Homework #1

Problem 1.8

1. $5\ddot{x} + 6\dot{x} + \sin(t)x = \cos(t^2)$, $x(0)=1$, $\dot{x}(0)=\pi$

ordinary, linear, variable-coefficient, inhomogeneous

2^{nd} order, IVP

$\begin{cases} \text{independent variable} : t \\ \text{dependent variable} : x \end{cases}$

3. $5\ddot{x} + 6\dot{x} + \sin(t)x = \cos(x)$, $x(0)=1$, $\dot{x}(0)=\pi$

ordinary, nonlinear, 2^{nd} order, IVP

$\begin{cases} \text{independent variable} : t \\ \text{dependent variable} : x \end{cases}$

5. $\cos(x)\cdot\dot{x} + e^t x = x^2$, $x(2)=e$

$\begin{cases} \text{independent variable} : t \\ \text{dependent variable} : x \end{cases}$

ordinary, nonlinear, 1^{st} order, IVP

7. $\cos(t)\cos(\dot{x}) + e^t x = x$, $x(7)=e$

$\begin{cases} \text{independent variable} : t \\ \text{dependent variable} : x \end{cases}$

ordinary, nonlinear, 1^{st} order, IVP

9. $\dot{x} + e^{\pi}x = 2$, $x(0)=1$

$\begin{cases} \text{independent variable} : \text{(maybe) } t \\ \text{dependent} : x \end{cases}$, ordinary, linear, constant-coefficient inhomogeneous, 1^{st} order, IVP

Problem 1.12

1. $x(t) = \cos(t)$, $\ddot{x} + x = 0$

$\ddot{x} + x = -\cos(t) + \cos(t) = 0$ ✓

2. $x(t) = 3\cos(t) + 2\sin(t)$, $\ddot{x} + x = 0$

$\ddot{x} + x = -3\cos(t) - 2\sin(t) + 3\cos(t) + 2\sin(t) = 0$ ✓

5. $x(t) = c_1 t + \frac{c_2}{t}$, $t^2\ddot{x} + t\dot{x} - x = 0$, $t \neq 0$

$t^2\ddot{x} + t\dot{x} - x = t^2\left(2\frac{c_2}{t^3}\right) + t\left(c_1 - \frac{c_2}{t^2}\right) - \left(c_1 t + \frac{c_2}{t}\right)$

$= \frac{2c_2}{t} + c_1 t - \frac{c_2}{t} - c_1 t - \frac{c_2}{t} = 0$ ✓

7. $x(t) = t + \frac{1}{t} + \frac{t}{2\ln t}$, $t^2\ddot{x} + t\dot{x} - x = t$, $t \neq 0$

$t^2\ddot{x} + t\dot{x} - x = t^2\left(\frac{2}{t^3} + \frac{2-\ln t}{2t\ln^3 t}\right) + t\left(1 - \frac{1}{t^2} + \frac{\ln t - 1}{2\ln^2 t}\right) - \left(t + \frac{1}{t} + \frac{t}{2\ln t}\right)$

$= \cancel{\frac{2}{t}} + \frac{t(2-\ln t)}{2\ln^3 t} + \cancel{t} - \cancel{\frac{1}{t}} + \frac{t(\ln t - 1)}{2\ln^2 t} - \cancel{t} - \cancel{\frac{1}{t}} - \frac{t}{2\ln t}$

$= \frac{2t - \cancel{t\ln t} + \cancel{t\ln^2 t} - \cancel{t\ln t} - \cancel{t\ln^2 t}}{2\ln^3(t)} = \frac{t(1-\ln t)}{\ln^3(t)} \neq t$

9. $x(t) = 3\cos 2t$, $x^2\ddot{x} + \frac{\dot{x}}{x} = -4\cos^3(2t) - 2\tan 2t$

$x^2\ddot{x} + \frac{\dot{x}}{x} = 9\cos^2 2t \cdot (-12\cos 2t) + \frac{-6\sin 2t}{3\cos 2t}$

$= -108\cos^3 2t - 2\tan 2t \neq -4\cos^3(2t) - 2\tan 2t$

Problem 2.3

$\frac{dx}{dt} = -\alpha x$, $x_{\frac{1}{2}} = 5600$ years

$\cancel{\frac{dx}{dt} + \alpha x = 0}$

1. $\begin{cases} \frac{dx}{dt} + \alpha x = 0 & (\alpha \text{ is a constant}) \\ x(0) = x(0) \end{cases}$

2. $\cancel{x(5600) = \frac{1}{2}(0)}$

$$\frac{dx}{dt} + \alpha x = 0 \Rightarrow x = Ce^{-\alpha t}$$

$$x(0) = x(0) \Rightarrow x(0) = C_1$$

$$\text{So, } \underline{x(t) = x(0) \cdot e^{-\alpha t}}$$

$$x(5600) = \frac{1}{2}x(0) \Rightarrow \frac{1}{2}x(0) = x(0)e^{-5600\alpha}$$

$$\Rightarrow \alpha = \frac{\ln 2}{5600}$$

$$\text{So, } x(t) = x(0)e^{-\frac{\ln 2}{5600}t}$$

$$x(t_1) = \cancel{30\%}\ 0.3x(0) \Rightarrow 0.3x(0) = x(0)e^{-\frac{\ln 2}{5600}t_1}$$

$$\Rightarrow t_1 = 9727 \text{ years}$$

Problem 2.4

$\frac{dx}{dt} + t_0 x = 0 \Rightarrow \frac{dx}{dt} = -t_0 x \Rightarrow \frac{dx}{x} = -t_0\, dt \Rightarrow \ln x = -t_0 t + C$

