AME40541/60541: Finite Element Methods
Homework 4 Solutions

Problem 1: The weak formulation of the PDE reads: find u such that for all w,

$$\int_0^L \left[\frac{dw}{dx}\frac{du}{dx} + w(x^2 - u)\right] dx - w(1) = 0.$$

Restriction to element Ω^e and using the basis $\{\phi_i^e\}$ to represent the solution u and test function w, we have the element stiffness matrix and force vector

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left[\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx} - \phi_i^e\phi_j^e\right] dx$$
$$F_i^e = -\int_{x_1^e}^{x_2^e} x^2\phi_i^e \, + \phi_i^e(1).$$

The boundary term is only included in the element force vector for the element that touches the right boundary ($x = 1$). Substitution of the analytical form of the basis functions (and simplification with MAPLE) yields

$$\boldsymbol{K}^e = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$\boldsymbol{F}^e = -\frac{h_e}{12}\begin{bmatrix} 3(x_1^e)^2 + 2x_1^e x_2^3 + (x_2^e)^2 \\ (x_1^e)^2 + 2x_1^e x_2^e + 3(x_2^e)^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where the second term in the force vector comes from the boundary term and is only included for the rightmost element of the mesh. See attached code.

Problem 2: The weak formulation of the PDE reads: find T such that for all w,

$$\int_\Omega w_{,i}T_{,i}\, dV - \int_{\partial\Omega_1} wT_{,i}n_i\, dS = 0.$$

Restriction to element Ω^e and using the basis $\{\phi_i^e\}$ to represent the solution T and test function w, we have the element stiffness matrix and force vector

$$K_{ij}^e = \int_{x_1^e}^{x_2^e}\int_{y_1^e}^{y_2^e} \phi_{i,k}^e\phi_{j,k}^e\, dy\, dx$$
$$F_i^e = \int_{y_1^e}^{y_2^e} \phi_i^e(x=0,y)\, dy,$$

where summation notation is used to sum over k (no sum over e). The element force vector is only non-zero (and equal to the vector given above) for the elements that touch $\partial\Omega_1$, i.e., all element lying on the left boundary. Substitution of the analytical form of the basis functions (and simplification with MAPLE) yields

$$\boldsymbol{K}^e = \frac{1}{6\Delta x_e\Delta y_e}\begin{bmatrix} 2(\Delta x_e^2 + \Delta y_e^2) & \Delta x_e^2 - 2\Delta y_e^2 & -2\Delta x_e^2 + \Delta y_e^2 & -\Delta x_e^2 - \Delta y_e^2 \\ \Delta x_e^2 - 2\Delta y_e^2 & 2(\Delta x_e^2 + \Delta y_e^2) & -\Delta x_e^2 - \Delta y_e^2 & -2\Delta x_e^2 + \Delta y_e^2 \\ -2\Delta x_e^2 + \Delta y_e^2 & -\Delta x_e^2 - \Delta y_e^2 & 2(\Delta x_e^2 + \Delta y_e^2) & \Delta x_e^2 - 2\Delta y_e^2 \\ -\Delta x_e^2 - \Delta y_e^2 & -2\Delta x_e^2 + \Delta y_e^2 & \Delta x_e^2 - 2\Delta y_e^2 & 2(\Delta x_e^2 + \Delta y_e^2) \end{bmatrix}, \quad \boldsymbol{F}^e = \frac{\Delta y_e}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

where the element force vector is only non-zero for elements that touch the left boundary. See attached code.