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The maximum flow problem of uncertain network ☆

Shengwei Han [a,*], Zixiong Peng [b], Shunqin Wang [c]

[a] Department of Mathematics, Shaanxi Normal University, Xi'an 710062, PR China
[b] School of Applied Mathematics, Central University of Finance and Economics, Beijing 100081, PR China
[c] School of Mathematics and Statistics, Nanyang Normal University, Nanyang 473061, PR China

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A B S T R A C T

The maximum flow problem is one of the classic combinatorial optimization problems with many applications in electrical power systems, communication networks, computer networks and logistic networks. The goal of the problem is to find the maximum amount of flow from the source to the sink in a network. A network is called uncertain if the arc capacities of the network are uncertain variables. The main purpose of this paper is to solve the maximum flow in an uncertain network by under the framework of uncertainty theory.

1. Introduction

The maximum flow problem is one of the classic problems of network optimization. The motivation for the maximum flow problem came from the Soviet railway system [2]. As Ford and Fulkerson [12] explained, the maximum flow problem, formulated by Harris, is as follows: Consider a railway network connecting two cities by way of a number of intermediate cities, where each link of the network has an assigned number to represent its capacity. Assuming a steady state condition, we find a maximal flow from one given city to the other. Due to its many applications in electrical power systems, communication networks, computer networks and logistic networks, the maximum flow problem has been widely studied over the last 50 years. Therefore, from the viewpoint of application value, the study of maximum flow problem is of great significance.

The maximum flow problem has a long history. In the past five decades, many efficient algorithms for the maximum flow problem have emerged [36]. Representative methods in maximum flow algorithms are based on either augmenting paths or preflows [20,30]. Augmenting-path algorithms push flow along a path from the source to the sink in the residual network, and include Ford–Fulkerson's labeling algorithm [11,12] and Dinic's blocking flow algorithm [8]. Preflow-based algorithms push flow along edges in the residual network, and include Karzanov's blocking flow algorithm [29] and Goldberg–Tarjan's push-relabeling algorithm [38]. A class of graph algorithms has recently emerged, referred to as symbolic graph algorithms [9,17], based on which Gu and Xu presented the symbolic algorithms for the maximum flow in general networks [16]. In short, for a fixed network, the maximum flow by the above algorithms can be calculated when the arc capacities of the network are given. However, different types of uncertainty are frequently encountered in practice for a variety of reasons. For a

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* Corresponding author. Tel.: +86 29 85310232.
E-mail address: hansw@snnu.edu.cn (S. Han).

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new network, the arc capacities are uncertain in many situations. How the maximum flow of a network can be modeled and solved when in an uncertain environment is the main concern of this paper.

How do we consider the maximum flow of an uncertain network? Since under the uncertain environment the arc capacities of a network are uncertain, some researchers regarded the arc capacities as random variables [10,15,33] or fuzzy variables [3–5,18,19,37]. Such researchers employed probability theory or fuzzy theory to study the maximum flow problem. In other words, they mainly used stochastic optimization, chance constrained programming and robust optimization to solve the maximum flow problem in an uncertain network [1,31,32,34]. This paper will introduce the concept of the maximum flow function and apply uncertainty theory to the maximum flow problem in an uncertain network. This is a distinct contribution from other optimization methods. The similarities and differences between Liu's uncertainty concept and standard probabilistic concept, as well as other concepts of uncertainty could be found in [25].

In order to deal with some uncertain phenomena, uncertainty theory was proposed by Liu in 2007 and refined by Liu in 2010, and it has become a branch of mathematics for modeling human uncertainty. Uncertainty theory is different from probability theory and fuzzy theory. In uncertainty theory, uncertain measure is defined based on four axioms: normality axiom, duality axiom, subadditivity axiom and product axiom. Up to now, theory and practice have shown that uncertainty theory is an efficient tool to deal with nondeterministic information, especially expert data and subjective estimation. From a theoretical aspect, uncertain process [22,40], uncertain differential equation [6,41], and uncertain logic [27] have been established. From a practical aspect, uncertain programming [13,14,24,39], uncertain calculus [7,23], and uncertain risk analysis [26] have also developed quickly. In short, uncertainty theory is increasingly being researched and used. To explore the recent developments of uncertainty theory, readers may consult the book of Liu [25].

