AME40541/60541: Finite Element Methods
Homework 2 Solutions

Problem 1: First, we compute the required element quantities:

$$\begin{aligned} h &= \begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \end{bmatrix}^T \\ EA/h &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5/\sqrt{2} \end{bmatrix}^T \\ \sin\theta &= \begin{bmatrix} 0 & 1 & 1 & 0 & 1/\sqrt{2} \end{bmatrix}^T \\ \cos\theta &= \begin{bmatrix} 1 & 0 & 0 & 1 & 1/\sqrt{2} \end{bmatrix}^T . \end{aligned} \tag{1}$$

The expression for the element stiffness matrix is identical to those in the notes (intrinstic to a truss element), which leads to the following element stiffness matrices

$$\begin{aligned} K^1 &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}, \quad K^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix}, \\ K^4 &= \begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K^5 = \frac{5}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}. \end{aligned} \tag{2}$$

Combining the element equations, equilibrium, and compatibility leads to the global system (without Dirichlet boundary conditions)

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}, \tag{3}$$

where the global displacements and forces are

$$\begin{aligned} \boldsymbol{u} &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{bmatrix}^T \\ \boldsymbol{F} &= \begin{bmatrix} r_1 & r_2 & 0 & r_4 & 0 & 0 & f & 0 \end{bmatrix}^T . \end{aligned} \tag{4}$$

The local-to-global degree of freedom mapping is

$$\boldsymbol{\Xi} = \begin{bmatrix} 1 & 1 & 3 & 5 & 1 \\ 2 & 2 & 4 & 6 & 2 \\ 3 & 5 & 7 & 7 & 7 \\ 4 & 6 & 8 & 8 & 8 \end{bmatrix},$$

which we can use to identify that $\boldsymbol{K}^3$ contributes to K_{ij} for $i, j = 3, 4, 7, 8$ and $\boldsymbol{K}^5$ contributes to K_{ij} for $i, j = 1, 2, 7, 8$. We can further use $\boldsymbol{\Xi}$ to fill the entire stiffness matrix:

$$\begin{aligned}
&K_{11} = K_{11}^1 + K_{11}^2 + K_{11}^5, && K_{12} = K_{12}^1 + K_{12}^2 + K_{12}^5, && K_{13} = K_{13}^1, && K_{14} = K_{14}^1,\\
&K_{15} = K_{13}^2, && K_{16} = K_{14}^2, && K_{17} = K_{13}^5, && K_{18} = K_{14}^5,\\
&K_{21} = K_{21}^1 + K_{21}^2 + K_{21}^5, && K_{22} = K_{22}^1 + K_{22}^2 + K_{22}^5, && K_{23} = K_{23}^1, && K_{24} = K_{24}^1,\\
&K_{25} = K_{23}^2, && K_{26} = K_{24}^2, && K_{27} = K_{23}^5, && K_{28} = K_{24}^5,\\
&K_{31} = K_{31}^1, && K_{32} = K_{32}^1, && K_{33} = K_{11}^3 + K_{33}^1, && K_{34} = K_{12}^3 + K_{34}^1,\\
&K_{35} = 0, && K_{36} = 0, && K_{37} = K_{13}^3, && K_{38} = K_{14}^3,\\
&K_{41} = K_{41}^1, && K_{42} = K_{42}^1, && K_{43} = K_{21}^3 + K_{43}^1, && K_{44} = K_{22}^3 + K_{44}^1,\\
&K_{45} = 0, && K_{46} = 0, && K_{47} = K_{23}^3, && K_{48} = K_{24}^3,\\
&K_{51} = K_{31}^2, && K_{52} = K_{32}^2, && K_{53} = 0, && K_{54} = 0,\\
&K_{55} = K_{33}^2 + K_{11}^4, && K_{56} = K_{34}^2 + K_{12}^4, && K_{57} = K_{13}^4, && K_{58} = K_{14}^4,\\
&K_{61} = K_{41}^2, && K_{62} = K_{42}^2, && K_{63} = 0, && K_{64} = 0,\\
&K_{65} = K_{43}^2 + K_{21}^4, && K_{66} = K_{44}^2 + K_{22}^4, && K_{67} = K_{23}^4, && K_{68} = K_{24}^4,\\
&K_{71} = K_{31}^5, && K_{72} = K_{32}^5, && K_{73} = K_{31}^3, && K_{74} = K_{32}^3,\\
&K_{75} = K_{31}^4, && K_{76} = K_{32}^4, && K_{77} = K_{33}^3 + K_{33}^4 + K_{33}^5, && K_{78} = K_{34}^3 + K_{34}^4 + K_{34}^5,\\
&K_{81} = K_{41}^5, && K_{82} = K_{42}^5, && K_{83} = K_{41}^3, && K_{84} = K_{42}^3,\\
&K_{85} = K_{41}^4, && K_{86} = K_{42}^4, && K_{87} = K_{43}^3 + K_{43}^4 + K_{43}^5, && K_{88} = K_{44}^3 + K_{44}^4 + K_{44}^5.
\end{aligned}$$

