Logic and Computability Exploration

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1 The Ceiling Fan Problem

I came up with this problem while trying to turn off the ceiling fan in my apartment.

Problem
You are in a room with a chain-operated ceiling fan that is currently spinning. This fan has an unknown number of speed settings, and it can be controlled in the usual way by pulling its chain. When the fan is turned off, it takes an unknown number of seconds for the fan to stop spinning due to momentum. Before the fan stops spinning you cannot be sure it is actually off. Furthermore, you cannot detect which speed the fan is moving, only whether or not the blades are spinning. Can you devise a procedure that will turn off any fan in a finite amount of time?

Solution
Let the fan run for one second then pull the chain, let the fan run for two seconds then pull the chain, repeat with three seconds, four seconds, et cetera until the fan visibly stops. This procedure ensures that the fan will be in the off state for an arbitrarily large number of seconds. Then eventually the fan blades will stop spinning and we will know the fan is off.

2 Investigating the Turing degrees of finitely-axiomatizable theories of first order logic

This summer I worked my way through *Computability and Logic* by Boolos and *Introduction to Recursive Function Theory* by Cutland. I noticed similarities between sentences of first order logic ordered by logical consequence and recursively enumerable sets of natural numbers ordered by Turing reducability. I wanted to see if I could turn results about Turing degrees into results about sentences of first order logic. This investigation is the result.

2.1 Preliminaries

I will use capital letters such as A, B, Γ to denote finite sets of sentences or a single sentence of first order logic. A finite set of sentences of first order logic can be converted into a single sentence by taking the conjunction of all sentences, so I will consider finite sets of sentences to be a single sentence when necessary.

I will use $A \leq_T B$ to denote that set A is Turing reducible to set B.
I will use $\implies$ to denote the semantic relation of logical implication.
I will use $\longrightarrow$ to denote the syntactic implication symbol (so $A \longrightarrow B$ will be short for $\neg A \vee B$).
I will use $Theory(A)$ to denote the smallest set of sentences of first order logic closed under logical implication that contains A. I will only consider theories that are *finitely-axiomatizable*, (i.e. theories that can be axiomatized by a finite set of sentences, A). When talking about Turing reductions on sets of logical sentences, we suppose an effectively computable coding of the sets of logical sentences onto the set of natural numbers $\mathbb{N}$.

I will operate in first order logic over a language L containing a constant 0, four dyadic predicates $<, S, M, @$, and enumerably many monadic predicates $Q_0, Q_1, Q_2, ...$

2.2 Building the framework

My first three lemmas will build up the framework for converting statements about Turing degrees to statements about logical implication on sentences.

2.1 Lemma. *Let A, B be finite sets of sentences in a language L of first order logic. If $A \implies B$, then $Theory(A) \leq_T Theory(B)$.*

Proof. Assume $A \implies B$. Let Γ be a sentence of first order logic. I will demonstrate that $\Gamma \in \mathit{Theory}(A)$ if and only if $A \longrightarrow \Gamma \in \mathit{Theory}(B)$. This will yield a Turing reduction from $\mathit{Theory}(A)$ to $\mathit{Theory}(B)$. First, note that for any pair of sentences X, Y of first order logic, $X \in \mathit{Theory}(Y)$ iff $Y \longrightarrow X$ is a logically valid sentence (ie true in every model).

Only if direction:
Suppose $\Gamma \in \mathit{Theory}(A)$. Then $A \longrightarrow \Gamma$ is logically valid, so it is true in every model including every model of B. Therefore, $A \longrightarrow \Gamma \in \mathit{Theory}(B)$.