In this paper, we will illustrate that uncertainty theory can serve as a powerful tool to deal with the maximum flow in an uncertain network. The question is in what condition a network is suitable for application of the uncertainty theory? When we lack observed data for the arc capacities of a network, we often invite some domain experts to evaluate their belief degree that each event will occur, not only for economic reasons, but also for technical difficulties. However, since human beings usually tend to overweight unlikely events, the belief degree may have much larger variance than the real frequency, in which case probability theory is no longer valid. In this situation, uncertainty theory could be applied in network research.

2. Preliminaries

In this section, we will introduce some basic concepts and results of uncertainty theory.

Definition 1 (see [21]). Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ. A set function $\mathbb{M}:\mathcal{L}\to[0,1]$ is called an *uncertain measure* if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $\mathbb{M}\{\Gamma\}=1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathbb{M}\{\Lambda\}+\mathbb{M}\{\Lambda^c\}=1$ for any event $\Lambda\in\mathcal{L}$.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\ldots$, we have

$$\mathbb{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\leqslant\sum_{i=1}^{\infty}\mathbb{M}\{\Lambda_i\}.$$

The triple $(\Gamma,\mathcal{L},\mathbb{M})$ is called an *uncertainty space*. Besides, in order to provide an operational law, Liu defined the product uncertain measure on the product σ-algebra $\mathcal{L}$ as follows.

Axiom 4: (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathbb{M}_k)$ be uncertainty spaces for $k=1,2,\ldots$. Then the product uncertain measure $\mathbb{M}$ is an uncertain measure satisfying

$$\mathbb{M}\left\{\prod_{i=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathbb{M}\{\Lambda_k\}$$

where Λ_k are arbitrary chosen events from $\mathcal{L}_k$ for $k=1,2,\ldots$, respectively.

Definition 2 (see [21]). An uncertain variable is a measurable function ξ from an uncertainty space to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi\in B\}=\{\gamma\in\Gamma|\xi(\gamma)\in B\}$$

is an event.

The uncertain variables we often used include linear uncertain variable, zigzag uncertain variable, normal uncertain variable and lognormal uncertain variable. As uncertainty theory has been widely applied in many fields, some new problems correspondingly emerge, one of which is to specify an uncertain variable definitely and conveniently. Liu suggested the concept of an uncertainty distribution, which is an important way to specify an uncertain variable.

Definition 3 (see [21]). The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathbb{M}\{\gamma \in \Gamma | \xi(\gamma) \leqslant x\}.$$

for any real number x.

The uncertainty distributions with respect to uncertain variables include linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution.

Linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant a \\ (x-a)/(b-a), & \text{if } a \leqslant x \leqslant b \\ 1, & \text{if } x \geqslant b \end{cases}$$

denoted by $\mathcal{L}(a,b)$ where a and b are real numbers with $a < b$.

Zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant a \\ (x-a)/2(b-a), & \text{if } a \leqslant x \leqslant b \\ (x+c-2b)/2(c-b), & \text{if } b \leqslant x \leqslant c \\ 1, & \text{if } x \geqslant c \end{cases}$$

denoted by $\mathcal{Z}(a,b,c)$ where a,b,c are real numbers with $a < b < c$.

Normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

Lognormal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-\ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x > 0$$

denoted by $\mathcal{LOGN}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$. For the properties of uncertainty distribution, please refer to the work of Peng and Iwamura (see [35]).

An uncertainty distribution Φ is said to be *regular* if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. In this paper, we always assume that the given uncertainty distributions are regular. Otherwise, we may give the uncertainty distribution a small perturbation so that it becomes regular.

A real-valued function $f(x_1,x_2,\ldots,x_n)$ is said to be *strictly increasing* provided that f satisfies the following conditions:

(1) if $x_i \leqslant y_i$ for $i = 1,2,\ldots,n$, then $f(x_1,x_2,\ldots,x_n) \leqslant f(y_1,y_2,\ldots,y_n)$;
(2) if $x_i < y_i$ for $i = 1,2,\ldots,n$, then $f(x_1,x_2,\ldots,x_n) < f(y_1,y_2,\ldots,y_n)$.

