AME40541/60541: Finite Element Methods
Homework 3 Solutions

Problem 1:

$$\sigma_{ji,j} + F_i = 0$$

Problem 2:

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + \mu(\epsilon_{ij} + \epsilon_{ji}) = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$$

$$\sigma_{kk} = \lambda \epsilon_{ss}\delta_{kk} + 2\mu\epsilon_{kk} = (3\lambda + 2\mu)\epsilon_{kk}$$

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - \frac{(3\lambda+2\mu)\epsilon_{kk}}{3}\delta_{ij} = 2\mu\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}\right)$$

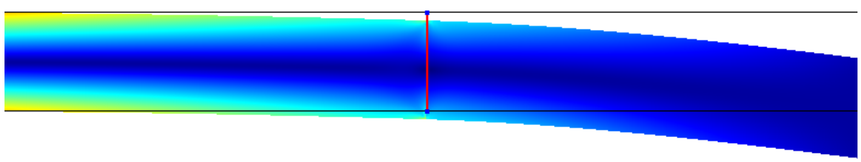
Problem 3: Multiply by test function, integrate over domain, and perform integration-by-parts

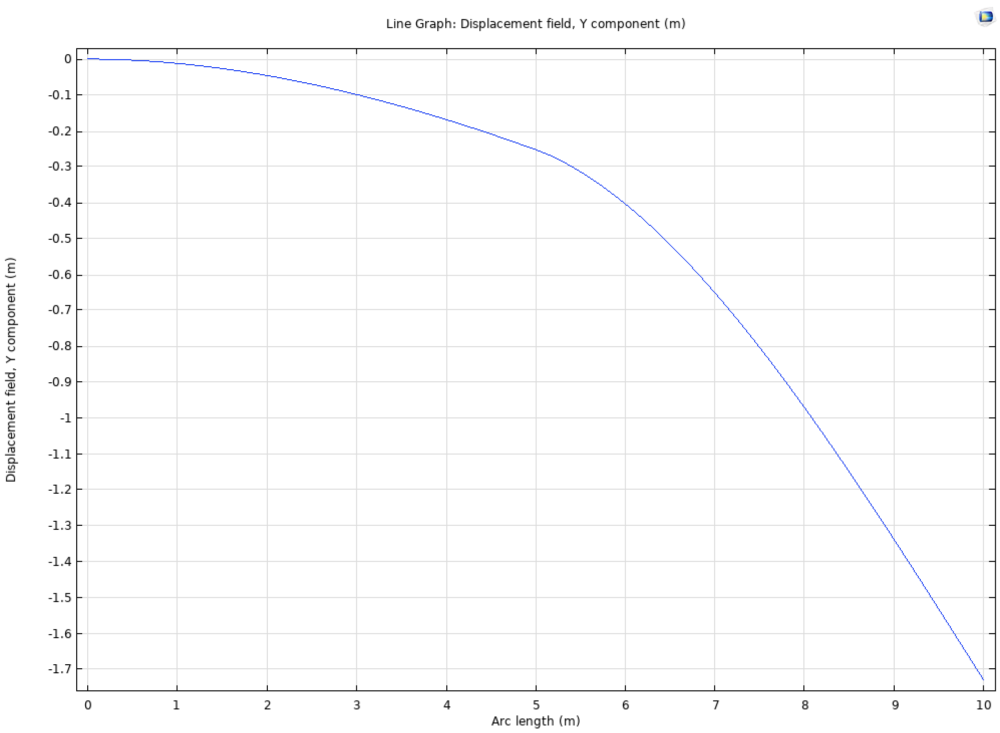
$$\int_0^L \left(\frac{dw}{dx}u\frac{du}{dx} + wf\right)dx - \left[wu\frac{du}{dx}\right]_0^L = 0$$

Since Dirichlet boundary condition specified at $x = L$, we have $w(L) = 0$ and apply the natural boundary condition at $x = 0$ to obtain the weak form

$$\int_0^L \left(\frac{dw}{dx}u\frac{du}{dx} + wf\right)dx = 0$$

Problem 4: COMSOL




Line Graph: Displacement field, Y component (m)
Displacement field, Y component (m)
Arc length (m)
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
-1.7
0
1
2
3
4
5
6
7
8
9
10

Problem 5:

(a) The conservation law in indicial notation reads

$$F_{ij,j} = S_i \quad \text{in } \Omega$$

where boundary conditions $U_i = \bar{U}_i$ on $\partial\Omega_D$ and $F_{ij}n_j = \bar{q}_i$ on $\partial\Omega_D$.

