monetary value of a forest stand is the principal determinant of the optimal harvesting policy and its value. Next we characterize the properties of the present value of amenity services in the ongoing rotation case and the expected total value of a forest stand in the single rotation case.

**Theorem 3.5.** *Assume that the mappings $\mu(x)$, $\sigma(x)$ are continuously differentiable with Lipschitz-continuous derivatives, that $\sigma'(x)$ is bounded, that the drift $\mu(x)$ is concave, and that the flow of rev-*

*enues $\pi(x)$ is nondecreasing and concave. Then, the expected cumulative present value accrued from amenity services is nondecreasing and concave, that is, $(R_r\pi)'(x) \geq 0$ and $(R_r\pi)''(x) \leq 0$ for all $x \in \mathbf{R}_+$ and higher forest stand value volatility decreases the expected cumulative present value $(R_r\pi)(x)$. Moreover, the expected total value of timber revenues over a single rotation cycle of length $t$*

$$v(t,x) = \mathbf{E}_x\left[e^{-rt}(X_t - c) + \int_0^t e^{-rs}\pi(X_s)\,ds\right]$$

*is increasing and concave in $x$, that is, $v_x(t,x) \geq 0$ and $v_{xx}(t,x) \leq 0$ for all $(t,x) \in \mathbf{R}_+^2$. Finally, higher forest stand value volatility decreases $v(t,x)$.*

*Proof.* See Appendix B. □

According to Theorem 3.5, under a set of typically satisfied conditions, the expected cumulative present value of the revenues accrued from amenity services is concave as a function of the current state $x$ and the sign of the relationship between volatility and $(R_r\pi)(x)$ is unambiguously negative. Moreover, we also find that the expected total value of revenues accrued over a single rotation cycle are concave as a function of the current state $x$ and, therefore, negatively dependent on the volatility of the forest stand value process.

The precise nature of the amenity valuation function, which turns out to be important later on, depends on the nature of amenity services like timber and deer, timber and water, wildlife diversity and visual aesthetics which might differ depending on the specific situation, cf. Calish et al. [1978] and Swallow et al. [1990]. Therefore, we next characterize how, under a certain set of conditions, the properties of both the expected cumulative value and the expected marginal cumulative value, accrued from amenity services, depend on the precise nature of the monetary valuation of amenities. Moreover, and importantly, we also demonstrate the impact of increased forest stand value volatility on these concepts. First, we present the following

**Theorem 3.6.** *Assume that $\sigma(x) = \sigma x$, where $\sigma > 0$ is a known constant, and that the mapping $\mu(x)$ is concave and continuously differentiable with a Lipschitz-continuous derivative. Assume also that $\mu'(x)$*

*is convex, that the flow of revenues* $\pi(x)$ *is nonincreasing and convex, and that* $\pi'(x)$ *is concave. Then,*

(i) *the expected cumulative present value accrued from amenity services* $(R_r\pi)(x)$ *is nonincreasing and convex, and higher volatility increases its value* $(R_r\pi)(x)$*, while*

(ii) *the marginal expected cumulative present value accrued from amenity services* $(R_r\pi)'(x)$ *is nondecreasing and concave, and higher volatility decreases its value. Moreover, higher volatility decreases the value* $(R_r\pi)(x) - (R_r\pi)(x_0)$ *for all* $x \in [x_0, \infty)$*.*

*Proof.* See Appendix C. □

Theorem 3.6 states a set of conditions under which increased volatility unambiguously increases the expected cumulative present value of the returns accrued from amenity services but simultaneously decreases its marginal value. This result is of interest since it demonstrates that, even when the sign of the relationship between increased volatility and the expected cumulative revenues may be positive, the impact on the difference $(R_r\pi)(x) - (R_r\pi)(x_0)$ determining the nature of the optimal ongoing rotation policy, may be negative. This means that the properties of the optimal ongoing rotation policy may be opposite to the ones of an optimal single rotation strategy. This result emphasizes the impact of the sequentiality of the ongoing rotation policies on their comparative static properties. Since the opportunity to harvest later in the future is lost in the single rotation case immediately after the irreversible decision has been made, the decision maker cannot pursue a policy where the value process would be sustained at a level where the flow of returns accrued from amenity services is high. Since the decision maker can, however, implement such a policy in the ongoing rotation case, we observe that the impact of increased volatility on the optimal policy is very different in cases where volatility affects the expected cumulative present value of the returns accrued from amenity services in an opposite way it affects its marginal value. Later on in Section 5 we demonstrate numerically that increased forest value volatility, under the specifications which lie in conformity with the assumptions presented in Theorem 3.6, will decrease the Faustmannian threshold unlike what we know about the current literature, see e.g., Reed [1993]. A second result which is reverse to Theorem 3.6 findings is now established in the following

**Theorem 3.7.** *Assume that $\sigma(x) = \sigma x$, where $\sigma > 0$ is a known constant, and that the mapping $\mu(x)$ is concave and continuously differentiable with Lipschitz-continuous derivative. Assume also that $\mu'(x)$ is convex, that the flow of revenues $\pi(x)$ is nondecreasing and concave, and that $\pi'(x)$ is convex. Then,*

(i) *the expected cumulative present value accrued from amenity services $(R_r\pi)(x)$ is nondecreasing and concave, and higher volatility decreases its value $(R_r\pi)(x)$, while*

