Homework #2

Problem 2.10

14. $\dot{x} + 2x = 2e^{-2t} + \cos t + t^3$

$x_n = C_1 e^{-2t}$

$x_p = C_2 t e^{-2t} + C_3 \cos t + C_4 \sin t + C_5 t^3 + C_6 t^2 + C_7 t + C_8$

$\dot{x}_p + 2x_p = C_2 e^{-2t} + (2C_4 - C_3)\sin t + (2C_3 + C_4)\cos t + 2C_5 t^3 + (3C_5 + 2C_6) t^2 + 2(C_6 + C_7) t + C_7 + 2C_8$

$$\Rightarrow \begin{cases} C_2 = 2 \\ C_3 = \frac{2}{5} \\ C_4 = \frac{1}{5} \\ C_5 = \frac{1}{2} \\ C_6 = -\frac{3}{4} \\ C_7 = \frac{3}{4} \\ C_8 = \frac{3}{8} \end{cases}$$

$\Rightarrow x_p = 2te^{-2t} + \frac{2}{5}\cos t + \frac{1}{5}\sin t + \frac{1}{2}t^3 - \frac{3}{4}t^2 + \frac{3}{4}t + \frac{3}{8}$

$\Rightarrow x(t) = x_n + x_p$

Promblem ②

solve $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$, $x > 0$

$M(x,y) = \frac{y}{x} + 6x$, $N(x,y) = \ln x - 2$

$\frac{\partial M}{\partial y} = \frac{1}{x}$, $\frac{\partial N}{\partial x} = \frac{1}{x}$ $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

$$\Rightarrow \begin{cases} \varphi(x,y) = \int M\,dx = y\ln x + 3x^2 + f(y) \\ \varphi(x,y) = \int N\,dy = y(\ln x - 2) + \cancel{f(x)}\, g(x) \end{cases}$$

$\Rightarrow \varphi(x,y) = y\ln x + 3x^2 - 2y$

so, $d\varphi = 0 \Rightarrow \varphi = C \Rightarrow y\ln x + 3x^2 - 2y = C$

Problem ③

Solve $(2x-y)dx + (2y-x)dy = 0$, $y(1)=3$

$M(x,y) = 2x-y$, $N(x,y) = 2y-x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

$\cancel{\Rightarrow \varphi(x,y) = \int N dy = 2xy - \frac{1}{2}y^2 + f(x)}$

$\Rightarrow \begin{cases} \varphi(x,y) = \int M dx = x^2 - xy + f(y) \\ \varphi(x,y) = \int N dy = y^2 - xy + g(x) \end{cases}$

$\Rightarrow \varphi(x,y) = x^2 + y^2 - xy$

So, $d\varphi = 0 \Rightarrow \varphi = c \Rightarrow x^2+y^2-xy = c$

substitude $y(1)=3 \Rightarrow 1+9-3 = c \Rightarrow c = 7$

So, $x^2+y^2-xy = 7$

Problem ④

Find an integrating factor to solve the give equation:

$y' = e^{2x} + y - 1$

$\frac{dy}{dx} = e^{2x}+y-1 \Rightarrow (e^{2x}+y-1)dx - dy = 0$

$M = e^{2x}+y-1$, $N = -1 \Rightarrow \frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0 \Rightarrow$ not exact

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{-1} = -1$ only x

So, $\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} = e^{\int(-1)dx} = e^{-x}$

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-1}{e^{2x}+y-1}$ not only y

Problem ⑤

$y' + \frac{2}{3}y = 1 - \frac{1}{2}t,\ y(0) = y_0$

$y_h(t) = Ce^{-\frac{2}{3}t}$

$y_p(t) = C_2 t + C_3$

$y_p' + \frac{2}{3}y_p = C_2 + \frac{2}{3}(C_2 t + C_3) = 1 - \frac{1}{2}t$

$\Rightarrow \begin{cases} C_2 = -\frac{3}{4} \\ C_3 = \frac{21}{8} \end{cases} \Rightarrow y_p(t) = -\frac{3}{4}t + \frac{21}{8}$