(b) The weighted residual formulation is: find $U : \Omega \to \mathbb{R}^m$ such that

$$\int_\Omega w_i(F_{ij,j} - S_i)\, dV = 0.$$

holds for all w. By moving a derivative from F to the test function, we have

$$\int_\Omega -(w_{i,j}F_{ij} + w_i S_i)\, dV + \int_{\partial\Omega} w_i F_{ij} n_j = 0.$$

Requiring the test function is zero on $\partial\Omega_D$ and applying the natural boundary conditions, we have the weak form of the conservation law

$$\int_\Omega -(w_{i,j}F_{ij} + w_i S_i)\, dV + \int_{\partial\Omega_N} w_i \bar{q}_i = 0.$$

(c) Any approximation U_h must posses two non-zeros derivatives to apply the method of weighted residuals and one non-zero derivative to apply the Ritz method (regardless of linearity of F). The method of weighted residuals requires the approximate solution U_h satisfies all boundary conditions, i.e., trial space

$$\mathcal{U}_h^{\text{wr}} = \{U_h \in \mathcal{F}_{\Omega\to\mathbb{R}^m} \mid U|_{\Omega_D} = \bar{U}, [F(U)n]_{\Omega_N} = \bar{q}\}$$

while the Ritz method only requires solution to satisfy the Dirichlet BCs, i.e., trial space

$$\mathcal{U}_h^{\text{ritz}} = \{U_h \in \mathcal{F}_{\Omega\to\mathbb{R}^m} \mid U|_{\Omega_D} = \bar{U}\}$$

requires the approximate solution satisfies the Dirichlet boundary conditions. While neither of these are linear spaces in general, $\mathcal{U}_h^{\text{ritz}}$ is an *affine* space regardless of the F; $\mathcal{U}_h^{\text{wr}}$ is only affine if F is linear in U and ∇U. This means it is much more difficult to construct approximations U_h for the method of weighted residuals for nonlinear PDEs because the trial space is a nonlinear (nonaffine) space.

(d) All PDEs are second-order

- Linear elliptic: $m = 1$, linear (F and S are linear in U and ∇U)

$$U_1 = u, \quad F_{1,j} = k_{js}u_{,s}, \quad S_1 = f, \quad \bar{U}_1 = \bar{u}, \quad \bar{q}_1 = \bar{t}$$

- Linear elasticity: $m = d$, linear (F and S are linear in U and ∇U)

$$U_i = u_i, \quad F_{ij} = \sigma_{ij} = \frac{1}{2}C_{ijkl}(u_{k,l} + u_{l,k}), \quad S_i = -f_i, \quad \bar{U}_i = \bar{u}_i, \quad \bar{q}_i = \bar{t}_i$$

- Incompressible Navier-Stokes: $m = d + 1$, nonlinear (F is linear in U and ∇U, S nonlinear)

$$U_i = \begin{cases} v_i & i < d+1 \\ p & i = d+1 \end{cases}, \quad F_{ij} = \begin{cases} -\rho\nu v_{i,j} + p\delta_{ij} & i < d+1 \\ v_{s,s} & i = d+1 \end{cases}, \quad S_i = \begin{cases} \rho v_j v_{i,j} & i < d+1 \\ 0 & i = d+1 \end{cases}$$

$$\bar{U}_i = \begin{cases} \bar{v}_i & i < d+1 \\ \bar{p} & i = d+1 \end{cases}, \quad \bar{t}_i = \begin{cases} -\rho\bar{t}_i & i < d+1 \\ 0 & i = d+1 \end{cases}$$

Problem 6: The incompressible Navier-Stokes equations in indicial notation are

$$\begin{aligned} -(\rho\nu v_{i,j})_{,j} + \rho v_j v_{i,j} + p_{,i} &= 0 \\ v_{j,j} &= 0 \end{aligned}$$

with boundary conditions $v_i = \bar{v}_i$ on $\partial\Omega_D$ and $\rho\nu v_{i,j} n_j - p n_i = \rho \bar{t}_i$ on $\partial\Omega_N$. The construction of the weak form begins by multiplying the momentum equations by the vector-valued test function w_i, the continuity equation by the scalar test function τ, summing over all equations, and integrating over the domain Ω

$$\int_\Omega \left[w_i(-(\rho\nu v_{i,j})_{,j} + \rho v_j v_{i,j} + p_{,i}) + \tau v_{j,j}\right] dV = 0$$

Then, using integration-by-parts to move a derivative off u in the first term and off pressure in the third term, we have

$$\int_\Omega \left[w_{i,j}\rho\nu v_{i,j} - w_{i,i}p + w_i\rho v_j v_{i,j} + \tau v_{j,j}\right] dV - \int_{\partial\Omega} w_i \left(\rho\nu u_{i,j} n_j - p n_i\right) dS = 0$$

Let $w_i(x) = 0$ for $x \in \partial\Omega_D$ (Dirichlet boundary conditions), then the weak form becomes

$$\int_\Omega \left[w_{i,j}\rho\nu v_{i,j} - w_{i,i}p + w_i\rho v_j v_{i,j} + \tau v_{j,j}\right] dV - \int_{\partial\Omega_N} w_i \left(\rho\nu v_{i,j} n_j - p n_i\right) dS = 0.$$

Using the expression for the natural boundary conditions, the weak form of the incompressible Naiver-Stokes equations is

$$\int_\Omega \left[w_{i,j}\rho\nu v_{i,j} - w_{i,i}p + w_i\rho v_j v_{i,j} + \tau v_{j,j}\right] dV - \int_{\partial\Omega_N} w_i \rho \bar{t}_i \, dS = 0.$$

