AME40541/60541: Finite Element Methods
Homework 5: Due Friday, October 28, 2022

Instructions: Upload solutions to Gradescope as a PDF; email code as a zip file to the instructor and grader.

Problem 1: (40 points) In this problem, you will implement a basic FEM code that we will enhance (substantially) in your final project. Before proceeding, carefully review the starter code, including all the comments, that has been provided on the course website in `hwk05−code−starter.zip`. I have provided the following functions:

- `create_mesh_hcube`: create `msh` (`xcg`, `e2vcg`) for uniform mesh of d-dimensional hypercube
- `create_ldof2gdof_cg`: create `ldof2gdof` matrix
- `assemble_nobc_mat_dense`: assemble global matrix from element matrices
- `create_dbc_strct`: create essential boundary condition structure (same as Homework 2)
- `create_femsp_cg`: create FE space structure
- `visualize_fem`: visualize FE mesh and solution

You are welcome to use your own version of `create_ldof2gdof_cg.m` and `assemble_nobc_mat_dense.m` you implemented in Homework 2 if you made them sufficiently general.

Problem 1.1 Implement a function `eval_unassembled_stiff_load.m` that evaluates and stores the element stiffness matrix and load vector for each element in the FE mesh. Starter code is provided on the course website in the Homework 5 code distribution.

Problem 1.2 Implement a function `assemble_nobc_vec.m` that assembles the element load vector into the global load vector without applying essential boundary conditions. Starter code is provided on the course website in the Homework 5 code distribution.

Problem 1.3 Implement a function `apply_bc_solve_fem.m` that applies essential boundary conditions via static condensation to the global FE system and solves the unknown solution coefficients. Starter code is provided on the course website in the Homework 5 code distribution.

Problem 1.4 Implement a function `solve_fem_dense.m` that uses the finite element method to approximate the unknown PDE solution at nodes using the functions created in Problems 1.1-1.3. Starter code is provided on the course website in the Homework 5 code distribution.

Problem 1.5 Use the element developed in Homework 4, Problem 1 to approximate the solution of

$$\begin{aligned} -\frac{d^2u}{dx^2} - u + x^2 = 0 \\ u(0) = 0, \left.\left(\frac{du}{dx}\right)\right|_{x=1} = 1 \end{aligned} \tag{1}$$

over the domain $\Omega = [0, 1]$ using the finite element method. Use a mesh consisting of three linear elements and plot against the exact solution

$$u(x) = \frac{2\cos(1-x) - \sin(x)}{\cos(1)} + x^2 - 2.$$

What do you notice about the accuracy of the FEM solution at the nodes vs. interior to elements? Repeat the analysis using a finite element mesh with 25 linear elements and plot the solution.

Problem 1.6 Use the element developed in Homework 4, Problem 2 to approximate the solution of

$$\begin{aligned} -\Delta T &= 0 \quad \text{in} \quad \Omega \\ \nabla T \cdot n &= 1 \quad \text{on} \quad \partial\Omega_1 \\ \nabla T \cdot n &= 0 \quad \text{on} \quad \partial\Omega_2 \\ T &= 0 \quad \text{on} \quad \partial\Omega_3 \cup \partial\Omega_4, \end{aligned} \tag{2}$$

over the domain $\Omega := [0,1] \times [0,1]$ using the finite element method. Use a mesh consisting of 3×3 linear elements and plot the solution. Repeat the analysis using a finite element mesh of 25×25 linear elements and plot the solution.

Problem 2: (25 points) Consider a single one-dimensional, $(p+1)$-node Lagrangian element with nodes located as $x_1^e, \dots, x_{p+1}^e$.

(a) What is the polynomial space associated with the element?

(b) Write the expressions for the element basis functions $\{\phi_1^e, \dots, \phi_{p+1}^e\}$ and their derivatives in terms of the nodal positions.