So, $y(t) = y_h + y_p = C_1 e^{-\frac{2}{3}t} - \frac{3}{4}t + \frac{21}{8}$

substitute $y(0) = y_0 \Rightarrow y_0 = C_1 + \frac{21}{8} \Rightarrow C_1 = y_0 - \frac{21}{8}$

$\Rightarrow y(t) = \left(y_0 - \frac{21}{8}\right)e^{-\frac{2}{3}t} - \frac{3}{4}t + \frac{21}{8}$

① $y_0 > \frac{21}{8}$, monotonically decreasing $(0, +\infty)$

② $y_0 = \frac{21}{8}$, 0

③ $y_0 < \frac{21}{8}$, monotonically increasing $(-\infty, 0)$

monotonically decreasing $(-\infty, +\infty)$

So, ①② must cross the t-axis

for ③

~~$y'(t) = -\frac{2}{3}\left(y_0 - \frac{21}{8}\right)e^{-\frac{2}{3}t} - \frac{3}{4} = 0 \Rightarrow$~~

when $\left(\left(y_0 - \frac{21}{8}\right)e^{-\frac{2}{3}t}\right)' = \frac{3}{4}$, $y(t)$ is of maximum value

$\Rightarrow t_0 = -\frac{3}{2}\ln\frac{9}{8\left(\frac{21}{8} - y_0\right)}$

let maximum value $= 0 \Rightarrow y(t_0) = 0 \Rightarrow$ (next page)

$y(t_0) = (y_0 - \frac{21}{8}) e^{-\frac{2}{3}(-\frac{3}{2}\ln\frac{9}{8(\frac{21}{8}-y_0)})} - \frac{3}{4}(-\frac{3}{2}\ln\frac{9}{8(\frac{21}{8}-y_0)}) + \frac{21}{8}$

$= (y_0 - \frac{21}{8}) \frac{9}{8(\frac{21}{8}-y_0)} + \frac{9}{8}\ln\frac{9}{8(\frac{21}{8}-y_0)} + \frac{21}{8}$

$= \frac{3}{2} + \frac{9}{8}\ln\frac{9}{8(\frac{21}{8}-y_0)} = 0$

$\Rightarrow y_0 = \frac{1}{8}(21 - 9e^{\frac{4}{3}})$

and now $y(t)$ touches but doesn't cross the t-axis

Problem ⑥

Find an integrating factor

$[4(\frac{x^3}{y^2}) + \frac{3}{y}]dx + [3(\frac{x}{y^2}) + 4y]dy = 0$

$M = 4(\frac{x^3}{y^2}) + \frac{3}{y}$, $N = 3(\frac{x}{y^2}) + 4y \Rightarrow \frac{\partial M}{\partial y} = -8\frac{x^3}{y^3} - \frac{3}{y^2}$, $\frac{\partial N}{\partial x} = \frac{3}{y^2}$

$\Rightarrow$ not exact

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-8\frac{x^3}{y^3} - \frac{6}{y^2}}{3(\frac{x}{y^2}) + 4y} \Rightarrow$ not only x

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{8\frac{x^3}{y^3} + \frac{6}{y^2}}{4\frac{x^3}{y^2} + \frac{3}{y}} = \frac{2}{y} \Rightarrow$ only y

so, $\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$

multi

~~Prob~~ Problem 2.28

$$(2x^2t^2+3x^3\sin^2 t)\frac{dx}{dt}+(2x^3t+2x^4\sin t\cdot\cos t)=0$$

$$\cancel{\Rightarrow (2x^3t+2x^4\sin t\cos t)dt+(2x^2t^2+3x^3\sin^2 t)dx=0}$$

$$\Rightarrow (2x^2t^2+3x^3\sin^2 t)dx+(2x^3t+2x^4\sin t\cos t)dt=0$$

let $M(x,t)=2x^2t^2+3x^2\sin^2 t$, $N(x,t)=2x^3t+2x^4\sin t\cos t$

$\frac{\partial M}{\partial t}=4x^2t+6x^2\sin t\cos t$, $\frac{\partial N}{\partial x}=6x^2t+8x^3\sin t\cos t$