(ii) *the marginal expected cumulative present value accrued from amenity services $(R_r\pi)'(x)$ is nonincreasing and convex, and higher volatility increases its value. Moreover, higher volatility increases the value $(R_r\pi)(x) - (R_r\pi)(x_0)$ for all $x \in [x_0, \infty)$.*

*Proof.* The proof is analogous with the proof of Theorem 3.6. □

In contrast to Theorem 3.6, Theorem 3.7 states a set of conditions under which increased volatility unambiguously decreases the expected cumulative present value of the returns accrued from amenity services but simultaneously increases its marginal value. The interpretation of this result is analogous to Theorem 3.6. However, we would like to emphasize again the role of the specific functional form of amenity as the principal determinant of the impact of volatility on both the expected cumulative present value and the marginal expected cumulative present value accrued from amenity services. Finally, in the ongoing rotation framework with amenity valuation we show the following.

**Theorem 3.8.** *If the value function $F^*(x)$ is convex, then increased volatility increases the value of the optimal rotation policy.*

*Proof.* The alleged result is a direct implication of the observation that if $\widehat{F}(x)$ denotes the value of the optimal rotation policy in the presence of a more volatile forest value dynamics characterized by the volatility coefficient $\hat{\sigma}(x) > \sigma(x)$ then $\widehat{F}(x) \geq \widehat{F}(x_0) + x - c$ for all

$x \in \mathbf{R}_+$ and $(\mathcal{A}\widehat{F})(x) - r\widehat{F}(x) + \pi(x) \leq 1/2(\sigma^2(x) - \hat{\sigma}^2(x))\widehat{F}''(x) \leq 0$ outside the optimal harvesting threshold. $\square$

Notice, however, that Theorem 3.8 does not characterize the relationship between volatility and the optimal rotation threshold $x^*$. As we will see in an explicit example later, the relationship between volatility and the optimal rotation threshold $x^*$ depends heavily on the nature of the amenity valuation function and the boundary behavior of the diffusion process modeling the value of the forest stand. Moreover, it is clear from Corollary 3.3, Theorem 3.6 and Theorem 3.7 that the form of the marginal value $V'(x)$ is actually the principal determinant of the sign of the relationship between increased volatility and the optimal exercise threshold. If $V'(x)$ is convex (concave), then increased volatility increases (decreases) its value and, therefore, increases (decreases) the threshold at which the irreversible harvesting opportunity is optimally exercised. This result is of interest since it illustrates that the impact of increased volatility on the value and on the exercise threshold of the optimal rotation policy may be opposite. Notice that the main reason for these findings is that the sign of the relationship between higher volatility and the value is a second order property depending on the sign of $V''(x)$ while the sign of the relationship between higher volatility and the optimal exercise threshold is a third order property depending on the sign of $V'''(x)$. This observation is explicitly illustrated in Section 5 using numerical calculations.

**4. Wicksellian stochastic single rotation problem with amenity valuation.** The analysis of the previous section has been devoted to the stochastic *Faustmannian ongoing rotation problem with amenity valuation*, i.e., the stochastic Hartmanian model. A closely related problem arising in the literature of optimal forest rotation is the so-called *Wicksellian single rotation problem.* In this section we explicitly characterize the associated Wicksellian single rotation problem with amenity valuation in terms of optimal exercise threshold, optimal rotation date and the resulting value function and compare the value function with the one from the Faustmannian ongoing rotation problem. The objective of the forest owner is now to determine the rotation date $\tau$ for which the maximum

$$\widehat{V}(x) = \sup_{\tau} \mathbf{E}_x\left[e^{-r\tau}(X_\tau - c) + \int_0^\tau e^{-rs}\pi(X_s)\,ds\right] \tag{4.1}$$

is attained. The strong Markov property of diffusions now implies that problem (4.1) can be re-expressed as

$$\widehat{V}(x) = (R_r\pi)(x) + \sup_{\tau} \mathbf{E}_x[e^{-r\tau}(X_\tau - c - (R_r\pi)(X_\tau))].$$

This demonstrates how the value of the optimal policy can be decomposed into the first RHS term describing the returns accrued from keeping the forest unharvested (the monetary value of the return accrued from amenity services) and into the second RHS term describing the expected returns accrued from exercising the harvesting opportunity later in the future. Our main result on this problem is summarized in the following

**Theorem 4.1.** *Assume that the expected growth rate of the total monetary value of a forest stand $\theta(x)$ satisfies the inequalities $\lim_{x\downarrow 0}\theta(x) > 0 > \lim_{x\to\infty}\theta(x)$. Then, there is a unique optimal exercise threshold $\tilde{x} = \arg\max\{x - c - (R_r\pi)(x)/\psi(x)\}$ satisfying the ordinary first order condition*

$$\frac{1 - (R_r\pi)'(\tilde{x})}{\psi(\tilde{x})} = \tilde{u}(\tilde{x}),$$

*where $\tilde{u}(x) = (y - c - (R_r\pi)(x))/\psi(x)$. Moreover, $\tau_W^* = \inf\{t \geq 0 : X(t) \geq \tilde{x}\}$ is the optimal rotation date, and the value function reads as*

$$\widehat{V}(x) = (R_r\pi)(x) + \psi(x)\sup_{y\geq x}\{\tilde{u}(y)\} = \begin{cases} x - c, & x \geq \tilde{x} \\ (R_r\pi)(x) + \tilde{u}(\tilde{x})\psi(x) & x < \tilde{x}, \end{cases} \tag{4.2}$$

*Proof.* The alleged result follows directly from Alvarez [2004b, Theorem 4, part (A)]. $\square$

Theorem 4.1 states a set of sufficient conditions under which the Wicksellian single rotation problem can be solved explicitly. It is worth emphasizing that the conditions of Theorem 4.1 are significantly weaker than the conditions of Theorem 3.4 for the Faustmannian ongoing rotation problem. Unfortunately, it is not easy (if possible at all) to extend the results of Theorem 4.1 to the ongoing rotation case, although we know that these values are closely related to each other.