(c) Implement a function that evaluates all one-dimensional Lagrange polynomials and their derivatives associated with nodes $x_1^e, \dots, x_{p+1}^e$. Starter code provided in Homework 5 code distribution. Check you code: all basis functions should possess the nodal/Lagrangian property; in addition, the basis functions should satisfy

$$\sum_{i=1}^{p+1} \phi_i^e(x) = 1, \qquad \sum_{i=1}^{p+1} \frac{d\phi_i^e}{dx}(x) = 0$$

for any $x \in [x_1^e, x_{p+1}^e]$.

(d) Plot the basis functions and their derivatives for $p = 1, \dots, 5$ nodes equally spaced in the domain $[-1, 1]$.

(e) Consider a one-dimensional domain, discretized by 3 of 5-node Lagrangian elements. Create a global numbering for the mesh and write the `e2vcg` matrix.

Problem 3: (30 points) The incompressible Navier-Stokes equations model low speed, viscous fluid flow:

$$-(\rho\nu v_{i,j})_j + \rho v_j v_{i,j} + P_{,i} = 0, \qquad v_{j,j} = 0$$

for $i = 1, \dots, d$, for all $x \in \Omega$ with the boundary conditions $(\rho\nu v_{i,j} - P\delta_{ij})n_j = \rho\bar{t}_i$ on $\partial\Omega$, where $v(x) \in \mathbb{R}^d$ is the velocity vector, $P(x) \in \mathbb{R}$ is the pressure, $\rho(x) \in \mathbb{R}$ is the density of the fluid, $\nu(x) \in \mathbb{R}$ is the kinematic viscosity of the fluid, $n(x) \in \mathbb{R}^d$ is the outward normal to $\partial\Omega$, and $\bar{t}(x) \in \mathbb{R}^d$ is the traction boundary condition. An important non-dimensional quantity in the study of fluid flow is the Reynolds number

$$Re = \frac{UL}{\nu}, \tag{3}$$

where U is the velocity of the fluid with respect to an object, L is the characteristic linear dimension, and ν is the kinematic viscosity of the fluid. The Reynolds number is the ratio of inertial-to-viscous forces and is used to predict flow patterns in different fluid flow situations.

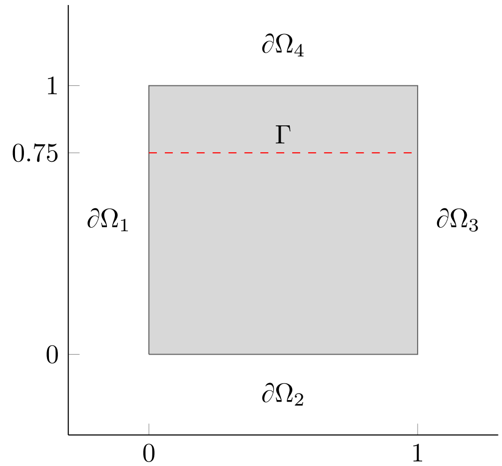

Figure 1: Square domain $\Omega = [0,1] \times [0,1]$ with boundary $\partial\Omega = \overline{\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4}$ and line along which to evaluate quantities (Γ).

The lid-driven cavity problem is a benchmark test in computational fluid dynamics. Consider flow through a square domain (Figure 1) with boundary conditions: stationary, no-slip walls ($v_1 = v_2 = 0$) on $\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$ and a moving, no-slip wall ($v_1 = 1$, $v_2 = 0$) on $\partial\Omega_4$. Since the pressure is only determined up to a constant, we need to prescribe it at one point on the boundary, e.g., take $P = 0$ at the point ($x_1 = 0$, $x_2 = 0$). Take the characteristic length scale to be $L = 1$ (length/height of domain), the characteristic velocity to be the velocity of the moving wall $U = 1$, the density to be $\rho(x) = 1$, and the viscosity to be $\nu(x) = 0.01$.

(a) Use COMSOL to solve for the velocity and pressure distribution of the lid-driven cavity problem; ensure your mesh is sufficiently refined.

(b) Plot the velocities v_1, v_2 along the line Γ shown in Figure 1 and the magnitude of the velocity throughout the domain.

(c) How does the flow change as you decrease the viscosity (ν) keeping the length L and lid velocity U fixed, i.e., increase the Reynolds number? Consider a sequence of decreasing viscosities; for each plot the velocity magnitude and vorticity throughout the domain.