The numeric value of the stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} 2.77 & 1.77 & -1 & 0 & 0 & 0 & -1.77 & -1.77 \\ 1.77 & 3.77 & 0 & 0 & 0 & -2 & -1.77 & -1.77 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 4 & 0 & -4 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1.77 & -1.77 & 0 & 0 & -4 & 0 & 5.77 & 1.77 \\ -1.77 & -1.77 & 0 & -3 & 0 & 0 & 1.77 & 4.77 \end{bmatrix}.$$

Problem 2: This problem is identical to Truss 0 with the lone exception that we know an expression for the vertical reaction force at node 2 in terms of the displacements ($r_4 = -ku_4$) and the vertical displacement at node 2 is non-zero and unknown. When we apply BCs, we cannot place r_4 on the right-hand side of the assembled equations because it depends on an unknown displacement. Instead, we use its expression and absorb the term into the the stiffness matrix. The only term in the stiffness matrix that will be changed from Problem 1 is K_{44}, which now reads

$$K_{44} = K_{22}^3 + K_{44}^1 + k. \tag{5}$$

Because the effect of the elastic boundary condition is included in the stiffness matrix (left-hand side of the equation), it does not appear in the forcing term (right-hand side of the equation)

$$\boldsymbol{F} = \begin{bmatrix} r_1 & r_2 & 0 & 0 & 0 & 0 & f & 0 \end{bmatrix}^T. \tag{6}$$

Finally, the application of the boundary conditions proceeds in a nearly identical manner to what we have seen thus far, except u_4 is unknown so it must be included in $\mathcal{I}_u$, i.e., $\mathcal{I}_u = \{3, 4, 5, 6, 7, 8\}$ and $\mathcal{I}_c = \{1, 2\}$. Following the remaining procedure from Problem 1, leads to the displacement and external/reaction forces at each nodes of the truss

$$\begin{aligned}
&u_1 = 0, && u_2 = 0, && f_1 = -0.1, && f_2 = -0.1,\\
&u_3 = 0, && u_4 = -0.1, && f_3 = 0, && f_4 = 0.1,\\
&u_5 = 0.1899, && u_6 = 0, && f_5 = 0, && f_6 = 0,\\
&u_7 = 0.1899, && u_8 = -0.1333, && f_7 = 0.1, && f_8 = 0.
\end{aligned}$$

Regarding a general procedure, we can consider the elastic boundary conditions prior to applying the displacement boundary conditions. Partition the global degrees of freedom using the index sets $\mathcal{I}_s$ and $\mathcal{I}_o$, where $\mathcal{I}_s$ are the global degrees of freedom with an elastic (spring) boundary condition and $\mathcal{I}_o$ are the remaining global degrees of freedom. Let k_i for $i \in \mathcal{I}_s$ be the spring stiffness at global degree of freedom i. We use this to partition the stiffness matrix $\boldsymbol{K}$, displacements $\boldsymbol{u}$, and force $\boldsymbol{F}$:

$$\begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{K}_{so} \\ \boldsymbol{K}_{os} & \boldsymbol{K}_{oo} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_s \\ \boldsymbol{u}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_s \\ \boldsymbol{F}_o \end{bmatrix}.$$