**5. Numerical illustrations.** After having characterized the value of both the Faustmannian ongoing rotation problem and the Wicksellian single rotation problem in the presence of amenity valuation under stochastic forest stand value, we now numerically illustrate our results in an explicitly parametrized model based upon logistic growth of the forest stand value. We study the role of amenity valuation function, interest rate and growth and volatility of forest value under much more general assumptions than in the earlier literature about the relationship between the forest value and the monetary value of amenity services. Our main emphasis here is to show the importance of the precise nature of amenity valuation for the relationship between forest value volatility and the optimal harvesting strategy.

In order to consider the impact of stochastic fluctuations of the value of the forest stand on the optimal rotation policy and its value, we now assume that in the absence of interventions the controlled diffusion evolves according to a stochastic logistic growth model described by the stochastic differential equation

$$dX_t = \mu X_t(1-\gamma X_t)\,dt + \sigma X_t\,dW_t, \quad X_0 = x, \tag{5.1}$$

where $\mu$, $\gamma$ and $\sigma$ are exogenously given positive constants and $dW_t$ is the increment of a Wiener process driving the underlying stochastic dynamics. In order to get an explicit solution, we also assume that $\pi(x) = \pi/x$. Applying now Itô's theorem to the mapping $Y_t = 1/X_t$ yields

$$dY_t = (\mu\gamma - (\mu-\sigma^2)Y_t)\,dt - \sigma Y_t\,dW_t, \quad Y0 = 1/x.$$

Consequently, we find that

$$\mathbf{E}_x\left[\frac{1}{X_t}\right] = \frac{\mu\gamma}{\mu-\sigma^2} + e^{-(\mu-\sigma^2)t}\left[\frac{1}{x} - \frac{\mu\gamma}{\mu-\sigma^2}\right]$$

which provided that the absence of speculative bubbles condition $r + \mu > \sigma^2$ guaranteeing the finiteness of the value of the optimal policy is satisfied, implies that the expected cumulative present value of the returns accrued from amenity services can be expressed as

$$(R_r\pi)(x) = \frac{\pi}{(r+\mu-\sigma^2)x} + \frac{\mu\gamma\pi}{r(r+\mu-\sigma^2)}. \tag{5.2}$$

Standard differentiation yields that

$$\frac{\partial (R_r\pi)(x)}{\partial\sigma} = \left[\frac{\pi}{x} + \frac{\mu\gamma\pi}{r}\right]\frac{2\sigma}{(r+\mu-\sigma^2)^2} > 0, \tag{5.3}$$

$$\frac{\partial (R_r\pi)'(x)}{\partial\sigma} = -\frac{\pi}{x^2}\,\frac{2\sigma}{(r+\mu-\sigma^2)^2} < 0, \tag{5.4}$$

and for all $x \in (x_0, \infty)$,

$$\frac{\partial}{\partial\sigma}\left((R_r\pi)(x) - (R_r\pi)(x_0)\right) = \frac{\pi(x_0 - x)2\sigma}{(r+\mu-\sigma^2)^2 x x_0} < 0. \tag{5.5}$$

This means, as was proved in Theorem 3.6, that higher volatility increases the expected cumulative present value of the returns accrued from amenity services, but decreases its marginal value. Moreover, in line with the findings of Theorem 3.6, we find that higher volatility decreases the difference $(R_r\pi)(x) - (R_r\pi)(x_0)$ and, therefore, increases the incentives to harvest. It is also worth noticing that since in the present example $\theta'(x) = -\pi/x^2 + \mu(1-2\gamma x) - r$ we observe that the conditions of our Theorem 3.4 under which a unique optimal harvesting threshold exists are satisfied whenever the inequality $27\mu^2\gamma^2\pi \geq (\mu - r)^3$ holds. It is clear from the analysis above that the auxiliary mapping $u(x)$ now reads as

$$u(x) = \frac{(r+\mu-\sigma^2)x_0 x(x-c) + \pi(x-x_0)}{(r+\mu-\sigma^2)x_0 x(\psi(x)-\psi(x_0))}, \tag{5.6}$$

where $\psi(x) = x^\delta M(\delta, 2\delta + (2\mu/\sigma^2), (2^{\mu\gamma}/\sigma^2)\,x)$, $\delta = 1/2 - \mu/\sigma^2 + \sqrt{((1/2) - (\mu/\sigma^2))^2 + 2r/\sigma^2}$, and $M$ is the confluent hypergeometric function, cf. Alvarez and Shepp [1998] and Willasen [1998]. In this example, the optimal Wicksellian threshold $\tilde{x}$ is the threshold maximizing the functional

$$u_W(x) = \frac{r(r+\mu-\sigma^2)x(x-c) - r\pi + \mu\gamma\pi x}{r(r+\mu-\sigma^2)x^{1+\delta}M(\delta, 2\delta+(2\mu/\sigma^2),(2\mu\gamma/\sigma^2)x)}. \tag{5.7}$$

