AME40541/60541: Finite Element Methods
Homework 3: Due Friday, September 30, 2022

Problem 1: (10 points) Re-write the Navier equations using indicial notation and Einstein summation convention.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + F_1 = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + F_2 = 0$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_3 = 0$$

Problem 2: (15 points) (AME 60541 only) The elasticity tensor for a St. Venant-Kirchhoff material is given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ, μ are the Lamé parameters. Calculate the stress tensor σ_{ij}, where $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and ϵ_{kl} is the strain tensor. Make sure to use the fact that the strain tensor is symmetric ($\epsilon_{ij} = \epsilon_{ji}$). Also, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$. In both cases, your answer should be in terms of λ, μ, and the strain tensor ϵ.

Problem 3: (10 points) From JNR 2.1: Construct the weak form of the nonlinear PDE

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \;\text{ in } (0, L), \qquad \left.\left(u\frac{du}{dx}\right)\right|_{x=0} = 0, \quad u(L) = \sqrt{2},$$

for $f : (0, L) \to \mathbb{R}$ is a given function.

Problem 4: (15 points) The linear elasticity equations model structural deformation in the limit of infinitesimal strain and a linear relationship between stress and stress

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega,$$
$$\sigma_{ij}n_j = \bar{t}_i \quad \text{on } \partial\Omega,$$

for $i = 1, \ldots, d$, where the stress tensor $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}$ and strain tensor $\boldsymbol{\epsilon} \in \mathbb{R}^{d\times d}$ are defined as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \qquad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and $u_i \in \mathbb{R}$ is the displacement in the ith direction for $i = 1, \ldots, d$. We will solely consider a homogeneous, isotropic material with $C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $\lambda(x)$ and $\mu(x)$ are the Lamé parameters. Considering the special case of $d = 2$ is equivalent to making the *plane strain* assumption.
Consider a multimaterial beam (Figure 1) with boundary conditions: clamped on $\partial\Omega_1$ ($u_1 = u_2 = 0$), no traction on $\partial\Omega_2 \cup \partial\Omega_4$ ($\bar{t}_1 = \bar{t}_2 = 0$), and a distributed force in the $-y$ direction of 0.1 on $\partial\Omega_3$ ($\bar{t}_1 = 0, \bar{t}_2 = -0.1$). Take the Lamé parameters for material 1 to be $\lambda_1(x) = 365$, $\mu_1(x) = 188$ and those for material 2 to be $\lambda_1(x) = 36.5$, $\mu_1(x) = 18.8$.

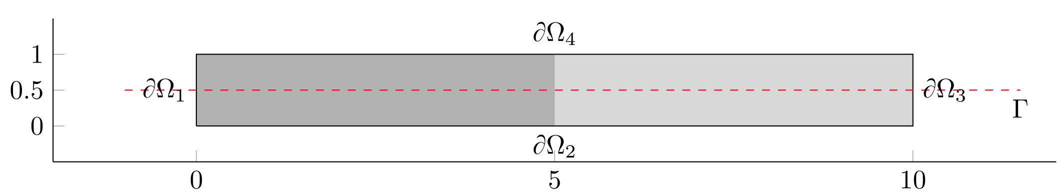

Figure 1: Multimaterial beam (Ω), boundaries ($\partial\Omega_i$), and line along which to evaluate quantities (Γ).

(a) Use COMSOL to approximate the solution to the linear elasticity equations on a sufficiently refined mesh.

(b) Evaluate the displacements u_1, u_2 along the line Γ (Figure 1) and plot the von Mises stress on the deformed geometry.

I suggest saving the numeric value of the slices as we will solve this problem using the FE code you develop during your final project; these values will provide a valuable test for your implementation.