Problem 7: <u>Exact solution</u>: integrate twice, divide by EI (assume constant), integrate twice (be sure to include constant of integration each time)

$$w(x) = \frac{1}{EI}\left(q_0\frac{x^4}{24} + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D\right).$$

Apply boundary conditions $w(0) = w(L) = EIw''(0) = EIw''(L) = 0$ to solve for constants:

$$A = -\frac{q_0 L}{2}, \quad B = 0, \quad C = \frac{q_0 L^3}{24}, \quad D = 0,$$

which gives

$$w(x) = \frac{q_0 x}{12EI}\left(\frac{x^3}{2} - Lx^2 + \frac{L^3}{2}\right).$$

<u>Weak form</u>: Multiply by test function $\psi(x)$ and integrate over domain

$$\int_0^L \psi\left(\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q_0\right) dx = 0.$$

Then move two derivatives from $w(x)$ onto $\psi(x)$

$$\int_0^L \left(-\frac{d\psi}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) - \psi q_0\right) dx + \left[\psi\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_0^L = 0$$

$$\int_0^L \left(\frac{d^2\psi}{dx^2}EI\frac{d^2w}{dx^2} - \psi q_0\right) dx + \left[\psi\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_0^L - \left[\frac{d\psi}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_0^L = 0.$$

From this we can identify the primary variables as w and $\frac{dw}{dx}$ and the corresponding secondary variables as $\frac{d}{dx}(EI\frac{d^2w}{dx^2})$ and $EI\frac{d^2w}{dx^2}$, respectively. From the boundary conditions $w(0) = w(L) = 0$, the test function satisfies $\psi(0) = \psi(L) = 0$ and the first boundary term drops out. In addition, from $EIw''(0) = EIw''(L) = 0$, the second boundary term drops out. We are left with the weak form

$$\int_0^L \left(\frac{d^2\psi}{dx^2} EI \frac{d^2 w}{dx^2} - \psi q_0 \right) dx = 0.$$

Method of weighted residuals: The weighted residual form of the PDE is: find $w(x)$ such that

$$\int_0^L \psi(x) \left(\frac{d^2}{dx^2} \left(EI \frac{d^2 w}{dx^2} \right) - q_0 \right) dx = 0$$

holds for all $\psi(x)$. Since we cannot practically enforce this equation for all $\psi(x)$, we consider all $\psi(x)$ in the subspace spanned by $\{\psi_1(x), \psi_2(x)\}$. Consider the two-term trigonometric expansion of the solution $w(x) \approx w_2(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$. It is easy to verify this satisfies the appropriate boundary conditions. Substitute this expansion into the weighted residual form and, for the Galerkin method, choose $\psi_k(x) = \sin\left(\frac{k\pi x}{L}\right)$ for $k = 1, 2$ and, for collocation, choose $\psi_k(x) = \delta(x - x_k)$ where $x_1 = 0.25$ and $x_2 = 0.75$. Using MAPLE to perform the calculus and algebra manipulations, we have

$$c_1 = -0.0129, \quad c_2 = 0$$

for Galerkin and

$$c_1 = -0.0143, \quad c_2 = 0$$

for collocation (where we have used $L = 1$ to simplify the collocation expression). Therefore, the approximate solutions are

$$w_{\text{res-gal}}(x) = -0.0129 \sin(\pi x)$$
$$w_{\text{res-col}}(x) = -0.0143 \sin(\pi x)$$

Ritz method: The Ritz method is based on the weak form of the PDE, derived previously which we only enforce force to hold for a subspace of twice continuously differentiable functions spanned by $\{\psi_1(x), \psi_2(x)\}$

$$\int_0^L \left(\frac{d^2\psi_i}{dx^2} EI \frac{d^2 w}{dx^2} - \psi_i q_0 \right) dx = 0, \qquad i = 1, 2.$$

Suppose we approximate our solution as $w(x) \approx c_1 w_1(x) + c_2 w_2(x)$, then we take $\psi_i(x) = w_i(x)$. Here, we consider two expansions

$$w_{\text{ritz-trig}}(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$$
$$w_{\text{ritz-poly}}(x) = c_1 x(x - L) + c_2 x^2 (x - L)^2.$$

It should be clear that these functions satisfy the essential boundary conditions ($w(0) = w(L) = 0$). Substituting these expressions into the weak form, we obtain $c_1 = -0.0131$, $c_2 = 0$ for the trigonometric basis and $c_1 = 0.041\bar{6}$, $c_2 = -0.041\bar{6}$, which gives

$$w_{\text{trig-ritz}}(x) = -0.0131 \sin\left(\frac{\pi x}{L}\right)$$
$$w_{\text{poly-ritz}}(x) = 0.041\bar{6} \left(x(x-1) - x^2(x-1)^2 \right).$$

Performance: Run code to generate plots. Errors: (a) $e_{\text{res-gal}} = 1.61 \times 10^{-8}$, (b) $e_{\text{res-col}} = 7.57 \times 10^{-7}$, (c) $e_{\text{ritz-trig}} = 1.87 \times 10^{-9}$, (d) $e_{\text{ritz-poly}} = 0.0$.