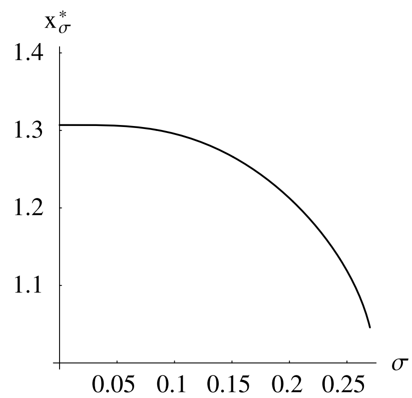


FIGURE 1. The optimal Faustmannian rotation threshold $x^*$.

The optimal Faustmannian ongoing rotation threshold $x^*$ is illustrated as a function of the volatility coefficient $\sigma$ of the forest stand value in Figure 1 (in the case when $\gamma = 0.2$, $r = 3.5\%$; $\mu = 4\%$, $c = 2$, $\pi = 1$ and $x_0 = 1$). The optimal Wicksellian single rotation threshold $\tilde{x}$ is, in turn, illustrated as a function of the volatility coefficient $\sigma$ of the forest stand value in Figure 2 (in the case when $\gamma = 0.2$, $r = 3.5\%$, $\mu = 4\%$, $c = 2$, $\pi = 1$ and $x_0 = 1$).

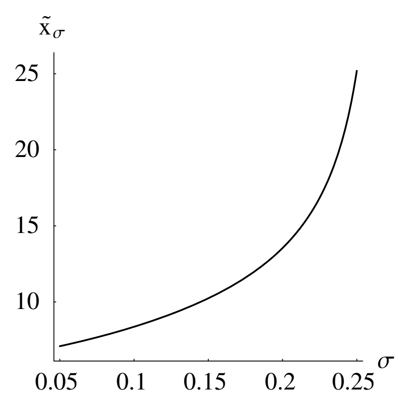


FIGURE 2. The optimal Wicksellian rotation threshold $\tilde{x}$.

TABLE 1. The rotation thresholds $x^*$ and $\tilde{x}$ and their difference.

| $\sigma$ | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 |
|---|---|---|---|---|---|
| $x^*$ | 1.33 | 1.32 | 1.29 | 1.23 | 1.15 |
| $\tilde{x}$ | 7.06 | 8.29 | 10.05 | 13.02 | 21.56 |
| $\tilde{x}-x^*$ | 5.73 | 6.97 | 8.76 | 11.79 | 20.41 |

In order to illustrate our results numerically as well, the optimal rotation thresholds $x^*$ and $\tilde{x}$ are illustrated as functions of the volatility coefficient $\sigma$ in Table 1 in the case when $\gamma = 0.2$, $r = \mu = 4\%$, $c = 2$, $\pi = 1$ and $x_0 = 1$.

According to these new findings presented in Table 1, higher volatility increases the Wicksellian threshold while, interestingly, it simultaneously decreases the Faustmannian threshold. This can be interpreted as follows. As we established in (5.3), higher volatility increases the expected cumulative revenues accrued from amenity services and, therefore, increases the incentives to postpone the harvesting decision in the Wicksellian case. In contrast, in (5.4) we found that increased volatility decreases the marginal value of the expected cumulative revenues accrued from amenity services and, therefore, increases the incentives to harvest by decreasing the expected marginal returns associated to the amenity services. We would like to point out that this new finding is the main qualitative difference between the Wicksellian and Faustmannian rotation policies in the presence of amenity services showing that the precise nature of the monetary value of amenities has an important qualitative effect as well.

**6. Concluding remarks.** We have provided a new and more general approach compared with the existing literature to study the optimal rotation policy with amenity valuation under forest stand value uncertainty both in the case of the ongoing rotation and single rotation frameworks. In the small existing literature the authors have assumed different formulations for stochastic forest stand value of amenity values without an explicit justification. First, we have followed a more plausible approach by postulating stochastic forest stand value growth and assuming that the monetary value of amenities is a continuous and nonnegative function of the forest stand value. Second, instead of relying on the ordinary Hamilton-Jacobi-Bellman approach, our analysis is based on the idea that the value of optimal harvesting problem can be derived explicitly from an associated functional equation in terms of exogenous parameters by relying on nonlinear programming techniques. Third, we have characterized how both the expected cumulative present value and the expected marginal cumulative present value, accrued from amenity services, depend on the precise nature of amenity valuation function and how forest stand value volatility affects these concepts in order to analyze the comparative static properties of the landowner's optimal policy. Finally, we have illustrated our results numerically in a stochastic model based on logistic forest stand value growth.

We have found that, under nonnegative monetary value accrued from amenity services, higher forest stand value volatility increases the Wicksellian harvesting threshold, while it decreases the Faustmannian harvesting threshold. This, to our knowledge, new finding is essentially based on the fact that although higher volatility may increase the expected cumulative monetary present value, affecting the Wicksellian single rotation optimal policy, it may simultaneously decrease the marginal expected cumulative monetary present value, which in turn affects the Faustmannian ongoing rotation optimal policy. Thus, in this case higher forest stand value volatility increases the incentives to postpone the single rotation harvesting decision, while it simultaneously decreases the incentives to postpone the ongoing harvesting decision.

