Problem 1

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad , \quad u(0) = 0 , \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 1$$

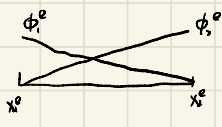$$\int_0^1 w\left(-\frac{d^2u}{dx^2} - u + x^2\right) dx = 0$$

$$\Rightarrow \int_0^1 -\frac{d}{dx}\left(w\frac{du}{dx}\right)dx + \int_0^1 \left[\frac{dw}{dx}\frac{du}{dx} + w(x^2 - u)\right]dx = 0$$

$$\Rightarrow \int_0^1 \left[\frac{dw}{dx}\frac{du}{dx} + w(x^2 - u)\right]dx - w\frac{du}{dx}\Big|_0^1 = 0$$

$$\xrightarrow{B.c} \int_0^1 \left[\frac{dw}{dx}\frac{du}{dx} + w(x^2 - u)\right]dx - w(1) = 0$$

$$B(w,u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu\right)dx$$

$$l(w) = -\int_0^1 wx^2\,dx + w(1) = \begin{cases} -\int_0^1 wx^2\,dx + w(1) & , \ e = N_e \\ -\int_0^1 wx^2\,dx & , \ \text{otherwise} \end{cases}$$

$$h = x_2^e - x_1^e$$
$$\phi_1^e = \frac{x_2^e - x}{x_2^e - x_1^e} \quad , \quad \frac{d\phi_1^e}{dx} = -\frac{1}{h}$$
$$\phi_2^e = \frac{x - x_1^e}{x_2^e - x_1^e} \quad , \quad \frac{d\phi_2^e}{dx} = \frac{1}{h}$$

$$k_{ij}^e = B_e(\phi_i^e, \phi_j^e) = \int_{x_1^e}^{x_2^e} \left[\frac{d\phi_i^e}{dx}\cdot\frac{d\phi_j^e}{dx} - \phi_i^e\phi_j^e\right]dx$$

$$f_i^e = l_e(\phi_i^e) = \begin{cases} \int_{x_1^e}^{x_2^e} \phi_i^e\,(-x^2)dx + \phi_i^e(1) & , \ e = N_e \\ \int_{x_1^e}^{x_2^e} \phi_i^e\,(-x^2)dx & , \ \text{otherwise} \end{cases}$$

Problem 2

$-T_{,ii} = 0$ in Ω BCs: $T_{,i} n_i = 1$ on $\partial\Omega_1$, $T_{,i} n_i = 0$ on $\partial\Omega_3$, $T = 0$ on $\partial\Omega_2, \partial\Omega_4$

$$\int_\Omega [w(-T_{,ii})] dV = 0$$

$$\Rightarrow \int_\Omega [-(w T_{,i})_{,i} + w_{,i} T_{,i}] dV = 0$$

$$\Rightarrow \int_\Omega w_{,i} T_{,i} \, dV - \int_{\partial\Omega} w T_{,i} n_i \, ds = 0$$

$$\xrightarrow{\text{BCs}} \int_\Omega w_{,i} T_{,i} \, dV - \int_{\partial\Omega_1} w \, ds = 0$$

$$B_e(w, T) = \int_\Omega w_{,i} T_{,i} \, dV$$

$$L_e(w) = \begin{cases} \int\int_{\partial\Omega_1} w \, ds & , \text{ if } x_i^e = x_2^e = 0 \\ \\ 0 & , \text{ else} \end{cases}$$

$$\phi_1^e = \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right)$$

$$\phi_2^e = \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right)$$

$$\phi_3^e = \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right)$$

$$\phi_4^e = \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right)$$

$$K_{ij}^e = B_e(\phi_i^e, \phi_j^e)$$

$$f_i^e = L_e(\phi_i^e)$$