Theorem 1 (see [25]). *Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

3. Maximum flow function

In this paper, we assume that the networks are directed with only one source and one sink. For a network, it is important to calculate the maximum flow when the arc capacities are given. With the Ford-Fulkerson's algorithm, if the arc capacities of a network are given, then we can calculate the maximum flow y of the network. For the different capacities of arcs, we will

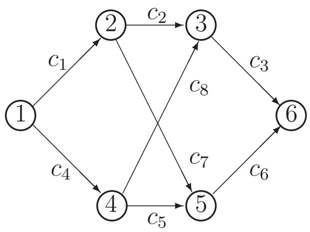

Fig. 1. Network G for Example 1.

obtain different but unique maximum flow y. In other words, the maximum flow y is a function of arc capacities. So we have the following definition.

Definition 4. Let G be a network with n arcs. Then the maximum flow y of the network G is a function of arc capacities, denoted by

$$y = f(c_1, c_2, \ldots, c_n),$$

we call $f(c_1, c_2, \ldots, c_n)$ the maximum flow function of the network G, where c_i denote the capacities of the ith arcs, $i = 1, 2, \ldots, n$, respectively.

Example 1. Let G be a network with eight arcs defined by Fig. 1.

Let $c_1 = 4, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = 2, c_6 = 5, c_7 = 3, c_8 = 2$, then by the Ford-Fulkerson's algorithm we can calculate the maximum flow $y = f(c_1, c_2, \ldots, c_8) = 8$.

Example 2. Let G be a network with five arcs defined by Fig. 2.

Then the maximum flow function $f(c_1, c_2, c_3, c_4, c_5) = (((c_1 - c_2) \vee 0) \wedge c_3 + c_4) \wedge c_5 + (c_1 \wedge c_2)$.

Someone maybe asks whether there is a need to know the expression of the maximum flow function $f(c_1, c_2, \ldots, c_n)$. In fact, it is not a must. For the maximum flow function $f(c_1, c_2, \ldots, c_n)$ we have the following theorem.

Theorem 2. *Let G be a network with n arcs and $f(c_1, c_2, \ldots, c_n)$ the maximum flow function of the network G. Then $f(c_1, c_2, \ldots, c_n)$ is a continuous and strictly increasing function with respect to c_i, where c_i denote the capacities of the ith arcs, $i = 1, 2, \ldots, n$, respectively.*

Proof. Assume that $f(c_1, c_2, \ldots, c_n)$ is the maximum flow function of the network G. Let $x_1 \leqslant y_1, x_2 \leqslant y_2, \ldots, x_n \leqslant y_n$ and $d = \min_{1 \leqslant i \leqslant n}\{y_i - x_i\}$, then $d \geqslant 0$. By the Ford-Fulkerson's algorithm, we can get that

$$f(x_1, x_2, \ldots, x_n) + d \leqslant f(y_1, y_2, \ldots, y_n),$$

that is

$$0 \leqslant f(y_1, y_2, \ldots, y_n) - f(x_1, x_2, \ldots, x_n).$$

If $x_1 < y_1, x_2 < y_2, \ldots, x_n < y_n$, then $d > 0$ and $0 < f(y_1, y_2, \ldots, y_n) - f(x_1, x_2, \ldots, x_n)$. Therefore, $f(c_1, c_2, \ldots, c_n)$ is a strictly increasing function.

In the following, we prove that the maximum flow function $f(c_1, c_2, \ldots, c_n)$ is continuous. For any point $(x_1, x_2, \ldots, x_n)$, we let $d = \max_{1 \leqslant i \leqslant n}\{|x'_i - x_i|\}$. Then by the Ford–Fulkerson's algorithm, we have

$$|f(x_1, x_2, \ldots, x_n) - f(x'_1, x'_2, \ldots, x'_n)| \leqslant nd,$$

which implies $f(c_1, c_2, \ldots, c_n)$ is continuous at the point $(x_1, x_2, \ldots, x_n)$. $\square$

Remark 1. For the maximum flow function $f(c_1, c_2, \ldots, c_n)$ of a network with n arcs, we have $f(mc_1, mc_2, \ldots, mc_n) = mf(c_1, c_2, \ldots, c_n)$ for any natural number $m \in \mathbb{N}$.