Then we use $\boldsymbol{F}_s = -\boldsymbol{k}\boldsymbol{u}_s$, where $\boldsymbol{k}$ is the diagonal matrix of the spring constants (ordered by global degree of freedom number), to obtain

$$\begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{K}_{so} \\ \boldsymbol{K}_{os} & \boldsymbol{K}_{oo} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_s \\ \boldsymbol{u}_o \end{bmatrix} = \begin{bmatrix} -\boldsymbol{k}\boldsymbol{u}_s \\ \boldsymbol{F}_o \end{bmatrix}.$$

Moving the displacement term to the left-hand side we arrive at

$$\begin{bmatrix} \boldsymbol{K}_{ss} + \boldsymbol{k} & \boldsymbol{K}_{so} \\ \boldsymbol{K}_{os} & \boldsymbol{K}_{oo} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_s \\ \boldsymbol{u}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}_o \end{bmatrix}.$$

This suggests the global stiffness matrix and force vector should be modified to

$$\tilde{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{K}_{ss} + \boldsymbol{k} & \boldsymbol{K}_{so} \\ \boldsymbol{K}_{os} & \boldsymbol{K}_{oo} \end{bmatrix}, \qquad \tilde{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}_o \end{bmatrix}. \tag{7}$$

i.e., identical to the standard stiffness matrix with the entries corresponding to degrees of freedom with an elastic constraint modified by the spring stiffness. With this modification to the stiffness matrix, the spring reaction is accounted for so the external load that would appear due to the spring in the $\boldsymbol{F}$ matrix is now zero. From this point, Dirichlet and force boundary conditions are applied to the system

$$\tilde{\boldsymbol{K}}\boldsymbol{u} = \tilde{\boldsymbol{F}} \tag{8}$$

in the usual way.

Problem 3: See Hwk 2 code

Problem 4: The displacements and external/reaction forces are:

$$\begin{aligned}
&u_1 = 0, && u_2 = 0, && f_1 = -0.1, && f_2 = -0.1,\\
&u_3 = 0, && u_4 = 0, && f_3 = 0, && f_4 = 0.1,\\
&u_5 = 0.0899, && u_6 = 0, && f_5 = 0, && f_6 = 0,\\
&u_7 = 0.0899, && u_8 = -0.0333, && f_7 = 0.1, && f_8 = 0
\end{aligned}$$

and the deformed truss is given in Figure 1.

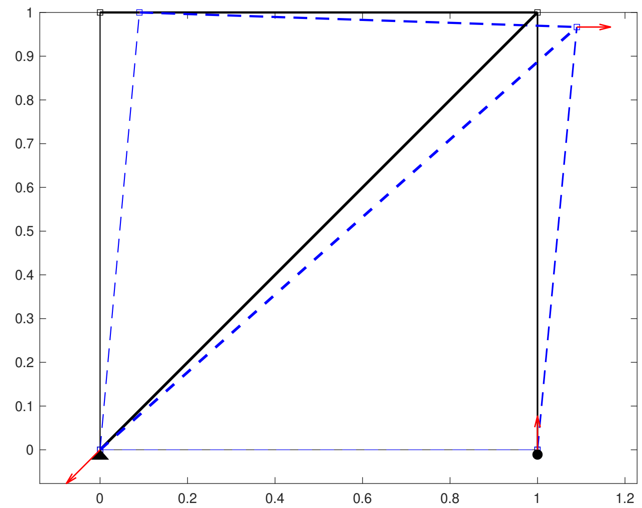


Figure 1: Solution (deformed and reference configuration) of Truss 0

Problem 5: The displacements and external/reaction forces are:

