Equation sheet - AME 30314 - Jiali Lu

Differential Equations classification and nomenclature

ODE (ordinary differential equation): unknown func depends on only one independent variable

PDE (partial diff eqn): unknown func depends on more than 1 independent variables

order of diff eqn: the order of highest derivative

linear / nonlinear D.E.: order of dependent variable

IVP (initial value problem): independent variable and its derivatives at the same value e.g. $y(0)=1$, $y'(0)=0$

BVP (Boundary Value problem): given more than one value of independent variable e.g. $y(0)=1$, $y(10)=10$

homogeneous: $\begin{cases} x''(t)+t^2\sin(t)\cdot x(t) = 5t\cdot x(t) \to \text{homogeneous} \\ x''(t)+t^2\sin(t)\cdot x(t) = \underline{5t} \to \text{inhomogeneous}\end{cases}$

1st order ODE

① linear, constant coefficient $x'+ax=g(t)$

- homogeneous solution $x=c_1e^{-at}$
- particular solution using undetermined coefficient

③ linear, variable coefficient $y'+p(x)y=g(x)$

$\Rightarrow$ multiply integrating factor: $\mu(x)=\exp(\int p(x)\,dx)$

② $\frac{dy}{dx}=f(x,y)$ or $M(x,y)dx+N(x,y)dy=0$

1) whether is separable

2) whether is in terms of $\frac{dy}{dx}=f(\frac{y}{x}) \Rightarrow v=\frac{y}{x}$, $\frac{dy}{dx}=\frac{d}{dx}(vx)=v+x\frac{dv}{dx}$

3) whether is exact $\Leftrightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\to$ if exact

$\Rightarrow \begin{cases}\psi=\int M dx \\ \psi=\int N\cdot dy\end{cases} \Rightarrow d\psi=0 \Rightarrow \psi=C$

$\to$ if not exact $\Rightarrow$ integrating factor

$\Rightarrow \begin{cases}\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} \text{ if only } x \Rightarrow \mu(x)=\exp\left(\int \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}dx\right) \\ \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} \text{ if only } y \Rightarrow \mu(y)=\exp\left(\int \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}dy\right)\end{cases}$

2nd order linear, constant coefficient ODE

- homogeneous solution

characteristic equation

$\to ay''+by'+c=0 \Rightarrow ar^2+br+c=0$

1) $b^2-4ac>0 \Rightarrow r_1, r_2$
$y=c_1e^{r_1t}+c_2e^{r_2t}$

2) $b^2-4ac=0 \Rightarrow r_1=r_2$
$y=c_1e^{r_1t}+c_2te^{r_1t}$

3) $b^2-4ac<0 \Rightarrow r=\lambda\pm i\mu$
$y=e^{\lambda t}(A\cos\mu t+B\sin\mu t)$

- inhomogeneous solution

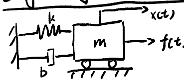
1) undetermined coefficients

2) method of variation of parameters (where y_1, y_2 two homogeneous soln) (e.g. e^{r_1t}, e^{r_2t})

$y_p(t)=-y_1(t)\int\frac{y_2(t)f(t)}{W(y_1,y_2)}dt+y_2(t)\int\frac{y_1(t)\cdot f(t)}{W(y_1,y_2)}dt$, $W(y_1,y_2)=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_1'y_2$

Single degree of freedom vibrations

damping ratio $\zeta=\frac{b}{\sqrt{4km}}$ natural frequency $w_n=\sqrt{\frac{k}{m}}$ damped natural frequency $w_d=w_n\sqrt{1-\zeta^2}$ (only relevant for $0<\zeta<1$)



$m\ddot{x}+b\dot{x}+kx=f(t) \Leftrightarrow \ddot{x}+2\zeta w_n\dot{x}+w_n^2x=\frac{f(t)}{m}$

① free vibration

1) undamped

$\Rightarrow m\ddot{x}+kx=0 \Rightarrow \ddot{x}+w_n^2x=0$

$\to x(t)=C_1\cos w_nt+C_2\sin w_nt$

$\xrightarrow{IC} x(t)=x_0\cos w_nt+\frac{\dot{x}_0}{w_n}\sin w_nt \Leftrightarrow x(t)=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{w_n}\right)^2}\cos(w_nt+\phi)$