4. Maximum flow of uncertain network

Dealing with practical problems, we have frequently encountered different types of uncertainty, which should be taken into our account. For a network (especially for a new network), we do not exactly know the capacity of each arc. In this situation, how shall we consider the maximum flow of the network? Since under the uncertain environment, the arc capacities of a network are uncertain, some researchers regard the arc capacities as random variables or fuzzy variables.

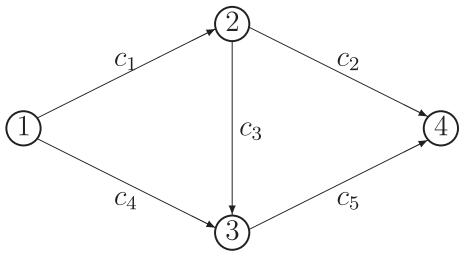

Fig. 2. Network G for Example 2.

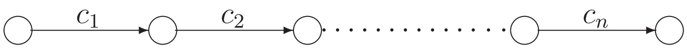

Fig. 3. Network G for Example 3.

In fact, for a given network, the arc capacities of the network are certain. Only for some reasons, their real values are beyond our study findings. Based on this view, we shall regard the arc capacities as uncertain variables in this paper.

Definition 5. A network G is called *uncertain* if the arc capacities of G are uncertain variables.

Uncertain network was first proposed by Liu in order to model the project scheduling (see [24]). In an uncertain network with n arcs, we will assume that the capacities of the ith arc are uncertain variable $\xi_i,\ i = 1, 2, \ldots, n$, respectively, and then see that the maximum flow of the network is also an uncertain variable ξ with $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$. For the uncertain variable ξ, it is very important to give its uncertainty distribution $\Psi(x)$ or its inverse uncertainty distribution $\Psi^{-1}(x)$. By Theorems 1 and 2, for the maximum flow of uncertain network, we have the following theorem.

Theorem 3. *Let G be an uncertain network with n arcs, $f(c_1, c_2, \ldots, c_n)$ the maximum flow function of the network G. Then the maximum flow of the network G is an uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ and its inverse uncertainty distribution is*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right)$$

where ξ_i denote the uncertain capacities of the ith arcs with uncertainty distributions $\Phi_i(x), i = 1, 2, \ldots, n$, respectively.

Remark 2. In Theorem 3, if the arc capacities of the network G are regarded as random variables or fuzzy variables, then it is very difficult to give the distribution of ξ or the inverse distribution of ξ. In other words, for random variables or fuzzy variables there is no such form of the distribution as in Theorem 3. This is also a reason why we choose uncertain theory to deal with the uncertain network.

Example 3. Let G be a series uncertain network with n arcs defined by Fig. 3. Then the maximum flow function $f(c_1, c_2, \ldots, c_n) = c_1 \wedge c_2 \wedge \ldots \wedge c_n$.

We assume that the capacities of the ith arcs are uncertain variables ξ_i with uncertainty distributions $\Phi_i(x),\ i = 1, 2, \ldots, n$, respectively. Then by Theorem 1 we have that the maximum flow of the network G is an uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ and its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \ldots \wedge \Phi_n^{-1}(\alpha).$$

Example 4. Let G be an uncertain network with four arcs defined by Fig. 4. Then the maximum flow function $f(c_1, c_2, c_3, c_4) = (c_1 \wedge c_2) + (c_3 \wedge c_4)$.

We assume that the capacities of the ith arc are uncertain variables ξ_i with uncertainty distributions $\Phi_i(x)\ i = 1, 2, 3, 4$, respectively. Then by Theorem 1 we have that the maximum flow of the network G is an uncertain variable $\xi = f(\xi_1, \xi_2, \xi_3, \xi_4)$ and its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = (\Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha)) + (\Phi_3^{-1}(\alpha) \wedge \Phi_4^{-1}(\alpha)).$$

99-Method In [25], Liu suggested that an uncertain variable ξ with uncertainty distribution $\Phi(x)$ is represented by a 99-table,

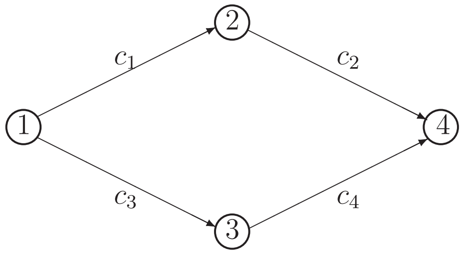

Fig. 4. Network G for Example 4.