$\frac{\partial M}{\partial t}\neq\frac{\partial N}{\partial x}\Rightarrow$ not exact

multiplied by $\frac{1}{x}$: $(2xt^2+3x^2\sin^2 t)dx+(2x^2t+2x^3\sin t\cos t)dt=0$

$M(x,t)=2xt^2+3x^2\sin^2 t$, $M(x,t)=2x^2t+2x^3\sin t\cos t$

$\frac{\partial M}{\partial t}=4xt+6x^2\sin t\cos t$, $\frac{\partial N}{\partial x}=4xt+6x^2\sin t\cos t$

$\frac{\partial M}{\partial t}=\frac{\partial N}{\partial x}\Rightarrow$ exact

so, $\varphi(x,t)=\int M\,dx=x^2t^2+x^3\sin^2 t+f(t)$

$\varphi(x,t)=\int N\,dt=x^2t^2+x^3\sin^2 t+f(x)$

$\Rightarrow \varphi(x,t)=x^2t^2+x^3\sin^2 t$

Problem 2.31

1. When velocity reaches the terminal velocity, $\frac{dv}{dt}=0$

so, $m\frac{dv}{dt}=mg-kv^2\Rightarrow mg-kV_{term}^2=0\Rightarrow V_{term}=\sqrt{\frac{mg}{k}}$

2. $\cancel{mv'=mg-kv^2\Rightarrow v'+\frac{k}{m}v^2.}$ → (next page)

2. ~~$\frac{dv(t)}{dt} =$~~

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 \Rightarrow \frac{dv}{g-\frac{k}{m}v^2} = dt \Rightarrow \frac{\sqrt{m}}{2\sqrt{gk}} \ln \frac{1+\frac{\sqrt{k}v}{\sqrt{mg}}}{1-\frac{\sqrt{k}v}{\sqrt{mg}}} = t + C$$

2/10

~~$\Rightarrow \ln \frac{1+\sqrt{k}v}{}$~~

$$\Rightarrow \ln \frac{1+\frac{\sqrt{k}v}{\sqrt{mg}}}{1-\frac{\sqrt{k}v}{\sqrt{mg}}} = \frac{2\sqrt{gk}\,t}{\sqrt{m}} + C \Rightarrow \frac{1+\frac{\sqrt{k}v}{\sqrt{mg}}}{1-\frac{\sqrt{k}v}{\sqrt{mg}}} = C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} \quad C\ (C_1>0)$$

$$\Rightarrow v = \sqrt{\frac{mg}{k}} \cdot \frac{C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} - 1}{C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} + 1}$$

3. $$v = v_{term} \cdot \frac{C_1 e^{2\sqrt{\frac{kg}{m}}t} - 1}{C_1 e^{2\sqrt{\frac{kg}{m}}t} + 1}$$

$$\begin{cases} v(0) = 600 \Rightarrow 600 = 300\,\frac{C_1 - 1}{C_1 + 1} \Rightarrow C_1 = -3 \\ v_{term} = 300 \end{cases}$$

so, $$v = 300 \cdot \frac{-3e^{2\sqrt{\frac{kg}{m}}t} - 1}{-3e^{2\sqrt{\frac{kg}{m}}t} + 1}$$

suppose $g = 10\,m/s^2$, from $v_{term} = \sqrt{\frac{mg}{k}} = 300 \Rightarrow k = \frac{5}{9}$

so, $$v = 300\,\frac{-3e^{\frac{t}{15}} - 1}{-3e^{\frac{t}{15}} + 1}$$

2. $\cancel{\frac{dv(t)}{dt} =}$ 2/10

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 \Rightarrow \frac{dv}{g - \frac{k}{m}v^2} = dt \Rightarrow \frac{\sqrt{m}}{2\sqrt{gk}} \ln \frac{1 + \frac{\sqrt{k}\,v}{\sqrt{mg}}}{1 - \frac{\sqrt{k}\,v}{\sqrt{mg}}} = t + C$$