Problem 5: (30 points) (AME 60541 only) Consider a system of m second-order conservation laws in a d-dimensional domain $\Omega \subset \mathbb{R}^d$

$$\nabla \cdot F(U, \nabla U) = S(U, \nabla U) \quad \text{in } \Omega,$$

where $U : \Omega \to \mathbb{R}^m$ is the state, $F(U, \nabla U) \in \mathbb{R}^{m \times d}$ is the flux function (operator), and $S(U, \nabla U) \in \mathbb{R}^m$ is a source term. The boundary conditions are $U = \bar{U}$ on $\partial\Omega_D$ and $F(U, \nabla U)n = \bar{q}$ on $\partial\Omega_N$, where $\bar{U} : \partial\Omega_D \to \mathbb{R}^m$ and $\bar{q} : \partial\Omega_N \to \mathbb{R}^m$ are known boundary functions, $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$, and $n \in \mathbb{R}^d$ is the outward normal.

(a) (5 points) Write the conservation law in indicial notation. Drop the arguments to the flux function and source term.

(b) (10 points) Construct both the weighted residual and weak formulation of the governing equations.

(c) (5 points) What conditions must an approximate solution $U_h(x) \approx U(x)$ satisfy if applying (1) the method of weighted residuals or (2) the Ritz method? Why is it difficult to construct U_h if F is nonlinear in U or ∇U if using the method of weighted residuals?

(d) (10 points) Write each of the following PDEs as a general conservation law, i.e., identify the state (U), flux function ($F(U, \nabla U)$), source term ($S(U, \nabla U)$), and boundary conditions ($\bar{U}$, $\bar{q}$). Also identify the PDE as linear or nonlinear (justify your answer) and the number of solution components (m).

- The second-order, linear elliptic PDE over the domain $\Omega \subset \mathbb{R}^d$

$$(k_{ij} u_{,j})_{,i} = f \ \text{ in } \Omega, \qquad u = \bar{u} \ \text{ on } \partial\Omega_D, \qquad k_{ij} u_{,j} n_i = \bar{t} \ \text{ on } \partial\Omega_N,$$

  where $u : \Omega \to \mathbb{R}$ is the unknown solution, $k : \Omega \to M_{n,n}(\mathbb{R})$ are the elliptic coefficients, $f : \Omega \to \mathbb{R}$ is the source term, $\bar{u} : \partial\Omega_D \to \mathbb{R}$ and $\bar{t} : \partial\Omega_N \to \mathbb{R}$ are boundary conditions, $n : \partial\Omega \to \mathbb{R}^d$ is the outward normal, and the boundary is $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$.

- The linear elasticity equations govern the deformation of a domain $\Omega \subset \mathbb{R}^d$ subject to loads $f : \Omega \to \mathbb{R}^d$

$$\sigma_{ij,j} + f_i = 0 \ \text{ in } \Omega, \qquad u_i = \bar{u}_i \ \text{ on } \partial\Omega_D, \qquad \sigma_{ij} n_j = \bar{t}_i \ \text{ on } \partial\Omega_N$$

  where $\sigma : \Omega \to M_{d,d}(\mathbb{R})$ is the stress field, $\bar{u} : \partial\Omega_D \to \mathbb{R}^d$ is the prescribed boundary displacement field and $\bar{t} : \partial\Omega_N \to \mathbb{R}^d$ is the prescribed traction, $n : \partial\Omega \to \mathbb{R}^d$ is the outward normal, and the boundary is $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$. The stress is related to the strain field $\epsilon : \Omega \to M_{d,d}(\mathbb{R})$ using Hooke's law (linear elastic material) and the strains are assumed infinitesimal

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \qquad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

- The incompressible Navier-Stokes equations (see equations in Problem 2)

Problem 6: (20 points) Consider the incompressible Navier-Stokes equations that govern the flow of an incompressible fluid with density $\rho : \Omega \to \mathbb{R}_{>0}$ and viscosity $\nu : \Omega \to \mathbb{R}_{>0}$ through a domain $\Omega \subset \mathbb{R}^3$