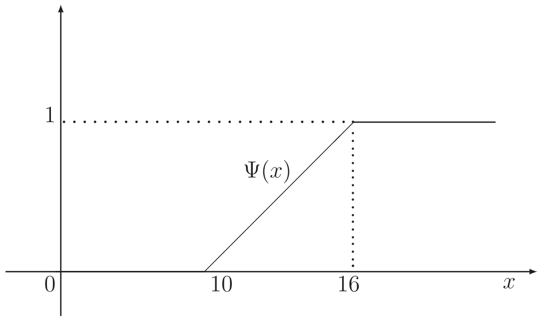

Fig. 5. The uncertainty distribution $\psi(x)$.

0.01	0.02	0.03	$\cdots$	0.99
x_1	x_2	x_3	$\cdots$	x_{99}

where $0.01, 0.02, \ldots, 0.99$ in the first row are the values of uncertainty distribution $\Phi(x)$, and $x_1, x_2, \ldots, x_{99}$ in the second row are the corresponding values of $\Phi^{-1}(0.01), \Phi^{-1}(0.02), \ldots, \Phi^{-1}(0.99)$. The 99-method may be extended to any fixed scale if there is a need. By the 99-table, we may express an empirical uncertainty distribution $\Psi(x)$ to approximate the uncertainty distribution $\Phi(x)$ of the uncertain variable ξ as

$$\Psi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + (\alpha_{i+1} - \alpha_i)(x - x_i)/(x_{i+1} - x_i), & \text{if } x_i \leqslant x \leqslant x_{i+1}, 1 \leqslant i \leqslant 99 \\ 1, & \text{if } x > x_{99}. \end{cases} \tag{4.1}$$

Let G be an uncertain network with n arcs, $f(c_1, c_2 \ldots, c_n)$ the maximum flow function of the network G, and ξ_i denote the uncertain capacities of the ith arcs with uncertainty distributions $\Phi_i(x), i = 1, 2, \ldots, n$, respectively. By the 99-table, we can give the empirical uncertainty distribution $\Psi(x)$ of the maximum flow ξ

Step 1 By Theorem 1, we can calculate $x_1, x_2, \ldots, x_{99}$.

$$\alpha_1 = 0.01, x_1 = f\left(\Phi_1^{-1}(0.01), \Phi_2^{-1}(0.01), \ldots, \Phi_n^{-1}(0.01)\right);$$
$$\alpha_2 = 0.02, x_2 = f\left(\Phi_1^{-1}(0.02), \Phi_2^{-1}(0.02), \ldots, \Phi_n^{-1}(0.02)\right);$$
$$\vdots$$
$$\alpha_{99} = 0.99, x_{99} = f\left(\Phi_1^{-1}(0.99), \Phi_2^{-1}(0.99), \ldots, \Phi_n^{-1}(0.99)\right).$$

Step 2 By the 99-table, we can give the empirical uncertainty distribution $\Psi(x)$.

Example 5. In Example 2, we assume that the capacities of the ith arcs are uncertain variables ξ_i with linear uncertainty distributions $\Phi_i, i = 1, 2, 3, 4, 5$, respectively, in which

$$\Phi_1(x) = \begin{cases} 0, & \text{if } x < 8 \\ (x - 8)/2, & \text{if } 8 \leqslant x \leqslant 10 \\ 1, & \text{if } x > 10 \end{cases}$$

$$\Phi_2(x) = \begin{cases} 0, & \text{if } x < 5 \\ (x-5)/2, & \text{if } 5 \leqslant x \leqslant 7 \\ 1, & \text{if } x > 7 \end{cases}$$

$$\Phi_3(x) = \begin{cases} 0, & \text{if } x < 1 \\ (x-1)/2, & \text{if } 1 \leqslant x \leqslant 3 \\ 1, & \text{if } x > 3 \end{cases}$$

$$\Phi_4(x) = \begin{cases} 0, & \text{if } x < 4 \\ (x-4)/2, & \text{if } 4 \leqslant x \leqslant 6 \\ 1, & \text{if } x > 10 \end{cases}$$

$$\Phi_5(x) = \begin{cases} 0, & \text{if } x < 7 \\ (x-7)/2, & \text{if } 7 \leqslant x \leqslant 9 \\ 1, & \text{if } x > 9. \end{cases}$$