$\begin{cases}\sin\phi=\dfrac{-\dot{x}_0/w_n}{\sqrt{x_0^2+\left(\frac{\dot{x}_0}{w_n}\right)^2}}\\ \cos\phi=\dfrac{x_0}{\sqrt{x_0^2+\left(\frac{\dot{x}_0}{w_n}\right)^2}}\end{cases}$

2) damped

$\Rightarrow \ddot{x}+2\zeta w_n\dot{x}+w_n^2x=0$

$\lambda=-\zeta w_n\pm w_n\sqrt{\zeta^2-1}$ $(\lambda_1=-\zeta w_n+w_n\sqrt{\zeta^2-1}$, $\lambda_2=-\zeta w_n-w_n\sqrt{\zeta^2-1})$

- $\zeta>1$

$x(t)=\frac{\dot{x}_0+x_0w_n(\zeta+\sqrt{\zeta^2-1})}{2w_n\sqrt{\zeta^2-1}}e^{\lambda_1t}-\frac{\dot{x}_0-x_0w_n(\zeta+\sqrt{\zeta^2-1})}{2w_n\sqrt{\zeta^2-1}}e^{\lambda_2t}$

- $\zeta=1$

$x(t)=c_1e^{-w_nt}+c_2te^{-w_nt}$

- $\zeta<1$

$\Rightarrow \lambda=-\zeta w_n\pm iw_n\sqrt{1-\zeta^2}$

$\Rightarrow x(t)=e^{-\zeta w_nt}(C_1\cos w_n\sqrt{1-\zeta^2}t+C_2\sin\sqrt{1-\zeta^2}t)$

② forced vibration

1) undamped

$m\ddot{x}+kx=F\cos wt \Leftrightarrow \ddot{x}+w_n^2x=\frac{F}{m}\cos wt$

$\boxed{\text{if } w\neq w_n}$ $x(t)=C_1\cos w_nt+C_2\sin w_nt+\frac{F}{m(w_n^2-w^2)}\cos wt$

$\xrightarrow{IC} x(t)=\left(x_0-\frac{F}{m(w_n^2-w^2)}\right)\cos w_nt+\frac{\dot{x}_0}{w_n}\sin w_nt+\frac{F}{m(w_n^2-w^2)}\cos wt$

$\Leftrightarrow x(t)=\left(x_0-\frac{\delta}{1-r^2}\right)\cos w_nt+\frac{\dot{x}_0}{w_n}\sin w_nt+\frac{\delta}{1-r^2}\cos wt$ $\left(\delta=\frac{F}{k},\ r=\frac{w}{w_n}\right)$

static deflection frequency ratio

$M=\frac{1}{1-r^2}$ magnification factor

if $x_0=0$, $\dot{x}_0=0$, or simply ignore ICs

$x(t)=\frac{\delta}{1-r^2}(\cos wt-\cos w_nt)$

$w\ll w_n \Rightarrow r\approx0 \Rightarrow x(t)\approx\delta(\cos wt-\cos w_nt)$ $0\sim2\delta$

$w\gg w_n \Rightarrow \frac{\delta}{1-r^2}\approx-\frac{\delta}{r^2}\approx0 \Rightarrow \approx0$

also, the response of the system is in phase with forcing function when $w<w_n$ and out of phase with forcing function when $w>w_n$

$\boxed{\text{if } w=w_n}$ $x(t)=x_0\cos w_nt+\frac{\dot{x}_0}{w_n}\sin w_nt+\frac{\delta w_nt}{2}\sin w_nt$ $(t\to\infty,\ x\to\infty)$

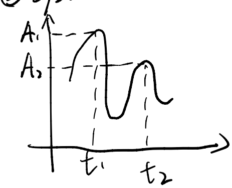
2) damped

$\ddot{x}+2\zeta w_n\dot{x}+w_n^2x=\frac{F}{m}\cos wt$

$\Rightarrow x_p(t)=\delta\sqrt{\frac{1}{(1-r^2)^2+(2\zeta r)^2}}\cos(wt+\phi)$

$\phi=\tan^{-1}\left(-\frac{2\zeta\frac{w}{w_n}}{1-\frac{w^2}{w_n^2}}\right)$

③ system identification (free damped vibration)



1) $T=t_2-t_1$

2) $w_d=\frac{2\pi}{T}=w_n\sqrt{1-\zeta^2}$

3) $\frac{A_2}{A_1}=e^{-\zeta w_n(t_2-t_1)}$

solve w_n & ζ

Variable coefficient linear ODEs (2nd order)