$\cancel{\Rightarrow \ln 1 + \sqrt{k}v}$

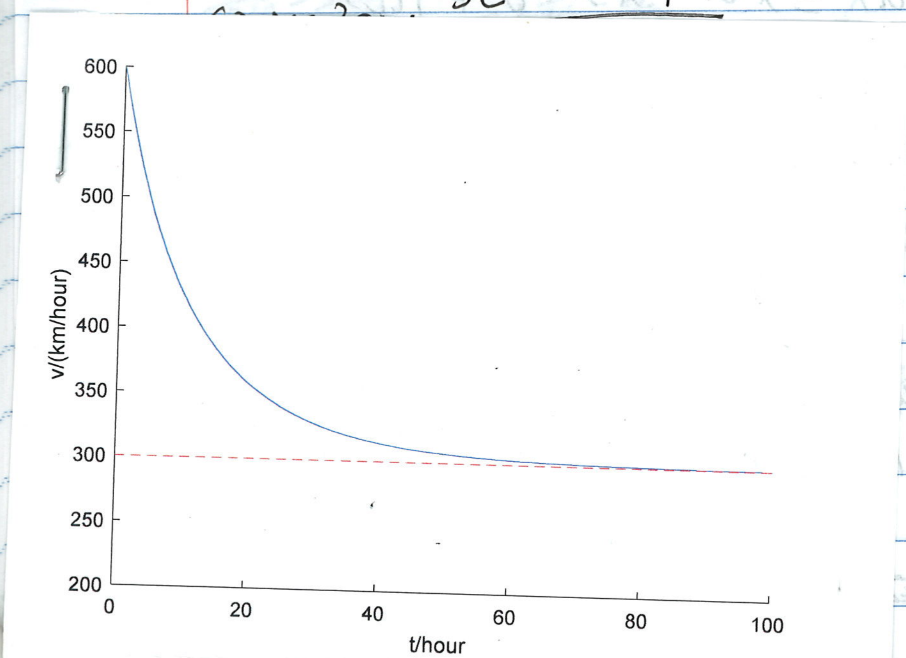
$$\Rightarrow \ln \frac{1 + \frac{\sqrt{k}\,v}{\sqrt{mg}}}{1 - \frac{\sqrt{k}\,v}{\sqrt{mg}}} = \frac{2\sqrt{gk}\,t}{\sqrt{m}} + C \Rightarrow \frac{1 + \frac{\sqrt{k}\,v}{\sqrt{mg}}}{1 - \frac{\sqrt{k}\,v}{\sqrt{mg}}} = C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} \quad (C_1 > 0)$$

$$\Rightarrow v = \sqrt{\frac{mg}{k}} \cdot \frac{C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} - 1}{C_1 e^{\frac{2\sqrt{gk}\,t}{\sqrt{m}}} + 1}$$

3. $$v = v_{term} \cdot \frac{C_1 e^{2\sqrt{\frac{kg}{m}}\,t} - 1}{C_1 e^{2\sqrt{\frac{kg}{m}}\,t} + 1}$$

$$\begin{cases} v(0) = 600 \Rightarrow 600 = 300 \dfrac{C_1 - 1}{C_1 + 1} \Rightarrow C_1 = -3 \\ v_{term} = 300 \end{cases}$$

$$-3e^{2\sqrt{\frac{kg}{m}}\,t} - 1$$



$300 \Rightarrow k = \frac{5}{9}$