By Theorem 1, we can see that the maximum flow of the network G is an uncertain variable and its inverse uncertainty distribution is

$$\begin{aligned} \Psi^{-1}(\alpha) &= f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \Phi_3^{-1}(\alpha), \Phi_4^{-1}(\alpha), \Phi_5^{-1}(\alpha)\right) \\ &= \left(\left((\Phi_1^{-1}(\alpha) - \Phi_2^{-1}(\alpha)) \vee 0\right) \wedge \Phi_3^{-1}(\alpha) + \Phi_4^{-1}(\alpha)\right) \wedge \Phi_5^{-1}(\alpha) + \left(\Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha)\right). \end{aligned}$$

By the 99-method, we can obtain the 99-table for the maximum flow ξ.

α	$\Phi_1^{-1}(\alpha)$	$\Phi_2^{-1}(\alpha)$	$\Phi_3^{-1}(\alpha)$	$\Phi_4^{-1}(\alpha)$	$\Phi_5^{-1}(\alpha)$	$\Psi^{-1}(\alpha)$
0.01	8.02	5.02	1.02	4.02	7.02	10.06
0.02	8.04	5.04	1.04	4.04	7.04	10.12
0.03	8.06	5.06	1.06	4.06	7.06	10.18
⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.99	9.98	6.98	2.98	5.98	8.98	15.94

By the 99-table, we can give the empirical uncertainty distribution $\Psi(x)$ of the maximum flow ξ (see Fig. 5).

5. Expected value of maximum flow

Expected value is the average value of an uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

In [21], Liu introduced the concept of an expected value and gave some algorithms for the expected value (see [25]). Let ξ be an uncertain variable. Then the *expected value* of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathbb{M}\{\xi \geqslant r\}dr - \int_{-\infty}^0 \mathbb{M}\{\xi \leqslant r\}dr$$

provided that at least one of the two integrals is finite. Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx.$$

Researches on the expected value of uncertain variables can be found in [28].

Theorem 4. *Let G be an uncertain network with n arcs, $f(c_1, c_2, \ldots, c_n)$ the maximum flow function of the network G and ξ_i denote the uncertain capacities of the ith arcs with uncertainty distributions $\Phi_i(x), i = 1, 2, \ldots, n$, respectively. Then the maximum flow $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right)d\alpha$$

provided that $E[\xi]$ exists.

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Proof. Since the maximum flow function $f(c_1, c_2, \ldots, c_n)$ is continuous and strictly increasing with respect to $c_1, c_2, \ldots, c_n$, it follows from Theorem 1 that the inverse uncertainty distribution of ξ is $\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right)$. By the definitions of expected valued operator and uncertainty distribution, we have

$$\begin{aligned} E[\xi] &= \int_0^{+\infty}(1-\Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx \\ &= \int_{\Psi(0)}^1 \Psi^{-1}(\alpha)d\alpha + \int_0^{\Psi(0)} \Psi(\alpha)d\alpha \\ &= \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))d\alpha. \quad \square \end{aligned}$$

In an uncertain network with n arcs, the maximum flow function $f(c_1, c_2, \ldots, c_n)$ of network cannot be directly expressed with respect to the arc capacities $c_1, c_2, \ldots, c_n$ for the complexity of network. Even though the maximum flow function $f(c_1, c_2, \ldots, c_n)$ is able to be expressed, it is still difficult to calculate the expected value $E[\xi]$ of the maximum flow ξ by the method in Theorem 4. In this case, how shall we calculate the expected value $E[\xi]$ of the maximal flow ξ? The following theorem will give the answer to the question.