$y'' + p(x)y(x) + q(x)y(x) = 0$

① power series solutions

assume $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $\Rightarrow$ plug back $\Rightarrow$ all adjust to highest order of x

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Also, $y(0) = a_0$

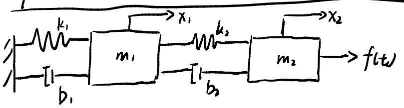
$y'(0) = a_1$

e.g. $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \Rightarrow \sum_{n=-2}^{\infty} (n+4)(n+3) a_{n+4} x^{n+2} + \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$

$\Rightarrow 2a_2 + 6a_3 t + a_1 + 2a_2 t + \sum_{n=0}^{\infty} [(n+4)(n+3)a_{n+4} + (n+3)a_{n+3} + a_n] x^{n+2} = 0$

$\Rightarrow \begin{cases} 2a_2 + 6a_3 = 0 \\ a_1 + 2a_2 = 0 \\ (n+4)(n+3)a_{n+4} + (n+3)a_{n+3} + a_n = 0 \end{cases}$

Systems of 1st order linear, constant coefficient ODEs



$\Rightarrow \begin{cases} m_1 \ddot{x}_1 + (b_1+b_2)\dot{x}_1 - b_2 \dot{x}_2 + (k_1+k_2)x_1 - k_2 x_2 = 0 \\ m\ddot{x}_2 - b_2 \dot{x}_1 + b_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = f(t) \end{cases}$

$\Rightarrow$ highest order $\begin{cases} x_1 : 2 \\ x_2 : 2 \end{cases} \Rightarrow \xi_1 = x_1, \ \xi_2 = \dot{x}_1, \ \xi_3 = x_2, \ \xi_4 = \dot{x}_2$

if $\begin{cases} x_1 : 3 \\ x_2 : 1 \end{cases} \Rightarrow \xi_1 = x_1, \ \xi_2 = \dot{x}_1, \ \xi_3 = \ddot{x}_1, \ \xi_4 = x_2$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \frac{1}{m_1}(-b_1\xi_2 - k_1\xi_1 + k_2\xi_3 - k_2\xi_1 + b_2\xi_4 - b_2\xi_2) \\ \xi_3 \\ \frac{1}{m_2}(-k_2\xi_3 + k_2\xi_1 - b_2\xi_4 + b_2\xi_2) \end{bmatrix} \Leftrightarrow \dot{\xi} = A\xi$$

1) $|A - \lambda I| = 0$

- $\lambda_1 \neq \lambda_2 = \text{real}$

$(A - \lambda_1 I)\xi = 0 \Rightarrow \xi^{(1)}$

$(A - \lambda_2 I)\xi = 0 \Rightarrow \xi^{(2)}$

$\Rightarrow \xi_1(t) = \xi^{(1)} e^{\lambda_1 t}, \ \xi_2(t) = \xi^{(2)} e^{\lambda_2 t}$

$\Rightarrow \xi(t) = c_1 \xi_1(t) + c_2 \xi_2(t)$

- $\lambda_1 = \lambda_2 = \text{real}$

$(A - \lambda_1 I)\xi = 0 \Rightarrow \xi^{(1)}, \xi^{(2)}$

$\Rightarrow \xi(t) = c_1 \xi^{(1)} e^{\lambda_1 t} + c_2 \xi^{(2)} e^{\lambda_1 t}$

- complex eigenvalues

$\lambda_1 = \mu + i\omega, \ \lambda_2 = \mu - i\omega$

$(A - \lambda_1 I)\xi = 0 \Rightarrow \xi^{(1)} \Rightarrow \xi^{(2)} = \overline{\xi^{(1)}}$

$\Rightarrow \xi(t) = k_1 e^{\mu t}(\xi_r^{(1)} \cos\omega t - \xi_i^{(1)} \sin\omega t) + k_2 e^{\mu t}(\xi_r^{(1)} \sin\omega t + \xi_i^{(1)} \cos\omega t)$

e.g. $\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \Rightarrow \xi_r^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \xi_i^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Partial Differential equations

$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} + d\frac{\partial z}{\partial x} + e\frac{\partial z}{\partial y} + fz = g$

$\begin{cases} b^2 - 4ac > 0 \Rightarrow \text{hyperbolic} \\ b^2 - 4ac < 0 \Rightarrow \text{elliptic} \\ b^2 - 4ac = 0 \Rightarrow \text{parabolic} \end{cases}$