$$\begin{aligned}
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_1}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_1}{\partial x_3}\right) + \rho v_1\frac{\partial v_1}{\partial x_1} + \rho v_2\frac{\partial v_1}{\partial x_2} + \rho v_3\frac{\partial v_1}{\partial x_3} + \frac{\partial p}{\partial x_1} &= 0 \\
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_2}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_2}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_2}{\partial x_3}\right) + \rho v_1\frac{\partial v_2}{\partial x_1} + \rho v_2\frac{\partial v_2}{\partial x_2} + \rho v_3\frac{\partial v_2}{\partial x_3} + \frac{\partial p}{\partial x_2} &= 0 \\
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_3}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_3}{\partial x_3}\right) + \rho v_1\frac{\partial v_3}{\partial x_1} + \rho v_2\frac{\partial v_3}{\partial x_2} + \rho v_3\frac{\partial v_3}{\partial x_3} + \frac{\partial p}{\partial x_3} &= 0 \\
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0
\end{aligned}$$

where $v : \Omega \rightarrow \mathbb{R}^3$ with components $v = (v_1, v_2, v_3)$ is the velocity of the fluid and $p : \Omega \rightarrow \mathbb{R}_{>0}$ is the pressure. The boundary is partitioned into two pieces: $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$, where $n : \partial\Omega \rightarrow \mathbb{R}^3$ is the outward unit normal. The flow velocity and pressure are prescribed along $\partial\Omega_D$

$$v = \bar{v}, \; p = \bar{p} \quad \text{on } \partial\Omega_D,$$

where $\bar{u} : \partial\Omega_D \rightarrow \mathbb{R}^3$ is the prescribed velocity with components $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $\bar{p} : \partial\Omega_D \rightarrow \mathbb{R}_{>0}$ is the prescribed pressure. The traction is prescribed as $\bar{t} = (\bar{t}_1, \bar{t}_2, \bar{t}_3)$ along $\partial\Omega_N$

$$\begin{aligned}
\rho\nu\left(\frac{\partial v_1}{\partial x_1}n_1 + \frac{\partial v_1}{\partial x_2}n_2 + \frac{\partial v_1}{\partial x_3}n_3\right) - pn_1 &= \rho\bar{t}_1 \\
\rho\nu\left(\frac{\partial v_2}{\partial x_1}n_1 + \frac{\partial v_2}{\partial x_2}n_2 + \frac{\partial v_2}{\partial x_3}n_3\right) - pn_2 &= \rho\bar{t}_2 \quad \text{on } \partial\Omega_N \\
\rho\nu\left(\frac{\partial v_3}{\partial x_1}n_1 + \frac{\partial v_3}{\partial x_2}n_2 + \frac{\partial v_3}{\partial x_3}n_3\right) - pn_3 &= \rho\bar{t}_3.
\end{aligned}$$

Re-write these equations in indicial notation and construct the weak form of the equations. Observe that in this form the equations can easily be generalized from three dimensions to d dimensions. This means you have also derived the weak formulation of the 1d, 2d (and higher dimensions!) incompressible Navier-Stokes equations as well.

Problem 7: (30 points) Consider the equations associated with a simply supported beam and subjected to a uniform transverse load $q = q_0$:

$$\begin{aligned}
\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) &= q_0 \quad \text{for} \quad 0 < x < L \\
w = EI\frac{d^2w}{dx^2} &= 0 \quad \text{at} \quad x = 0, L.
\end{aligned}$$

Take $L = 1$, $EI = 1$, and $q_0 = -1$ and approximate the solution using the following methods:

(a) the method of weighted residuals (Galerkin) using the two-term trigonometric basis $w(x) \approx w_2(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$,

(b) the method of weighted residuals (collocation) using the same trigonometric basis and collocation nodes $x_1 = 0.25$ and $x_2 = 0.75$,

(c) the Ritz method using the same trigonometric basis, and

(d) the Ritz method using a two-term polynomial basis ($w(x) \approx w_2(x) = c_1x(x-L) + c_2x^2(x-L)^2$).

For each method, verify the solution basis satisfies the appropriate conditions. Plot the approximate solution generated by each method as well as the analytical solution. In a separate figure, plot the error of each method $e(x) = |w(x) - \tilde{w}(x)|$, where $\tilde{w}$ is the approximate solution, and the residual over the domain. Finally, quantify the error of each approximation using the L^2-norm

$$e_{L^2(\Omega)} = \sqrt{\int_\Omega |e(x)|^2 \, dV}.$$

I recommend using some symbolic mathematics software (Maple, Mathematica, MATLAB, etc) to assist with the calculations.