Theorem 5. *Let G be an uncertain network with n arcs, $f(c_1, c_2, \ldots, c_n)$ the maximum flow function of the network G and ξ_i denote the uncertain capacities of the ith arcs with uncertainty distributions $\Phi_i(x), i = 1, 2, \ldots, n$, respectively. If $\Psi(x)$ is the empirical uncertainty distribution (4.1) of the maximum flow ξ obtained by the 99-method, then the maximum flow $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = 0.015x_1 + 0.01\sum_{i=2}^{98} x_i + 0.015x_{99}.$$

where $x_i = f\left(\Phi_1^{-1}(i \times 10^{-2}), \Phi_2^{-1}(i \times 10^{-2}), \ldots, \Phi_n^{-1}(i \times 10^{-2})\right),\ i = 1, 2, \ldots, 99.$

Proof. Let $\Psi(x)$ be the empirical uncertainty distribution (4.1) of the maximum flow ξ obtained by the 99-Method. Then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + 0.01(x - x_i)/(x_{i+1} - x_i), & \text{if } x_i \leqslant x \leqslant x_{i+1}, 1 \leqslant i \leqslant 99 \\ 1, & \text{if } x > x_{99} \end{cases}$$

where $\alpha_i - \alpha_{i-1} = 0.01, \alpha_0 = 0, x_i = f\left(\Phi_1^{-1}(i \times 10^{-2}), \Phi_2^{-1}(i \times 10^{-2}), \ldots, \Phi_n^{-1}(i \times 10^{-2})\right), i = 1, 2, \ldots, 99$. Thus the maximum flow $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^{+\infty}(1-\Psi(x))dx - \int_{-\infty}^0 \Psi(x)dx = 0.015x_1 + 0.01\sum_{i=2}^{98} x_i + 0.015x_{99}. \quad \square$$

Remark 3. In Theorem 5, we use the empirical uncertainty distribution instead of the uncertainty distribution of the maximum flow ξ to calculate the expected value $E[\xi]$ of the maximum flow ξ. If a more precise result is needed, then we shall use the 999-method or 9999-method to give the empirical uncertainty distribution of the maximum flow ξ.

6. Numerical example

In this section, we shall consider the maximum flow problem of the network G defined by Fig. 2 with the uncertain variables as given in Example 5.

In Example 2, we can see that the maximum flow function $f(c_1, c_2, c_3, c_4, c_5) = (((c_1 - c_2) \vee 0) \wedge c_3 + c_4) \wedge c_5 + (c_1 \wedge c_2)$. Thus, by Theorem 4 we can calculate the expected value $E[\xi]$ of the maximum flow ξ of the uncertain network G

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right)d\alpha$$

By the description in Section 5, it is difficult to calculate the integral for the complexity of the maximum flow function $f(c_1, c_2, c_3, c_4, c_5)$. In the following, we will calculate the expected value $E[\xi]$ of the maximum flow ξ in terms of the empirical uncertainty distribution obtained by the 99-method. In Example 5, we can obtain $x_1, x_2, \ldots, x_{99}$ from the 99-table

x_1	x_2	x_3	$\cdots$	x_{98}	x_{99}
10.06	10.12	10.18	$\cdots$	15.88	15.94

By Theorem 5, we can calculate the expected value $E[\xi]$ of the maximum flow ξ of the uncertain network G

$$E[\xi] = 0.015x_1 + 0.01\sum_{i=2}^{98} x_i + 0.015x_{99} = 13.$$

In Example 5, in terms of the uncertainty distributions $\Phi_i(x)$ of arc capacities $\xi_i, i = 1,2,3,4,5$, we can obtain the least (greatest) values 8, 5, 1, 4, 7 (10,7,3,6,9) of each arc capacities, respectively. Thus, we can calculate the least value $f(8,5,1,4,7) = 10$ and the greatest value $f(10,7,3,6,9) = 16$ of the maximum flow ξ, respectively. The expected value $E[\xi] = 13$ of the maximum flow ξ lies between 10 and 16. This case indicates that the result obtained by the empirical uncertainty distribution is acceptable.

7. Conclusion

Uncertain factors often appear in practical problems. Uncertainty theory provides a new approach to deal with uncertain factors. In this paper, we investigate the maximum flow problem of network in an uncertain environment. We first introduce the concept of maximum flow function of network, and then under the framework of uncertainty theory we use the 99-method to give the uncertainty distribution and the expected value of the maximum flow of uncertain network. It is hoped that these conclusions will be helpful for decision-makers of a project.

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