① 1-D wave equation

$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$\begin{cases} u(0,t) = 0, \ u(L,t) = 0 \\ u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$

$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T''(t)}{T(t)} = -\lambda \quad (\lambda > 0)$

$X = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x \xrightarrow{BCs} \lambda = \left(\frac{n\pi}{L}\right)^2 \ (n = 1, 2, \dots)$

$T = c_3 \sin\alpha\sqrt{\lambda}t + c_4 \cos\alpha\sqrt{\lambda}t \Rightarrow u_n \Rightarrow u = \sum_{n=1}^{\infty} u_n \Rightarrow \int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$

② 1-D heat conduction equation

- $u(0,t) = u(L,t) = 0, \ u(x,0) = f(x)$

$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda$

$\begin{cases} X(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x \\ T(t) = e^{-\alpha^2 \lambda t} \end{cases}$

- $u(0,t) = T_L, \ u(L,t) = T_R$

$u(x,t) = u_{tr}(x,t) + u_{ss}(x)$

$= \frac{T_R - T_L}{L}x + T_L + \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t}$

- $u(0,t) = 0, \ \frac{\partial u}{\partial x}(L,t) = 0 \ \& \ u(x,0) = f(x)$

$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) \cdot e^{-\left(\frac{\alpha(2n+1)\pi}{2L}\right)^2 t}$

where $c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{(2n+1)\pi x}{2L} dx$

③ Laplace equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda \quad (\lambda > 0)$

$\begin{cases} u(0,y) = 0 & u(x,0) = 0 \\ u(L_x,y) = 0 & u(x,L_y) = f(x) \end{cases} \Rightarrow u = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L_x}\left(e^{\frac{n\pi y}{L_x}} - e^{-\frac{n\pi y}{L_x}}\right)$

where $c_n = \frac{2}{L_x\left(e^{\frac{n\pi L_y}{L_x}} - e^{-\frac{n\pi L_y}{L_x}}\right)} \cdot \int_0^{L_x} f(x) \sin\frac{n\pi x}{L_x} dx$

Numerical solutions of ODE

① Euler method

$\frac{dy}{dt} = F(y,t) \Rightarrow \boxed{y_{n+1} = y_n + F(y_n, t_n) \cdot \Delta t}$

② Modified Euler method (Heun's method)

predictor equation

$y_{n+1} = y_n + F(y_n, t_n)\Delta t$

corrector equation

$y_{n+1} = y_n + \left\{\frac{F(y_n, t_n) + F(y_{n+1}, t_{n+1})}{2}\right\}\Delta t$

③ 4th order Runge-Kutta method

$y_{n+1} = y_n + \frac{\Delta t}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$

$\begin{cases} k_{n1} = F(t_n, y_n) \\ k_{n2} = F(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_{n1}) \\ k_{n3} = F(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_{n2}) \\ k_{n4} = F(t_n + \Delta t, y_n + \Delta t k_{n3}) \end{cases}$

④ RK4 for system

let $p = x'$

$\begin{cases} p' = f_p(x, p, t) & , \ p(0) = x'(0) = \dots \\ x' = p & \ x(0) = \dots \end{cases}$

$\begin{cases} p_{n+1} = p_n + \frac{\Delta t}{6}(k_{p1} + 2k_{p2} + 2k_{p3} + k_{p4}) \\ x_{n+1} = x_n + \frac{\Delta t}{6}(k_{x1} + 2k_{x2} + 2k_{x3} + k_{x4}) \end{cases}$

$k_{p1} = f_p(x_n, p_n, t_n)$

$k_{x1} = p_n$

$k_{p2} = f_p(x_n + \frac{\Delta t}{2} \cdot k_{x1}, p_n + \frac{\Delta t}{2} k_{p1}, t_n + \frac{\Delta t}{2})$ ...

$k_{x2} = p_n + \frac{\Delta t}{2} k_{p1}$

⑤ shooting method (for boundary value problem)

* equ should be linear

guess ① $p_1(0) \Rightarrow y_1(1)$

guess ② $p_2(0) \Rightarrow y_2(1)$

$\Rightarrow p(0) = p_1(0) + \frac{p_2(0) - p_1(0)}{y_2(1) - y_1(1)}(y(1) - y_1(1))$