There are naturally several directions towards which our analysis would be of interest to extend. First, it would be of interest to analyze the optimal rotation policy in the general case where the forest value is decomposed into timber price and volume. Such an approach would admit the separate analysis of the relationship between amenity value and timber volume. Second, given the perpetuity of the time horizon of the considered rotation problems, it would be of interest to analyze how interest rate uncertainty affects the optimal policy, cf. Alvarez and Koskela [2005]. Unfortunately, both of these extensions are outside the scope of the present paper and left for future research.

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## Appendix

### A. Proof of Theorem 3.4.

*Proof.* It is sufficient to establish that our assumptions imply that the functional $(1-(R_r\pi)'(x))/\psi'(x)$ is decreasing on $\mathbf{R}_+$ and that $x^* = \arg\max_{x\geq x_0}\{u(x)\}$ exists and is unique. In order to accomplish these tasks, consider first the mapping $\tilde{u}:\mathbf{R}_+\mapsto\mathbf{R}$ defined as

$$\tilde{u}(x) = \frac{x-c-(R_r\pi)(x)}{\psi(x)}. \tag{A.1}$$

As it is known from the literature on linear diffusions, the Green function $G_r(x,y)$ of the diffusion $X_t$ can be expressed in terms of the increasing fundamental solution $\psi(x)$ and the decreasing fundamental solution $\varphi(x)$ as (cf. Borodin and Salminen [2002, p. 19])

$$G_r(x,y) = \begin{cases} B^{-1}\psi(x)\varphi(y), & x<y \\ B^{-1}\psi(y)\varphi(x), & x\geq y \end{cases}$$

where $B = (\psi'(x)\varphi(x)-\varphi'(x)\psi(x))/S'(x) > 0$ denotes the constant Wronskian determinant of the fundamental solutions and

$$S'(x) = \exp\left(-\int\frac{2\mu(x)}{\sigma^2(x)}\,dx\right)$$

denotes the density of the scale function of $X$. Our assumption $\pi\in\mathcal{L}^1(\mathbf{R}_+)$ implies that the expected cumulative present value $(R_r\pi)(x)$ can be expressed in terms of the Green function $G_r(x,y)$ and the density

$m'(y) = 2/(\sigma^2(y)S'(y))$ of the speed measure as

$$(R_r\pi)(x) = B^{-1}\varphi(x)\int_0^x \psi(y)\pi(y)m'(y)\,dy + B^{-1}\psi(x)\int_x^\infty \varphi(y)\pi(y)m'(y)\,dy$$

implying that

$$\frac{\psi'(x)}{S'(x)}(R_r\pi)(x) - \frac{(R_r\pi)'(x)}{S'(x)}\psi(x) = \int_0^x \psi(y)\pi(y)m'(y)\,dy. \tag{A.2}$$

In order to analyze the mapping $(x-c)/\pi(x)$, we first observe that if $g : \mathbf{R}_+ \mapsto \mathbf{R}$ is a twice continuously differentiable mapping on $\mathbf{R}_+$, then applying Dynkin's theorem to $g(x)$ yields

$$\mathbf{E}_x\left[e^{-r\tau(a,b)}g(X_{\tau(a,b)})\right] = g(x) + \mathbf{E}_x\int_0^{\tau(a,b)} e^{-rs}L(X_s)\,ds, \tag{A.3}$$

where $L(x) = (\mathcal{A}g)(x) - rg(x)$ and $\tau(a,b) = \inf\{t \geq 0 : X_t \notin (a,b)\}$ denotes the first exit time of the underlying process $X_t$ from the open set $(a,b)$ for which $0 < a < b < \infty$, i.e., an open set with compact closure in $\mathbf{R}_+$. Since the left-hand side of (A.3) satisfies the ordinary differential equation $(\mathcal{A}u)(x) - ru(x) = 0$ subject to the boundary conditions $u(a) = g(a)$ and $u(b) = g(b)$, we observe that

$$\mathbf{E}_x\left[e^{-r\tau(a,b)}g(X_{\tau(a,b)})\right] = g(a)\,\frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} + g(b)\,\frac{\hat{\psi}(x)}{\hat{\psi}(b)},$$

where $\hat{\psi}(x) = \psi(x) - \psi(a)\varphi(x)/\varphi(a)$ and $\hat{\varphi}(x) = \varphi(x) - \varphi(b)\psi(x)/\psi(b)$. On the other hand, since the expected cumulative present value appearing in the right-hand side of (A.3) satisfies the ordinary differential equation $(\mathcal{A}v)(x) - rv(x) + L(x) = 0$ subject to the boundary conditions $v(a) = v(b) = 0$, we observe that

$$\mathbf{E}_x\int_0^{\tau(a,b)} e^{-r_s}L(X_s)\,ds = \widehat{B}^{-1}\hat{\varphi}(x)\int_a^x \hat{\psi}(y)L(y)m'(y)\,dy + \widehat{B}^{-1}\hat{\psi}(x)\int_x^b \hat{\varphi}(y)L(y)m'(y)\,dy,$$