$\Rightarrow x = Ce^{-t_0 t}$

So if $\lambda = -t_0$, then it works

Problem 2.6

1. $\frac{dx}{dt} + \frac{x}{100} = 0 \Rightarrow$ ~~$x = x(0)e$~~

$x = C_1 e^{-\frac{1}{100}t}$

$x(0) = 100 \Rightarrow 100 = C_1$

So, $x(t) = 100e^{-\frac{1}{100}t}$

let $x(t_1) = 50 \Rightarrow 50 = 100e^{-\frac{1}{100}t_1} \Rightarrow t_1 = 100\ln 2 = 69.3$ min

2. $\frac{dx}{dt} = -\frac{F}{W}x \Rightarrow \frac{dx}{dt} + \frac{F}{W}x = 0 \Rightarrow x = C_2 e^{-\frac{F}{W}t}$

$x(0) = C \Rightarrow C = C_2$

So, $x(t) = Ce^{-\frac{F}{W}t}$

~~Prom~~ Problem 2.8

1. $\frac{dx}{dt} = kx \quad k \to \frac{1}{time}$

$x = C \cdot e^{-kt} \quad k \to \frac{1}{time}$

$= x(0)e^{-kt}$

2. $\begin{cases} x(0) = 100000 \\ x(1) = 150000 \end{cases}$, $x = x(0)e^{-kt}$

$x(t) = 100000\, e^{-kt}$

$x(1) = 150000 = 100000\, e^{-k} \Rightarrow k = \ln\frac{2}{3} = -0.405$

3. $x(t) = e^{0.405t}$

suppose the number of people in the Earth is constant and equals to 70×10^9

$x(t_1) = e^{0.405 t_1} = 70 \times 10^9$

$\Rightarrow t_1 = 61.66$ days

Problem 2.10

3. $\dot{x} + x = \cos t + 2\sin t$

$x_h = C_1 e^{-t}$

$x_p = C_2 \cos t + C_3 \sin t$

$\Rightarrow \dot{x}_p + x_p = -C_2 \sin t + C_3 \cos t + C_2 \cos t + C_3 \sin t$

$= (C_2 + C_3)\cos t + (C_3 - C_2)\sin t = \cos t + 2\sin t$

$\Rightarrow \begin{cases} C_2 + C_3 = 1 \\ C_3 - C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_2 = -\frac{1}{2} \\ C_3 = \frac{3}{2} \end{cases}$

so, $x(t) = C_1 e^{-t} - \frac{1}{2}\cos t + \frac{3}{2}\sin t$

9. $\dot{x} + 3x = 3t^2 + \cos 2t$

$x_h = C_1 e^{-3t}$

$\cancel{x_p}$

$g(t) = 3t^2 + \cos 2t$

$g'(t) = 6t - 2\sin 2t$

$g''(t) = 6 - 4\cos 2t$

$g'''(t) = 8\sin 2t$

$g^{(4)}(t) = 16\cos 2t$

$g^{(5)}(t) = -32\sin 2t$ x

So, $x_p(t) = C_1 g(t) + C_2 g'(t) + C_3 g''(t) + C_4 g'''(t) + C_5 g^{(4)}(t)$

$$\cancel{\dot{x}_p + 3x_p = 3C_1 t^2 + C_1\cos 2t + 6C_2 t - 2C_2\sin 2t + 6C_3 - 4C_3\cos 2t + 8C_4\sin 2t + 16C_5\cos 2t = 3t^2 + \cos 2t}$$

$$\Rightarrow \begin{cases} C_1 = 1 \\ \cancel{C_2 = 0} \\ \cancel{C_3 = 0} \\ \cancel{C_4 = 0} \\ \cancel{C_5 = 0} \end{cases} \cancel{\Rightarrow x_p(t) = g(t) = 3t^2 + \cos 2t}$$

~~So, $x(t) = C_1 e^{-3t} + 3t^2 + \cos 2t$~~

$$\dot{x}_p + 3x_p = 9C_1 t^2 + (6C_1 + 18C_2)t + 6C_2 + 18C_3 + (-2C_2 + 16C_4 + 3C_1 - 12C_3 + 48C_5)\cos 2t + (-2C_1 + 8C_3 - 32C_5 - 6C_2 + 24C_4)\sin 2t$$

$$\Rightarrow \begin{cases} C_1 = \frac{1}{3} \\ C_2 = -\frac{1}{9} \\ C_3 = \frac{1}{27} \\ C_4 = -\frac{1}{4\times 20} \\ C_5 = \frac{17}{9\times 13\times 24} \end{cases} \qquad x(t) = C_1 e^{-3t} + x_p$$

13. $\dot{x}+2x=2e^{-2t}+\cos t$

$x_h=C_1e^{-2t}$

$x_p=C_2te^{-2t}+C_3\cos t+C_4\sin t$

$$\dot{x}_p+2x_p=\overset{C_2e^{-2t}-\cancel{2C_2te^{-2t}}}{\cancel{-2C_2e^{-2t}}}-C_3\sin t+C_4\cos t+\overset{\cancel{2C_2te^{-2t}}}{\cancel{2C_2e^{-2t}}}+2C_3\cos t+2C_4\sin t$$

$$=(2C_4-C_3)\sin t+(2C_3+C_4)\cos t+C_2e^{-2t}$$

$$=2e^{-2t}+\cos t$$

$$\Rightarrow\begin{cases}C_2=2\\C_3=\frac{2}{5}\\C_4=\frac{1}{5}\end{cases}\Rightarrow x_p=2te^{-2t}+\frac{2}{5}\cos t+\frac{1}{5}\sin t$$

so, $x(t)=x_h+x_p=C_1e^{-2t}+2te^{-2t}+\frac{2}{5}\cos t+\frac{1}{5}\sin t$