$$\begin{array}{llll} u_1 = 0, & u_2 = 0, & f_1 = 0, & f_2 = -0.25, \\ u_3 = -0.1054, & u_4 = 0, & f_3 = 0, & f_4 = 0.75, \\ u_5 = 0.0487, & u_6 = 0.0723, & f_5 = 0, & f_6 = 0, \\ u_7 = 0.0848, & u_8 = -0.0351, & f_7 = 0, & f_8 = 0, \\ u_9 = 0.1027, & u_{10} = -0.1914, & f_9 = 0, & f_{10} = -0.5 \end{array}$$

and the deformed truss is given in Figure 2.

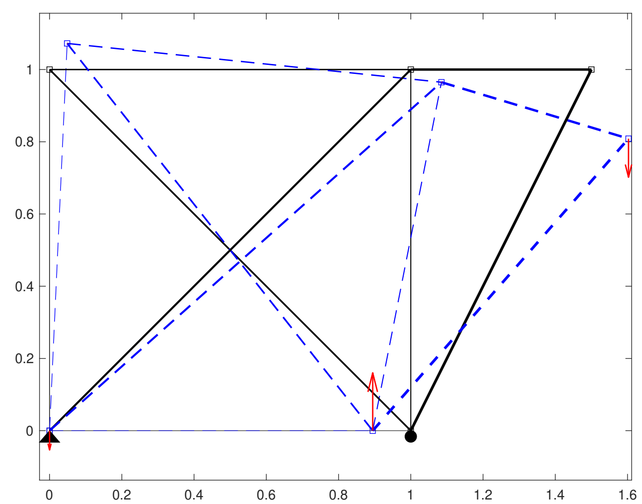


Figure 2: Solution (deformed and reference configuration) of Truss 2

Problem 6: The requested displacements and reaction forces are

$$u_{33} = -0.00615, \quad u_{34} = -0.238, \quad f_1 = -0.01, \quad f_2 = 0.004375$$

and the deformed truss is given in Figure 3.