where $\widehat{B} = B\hat{\varphi}(a)/\varphi(a) = B\hat{\psi}(b)/\psi(b)$ denotes the constant Wronskian of $\hat{\psi}(x)$ and $\hat{\varphi}(x)$. Combining these results then finally imply that (A.3) can be re-expressed as

$$\begin{aligned} g(x) = g(a)\,\frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} + g(b)\,\frac{\hat{\psi}(x)}{\hat{\psi}(b)} - \widehat{B}^{-1}\hat{\varphi}(x)\int_a^x \hat{\psi}(y)L(y)m'(y)\,dy \\ - \widehat{B}^{-1}\hat{\psi}(x)\int_x^b \hat{\varphi}(y)L(y)m'(y)\,dy, \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} \frac{g(x)}{\hat{\psi}(x)} = \frac{g(a)\hat{\varphi}(x)}{\hat{\varphi}(a)\hat{\psi}(x)} + \frac{g(b)}{\hat{\psi}(b)} - \widehat{B}^{-1}\frac{\hat{\varphi}(x)}{\hat{\psi}(x)}\int_a^x \hat{\psi}(y)L(y)m'(y)\,dy \\ - \widehat{B}^{-1}\int_x^b \hat{\varphi}(y)L(y)m'(y)\,dy. \end{aligned}$$

Standard differentiation and reordering terms then yields

$$\frac{g'(x)}{S'(x)}\,\hat{\psi}(x) - \frac{\hat{\psi}'(x)}{S'(x)}\,g(x) = \int_a^x \hat{\psi}(y)L(y)m'(y)\,dy - \frac{Bg(a)}{\varphi(a)}.$$

Thus, if $g(x)$ is bounded at the origin and $L \in \mathcal{L}^1(\mathbf{R}_+)$, then letting $a \downarrow 0$ yields

$$\text{(A.4)} \qquad \frac{g'(x)}{S'(x)}\,\psi(x) - \frac{\psi'(x)}{S'(x)}\,g(x) = \int_0^x \psi(y)L(y)m'(y)\,dy.$$

Applying now (A.4) to the mapping $(x-c)$ gives rise to

$$\text{(A.5)} \qquad \frac{\psi(x)}{S'(x)} - \frac{\psi'(x)}{S'(x)}\,(x-c) = \int_0^x \psi(y)(\mu(y) - r(y-c))m'(y)\,dy.$$

Combining (A.5) with (A.2) now implies that
(A.6)

$$\frac{1-(R_r\pi)'(x)}{S'(x)}\,\psi(x) - \frac{\psi'(x)}{S'(x)}\,(x-c-(R_r\pi)(x)) = \int_0^x \psi(y)\theta(y)m'(y)\,dy.$$

Consider now the functional $I : \mathbf{R}_+ \mapsto \mathbf{R}$ defined as

$$I(x) = \int_0^x \psi(y)\theta(y)m'(y)\,dy.$$

Our assumptions on the mapping $\theta(x)$ imply that $I(x) > 0$ for all $x \leq x_1$, where $x_1 = \sup\{x \in \mathbf{R}_+ : \theta(x) = 0\}$. Assume now that $x > K > x_1$. Then, the additivity of $I(x)$ and the monotonicity of $\theta(x)$ imply that

$$I(x) \leq I(K) + \frac{\theta(K)}{r}\left[\frac{\psi'(x)}{S'(x)} - \frac{\psi'(K)}{S'(K)}\right] \downarrow -\infty$$

as $x \to \infty$, since $\psi'(x)/S'(x) \to \infty$ as $x \to \infty$ (by the assumed boundary behavior of $X_t$ at $\infty$). Therefore, the continuity of the functional $I(x)$ implies that equation $I(x) = 0$ has at least one root $\tilde{x} \in (x_1, \infty)$. However, since $I'(x) = \psi(x)\theta(x)m'(x) < 0$ on $(x_1, \infty)$, we observe that $\tilde{x}$ is unique by the monotonicity of $I(x)$. Moreover, $I(x) \gtreqless 0$ for all $x \lesseqgtr \tilde{x}$ implying that $\tilde{x} = \arg\max\{\tilde{u}(x)\}$. Given this observation, consider the mapping

$$U_1(x) = \frac{(1-(R_r\pi)'(x))}{\psi'(x)}\,(\psi(x)-\psi(x_0))-(x-c-(R_r\pi)(x)+(R_r\pi)(x_0)).$$

It is now clear that $U_1(x_0) = c - x_0 > 0$ and $U_1(x) < 0$ for all $x \geq \tilde{x}$. Therefore, equation $U_1(x) = 0$ has at least one root $x^* \in (x_0, \tilde{x})$. Since

$$U_1'(x) = \frac{d}{dx}\left[\frac{(1-(R_r\pi)'(x))}{\psi'(x)}\right]\,(\psi(x) - \psi(x_0))$$

we find that the uniqueness of $x^*$ is guaranteed whenever $(1 - (R_r\pi)'(x))/\psi'(x)$ is decreasing on $\mathbf{R}_+$. To prove that this is indeed the case, we first observe that, since $(\mathcal{A}(R_r\pi))(x) - r(R_r\pi)(x) + \pi(x) = 0$ and $(\mathcal{A}\psi)(x) = r\psi(x)$, we find by combining (A.2) and (A.5) that

$$\frac{d}{dx}\left[\frac{1-(R_r\pi)'(x)}{\psi'(x)}\right] = \frac{2S'(x)}{\sigma^2(x)\psi'^2(x)}\left[\theta(x)\frac{\psi'(x)}{S'(x)} - r\int_0^x \psi(y)\theta(y)m'(y)\,dy\right]. \tag{A.7}$$

Invoking now the assumed monotonicity of $\theta(x)$ yields

$$\frac{d}{dx}\left[\frac{1-(R_r\pi)'(x)}{\psi'(x)}\right] \leq \frac{2S'(x)}{\sigma^2(x)\psi'^2(x)}\left[\theta(x)\frac{\psi'(x)}{S'(x)} - r\theta(x)\int_0^x \psi(y)m'(y)\,dy\right] = 0$$

demonstrating that $(1 - (R_r\pi)'(x))/\psi'(x)$ is decreasing on $\mathbf{R}_+$ and, therefore, completing the proof of our theorem. $\square$

**B. Proof of Theorem 3.5.**

*Proof.* Assume that our conditions are satisfied and denote as $X_t^x$ the (flow of the) solution of the stochastic differential equation

$$X_t^x = x + \int_0^t \mu(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dW_s.$$

Given the smoothness-assumptions on the drift $\mu(x)$ and the infinitesimal diffusion coefficient $\sigma(x)$ imply that the process $Y_t^1 = \partial X_t^x/\partial x$ constitutes the strong solution of the stochastic differential equation, cf. Øksendal [2003, pp. 68–71]

$$Y_t^1 = 1 + \int_0^t \mu'(X_s^x)Y_s^1\,ds + \int_0^t \sigma'(X_s^x)Y_s^1\,dW_s$$

and can, therefore, be expressed as

$$Y_t^1 = \exp\left(\int_0^t \mu'(X_s^x)\,ds\right)M_t,$$

where

$$M_t = \exp\left(\int_0^t \sigma'(X_s^x)\,dW_s - \frac{1}{2}\int_0^t \sigma'^2(X_s^x)\,ds\right)$$

is a positive exponential martingale due to the assumed boundedness of $\sigma'(x)$. Given this observation, we find by ordinary differentiation that

$$(R_r\pi)'(x) = \mathbf{E}^P \int_0^1 e^{\int^s +0(\mu'(X_t^x)-r)\,dt}\,\pi'(X_s^x)\,M_s\,ds. \tag{B.1}$$

Defining the equivalent measure $\mathbf{Q}$ by the likelihood ratio $d\mathbf{Q} = d\mathbf{P} = M_t$ implies that (B.1) can be re-expressed as

$$(R_r\pi)'(x) = \mathbf{E}^{\mathbf{Q}} \int_0^\infty e^{\int_0^s(\mu'(X_t^x)-r)\,dt}\,\pi'(X_s^x)\,ds \geq 0, \tag{B.2}$$

where the underlying stochastic dynamics evolve under the equivalent measure $Q$ according to the stochastic differential equation

$$X_t^x = x + \int_0^t (\mu(X_s^x) + \sigma'(X_s^x)\sigma(X_s^x))\, ds + \int_0^t \sigma(X_s^x)\, d\widetilde{W}_s.$$

Since the flow of the solution $X_t^x$ of the stochastic differential equation is monotonically increasing as a function of the initial state $x$, the assumed concavity of the drift $\mu(x)$ and the function $\pi(x)$ proves the alleged concavity of the expected cumulative present value $(R_r\pi)(x)$. Establishing that the expected present value $u(t,x) = \mathbf{E}^P[e^{-rt}(X_t^x - c)]$ is nondecreasing and concave as a function of $x$ is completely analogous. The monotonicity and concavity of $v(t,x)$ as a function of the current state $x$ follows then from the analogous properties for $(R_r\pi)(x)$ and $u(t,x)$.

It remains to establish that increased volatility decreases these values. To accomplish this task, denote now the expected cumulative present value of the revenue flow $\pi(x)$ in the presence of the more volatile process as $(\widetilde{R}_r\pi)(x)$ and the differential operator associated to the more volatile process, $\widetilde{X}_t$, as

$$\tilde{\mathcal{A}} = \frac{1}{2}\,\tilde{\sigma}^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx}.$$

Since $(R_r\pi)(x)$ satisfies the ordinary second order differential equation

$$\frac{1}{2}\,\sigma^2(x)(R_r\pi)''(x) + \mu(x)(R_r\pi)'(x) - r(R_r\pi)(x) + \pi(x) = 0,$$

we find by the concavity of $(R_r\pi)(x)$ that

$$(\tilde{\mathcal{A}}(R_r\pi))(x) - r(R_r\pi)(x) + \pi(x) = \frac{1}{2}\,(\tilde{\sigma}^2(x) - \sigma^2(x))(R_r\pi)''(x) \le 0.$$