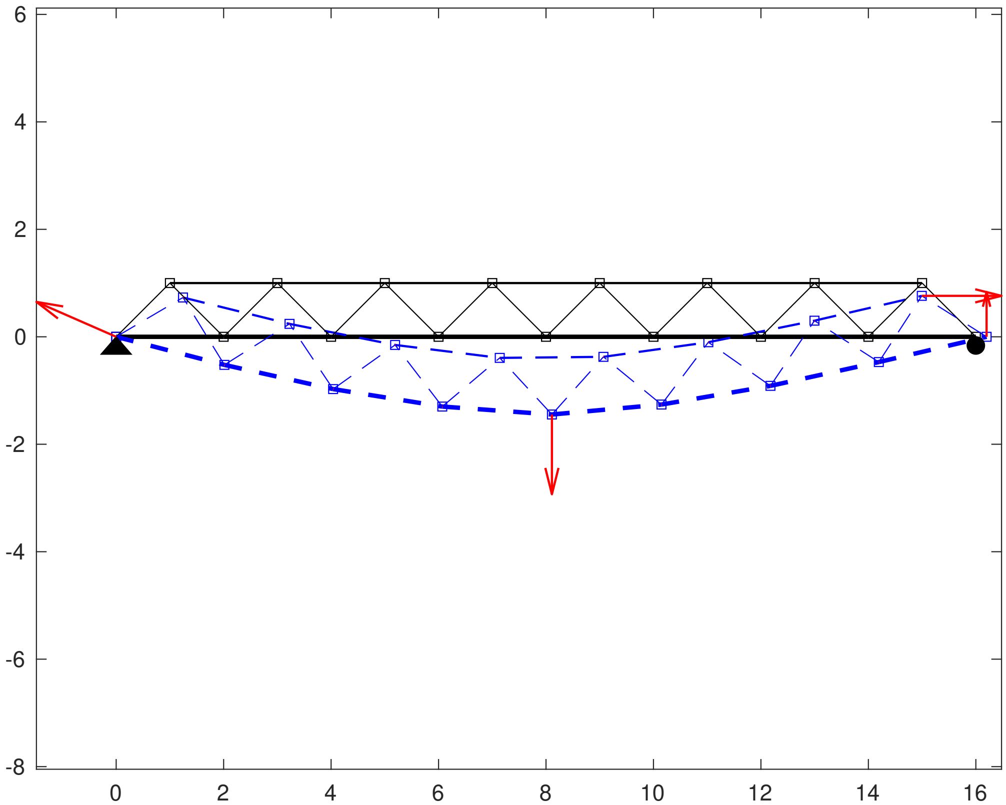


Figure 3: Solution (deformed and reference configuration) of Warren truss

Problem 7: The requested displacements and reaction forces are

$$\begin{aligned} u_1 &= 0, & u_2 &= 0, & f_1 &= -0.1, & f_2 &= -0.1, \\ u_3 &= 0, & u_4 &= -0.1, & f_3 &= 0, & f_4 &= 0.1, \\ u_5 &= 0.1899, & u_6 &= 0, & f_5 &= 0, & f_6 &= 0, \\ u_7 &= 0.1899, & u_8 &= -0.1333, & f_7 &= 0.1, & f_8 &= 0. \end{aligned}$$

and the deformed truss is given in Figure 4.

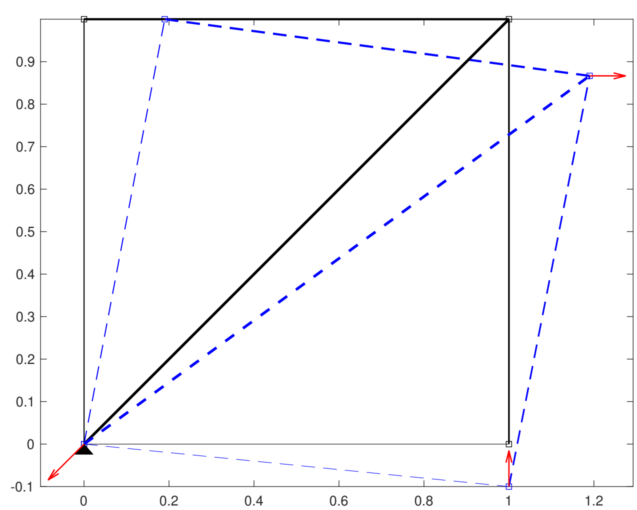
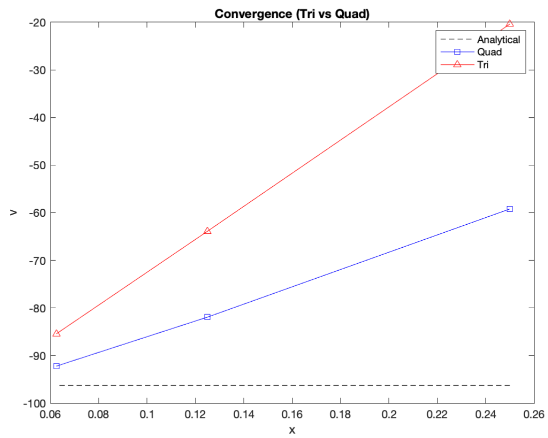


Figure 4: Solution (deformed and reference configuration) of Truss 1

Problem 8: I chose the following parameters

$$q_0 = -1, \quad c = 0.1, \quad L = 50, \quad E = 1e6, \quad \nu = 0.3,$$

which leads to an exact solution at $(x, y) = (25, 0)$ of $v(25, 0) = -96.246$.



Linear triangles are stiffer than linear quadrliateral elements in bending so the binomial $\xi\eta$ (bilinear term) in the quadrliateral element plays a significant role, at least in bending.