Consequently, we have

$$\begin{aligned}
0 &\le \mathbf{E}_x\left[e^{-rT_n}(R_r\pi)(\widetilde{X}_{T_n})\right] \\
&= (R_r\pi)(x) + \mathbf{E}_x \int_0^{T_n} e^{-r_s}\left[(\tilde{\mathcal{A}}(R_r\pi))(\widetilde{X}_s) - r(R_r\pi)(\widetilde{X}_s)\right] ds \\
&\le (R_r\pi)(x) - \mathbf{E}_x \int_0^{T_n} e^{-r_s}\,\pi(\widetilde{X}_s)\, ds,
\end{aligned}$$

where $T_n = n \wedge \inf\{t \geq 0 : \widetilde{X}_t \notin (n^{-1}, n)\}$, $n \in \mathbf{N}$, is an almost surely finite stopping time. Letting $n \to \infty$ and invoking monotone convergence then implies that $(R_r\pi)(x) \geq (\widetilde{R}_r\pi)(x)$. Thus, increased volatility decreases the expected cumulative present value accrued from amenity services. To establish that increased volatility decreases $u(t,x)$, we first denote as $\tilde{u}(t,x)$ the value in the presence of the more volatile process $\widetilde{X}_t$. It is now a direct consequence of the Feynman-Kač theorem that $u(t,x)$ is the solution of the boundary value problem, cf. Øksendal [2003, p. 143]

$$(\mathcal{A}u)(t,x) - ru(t,x) = \frac{\partial u}{\partial t}(t,x), \quad (t,x) \in \mathbf{R}^2_+,$$
$$u(0,x) = x - c, \quad x \in \mathbf{R}_+.$$

The concavity of $u(t,x)$ now implies that

$$(\mathcal{A}u)(t,x) - ru(t,x) - u_t(t,x) = \frac{1}{2}(\tilde{\sigma}^2(x) - \sigma^2(x))u_{xx}(t,x) \leq 0.$$

Applying Itô's theorem on the functional $e^{-rs}u(t-s, \widetilde{X}_s)$ and invoking the boundary condition $u(0,x) = x - c$ then yields

$$\begin{aligned}\mathbf{E}[e^{-rt}(\widetilde{X}_t - c)] &= u(t,x) + \mathbf{E}\int_0^t e^{-rs} \\ &\quad \times \Big((\tilde{\mathcal{A}}u)(t-s, \widetilde{X}_s) - ru(t-s, \widetilde{X}_s) - u_s(t-s, \widetilde{X}_s)\Big)\, ds \\ &\leq u(t,x) = \mathbf{E}[e^{-rt}(X_t - c)],\end{aligned}$$

proving the alleged negativity of the sign of the relationship between volatility and $u(t,x)$. Proving that increased volatility decreases $v(t,x)$ is analogous.

## C. Proof of Theorem 3.6.

*Proof.* It is clear that, in line with our observations in Theorem 3.5, our assumptions now imply that

$$\frac{\partial X_t^x}{\partial x} = e^{\int_0^t \mu'(X_s^x)\, ds} M_t,$$

where $M_t = Y_t^{-1} = e^{-1/2\sigma^2 t + \sigma W_t}$ is a positive exponential martingale. Combining this result with the concavity of the drift implies that $X_t^x$ is increasing and concave as a mapping of its initial state $x$. Hence, the assumed monotonicity and convexity properties of the revenue flow $\pi(x)$ imply that $\pi(X_t^x)$ is decreasing and convex as a function of the initial state $x$ and that $(R_r\pi)(x)$ is decreasing and convex as a function of the initial state $x$ as well. The positivity of the sign of the relationship between increased volatility and the expected cumulative present value, accrued from amenity services, can now be established by relying on analogous arguments with the ones presented in the proof of Theorem 3.5.

Consider now the marginal expected cumulative revenues accrued from amenity services. Standard differentiation now yields

$$(R_r\pi)'(x) = \mathbf{E}\int_0^\infty e^{-\int_0^s (r-\mu'(\overline{X}_t^x))\,dt}\,\pi'(\overline{X}_s^x)\,ds,$$

where $\overline{X}_t^x$ evolves under $\mathbf{P}$, by the strong uniqueness of solutions, according to the dynamics described by the stochastic differential equation

$$d\overline{X}_t^x = \left(\mu(\overline{X}_t^x) + \sigma^2\overline{X}_t^x\right)dt + \sigma\overline{X}_t^x\,dW_t, \quad \overline{X}_0^x = x.$$

It is now a straightforward consequence of the analysis above that the assumed concavity of the drift $\mu(x)$ implies that $\mu(x) + \sigma^2 x$ is concave as well and, therefore, that $\overline{X}_t^x$ is increasing and concave as a mapping of its initial state $x$. This finding together with the assumed monotonicity and convexity of the factor $\mu'(x)$ now implies that $e^{\int_0^t \mu'(\overline{X}_s^x)\,ds}$ is decreasing and convex as a function of the initial state $x$. Similarly, the assumed monotonicity and concavity of $\pi'(x)$ implies that the factor $\pi'(\overline{X}_t^x)$ is increasing and concave in $x$. Combining these two observations then finally implies that the mapping $(R_r\pi)'(x)$ is nondecreasing and concave in $x$. The negativity of the sign of the relationship between increased volatility and the marginal value $(R_r\pi)'(x)$ follows from the proof of Theorem 3.5. It remains to establish that increased volatility decreases the difference $(R_r\pi)(x) - (R_r\pi)(x_0)$. The fundamental theorem of integral calculus implies that

$$(R_r\pi)(x) - (R_r\pi)(x_0) = \int_{x_0}^x (R_r\pi)'(y)\,dy$$

from which the result follows by the negativity of the sign of the relationship between increased volatility and the marginal value $(R_r\pi)'(x)$.

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