If direction:
Suppose $A \longrightarrow \Gamma \in \mathit{Theory}(B)$. Then $B \longrightarrow (A \longrightarrow \Gamma)$ is logically valid. But since $A \implies B$, we know that $A \longrightarrow B$ is logically valid, so $A \longrightarrow (A \longrightarrow \Gamma)$ is logically valid, so $A \longrightarrow \Gamma$ is logically valid, so $\Gamma \in \mathit{Theory}(A)$. □

***2.2 Lemma.** *There exists a language L such that every recursively enumerable set has a Turing equivalent finitely-axiomatizable theory of first order logic in L.*

****I have not finished proving this lemma yet***

Proof. We must show that for every recursively enumerable set X, there is a finite set of sentences Y such that $X =_T \mathit{Theory}(Y)$. The proof of the undecidability of first order logic in section 11.1 of *Computability and Logic* demonstrates that for all recursively enumerable sets X, there exists a finite set of sentences Y such that $X \leq_T \mathit{Theory}(Y)$. This proof makes use of the predicates of our language L.

So what's left is to prove $\mathit{Theory}(Y) \leq_T X$. **I haven't proved this yet.** I've been trying to use the *Computability and Logic* proof to prove by induction on the length of sentences of L that we can decide $\mathit{Theory}(Y)$ using an X oracle. Anecdotally, I believe this is possible because I have not yet been able to find a sentence of L whose membership in $\mathit{Theory}(Y)$ cannot be decided by finite queries to an X oracle. □

2.3 Lemma. *Let A, B be recursively enumerable subsets of $\mathbb{N}$. If $A \not\leq_T B$ (A is not Turing reducible to B), then there exists sentences A', B' of first order logic in language L such that $A' \not\Longrightarrow B'$ (A' does not logically imply B').*

Proof. Let A, B be recursively enumerable sets such that $A \not\leq_T B$. Then by Lemma 3.2 there exist finitely-axiomatizable theories $\mathit{Theory}(A')$, $\mathit{Theory}(B')$ in language L that are Turing equivalent to A and B respectively. Suppose $A' \implies B'$. Then $\mathit{Theory}(A') \leq_T \mathit{Theory}(B')$ by Lemma 3.1, which implies that $A \leq_T B$, a contradiction. □

2.3 Results

Lemma 3.3 converts statements about the nonreducability relation ($\not\leq_T$) on Turing degrees to statements about nonimplication relation ($\not\Longrightarrow$) on sentences of first order logic in language L. I now use Lemma 3.3 to convert some classical results in computability theory to statements about sentences of first order logic.

1 Theorem. *There exists countably infinitely many pairwise logically nonequivalent sentences of language L.*

Proof. There are countably infinitely many recursively enumerable Turing degrees. Each of these Turing degrees has a corresponding Turing equivalent sentence in language L. By Lemma 3.3, no two of these sentences can be logically equivalent. $\square$

2 Theorem. *For sentences A, B of language L such that $A \not\Longrightarrow B$, there exists sentence C such that $A \not\Longrightarrow C$ and $C \not\Longrightarrow B$.*

Proof. This follows from the fact that recursively enumerable Turing degrees are dense under Turing reducibility (Sacks 1964). For all Turing degrees a, b such that $a <_T b$, there exists a Turing degree c such that $a <_T c <_T b$. Direct application of Lemma 3.3 yields this theorem. $\square$

3 Theorem. *There exists a pair of sentences A, B in language L where both $Theory(A)$ and $Theory(B)$ are nonrecursive recursively enumerable such that for every sentence C of L, $C \Longrightarrow A$ and $C \Longrightarrow B$ only if $Theory(C)$ is recursive.*

Proof. This follows from the fact that there are a pair of nonrecursive recursively enumerable Turing degrees whose greatest lower bound is recursive (Lachlan / Yates, 1966). $\square$

4 Theorem. *For every pair of sentences A, B in language L, there exists a sentence C such that at least one of the following statements is true:*

- $\boldsymbol{0} <_T Theory(C)$, $C \Longrightarrow A$, *and* $C \Longrightarrow B$
- $Theory(C) <_T$ ***0'***, $A \Longrightarrow C$, *and* $B \Longrightarrow C$

*where **0** denotes the Turing degree of recursive sets and **0'** denotes the least upper bound of recursively enumerable Turing degrees under the Turing reduction partial order.*

Proof. This follows from the so-called nondiamond theorem (Lachland 1966) that there are no pairs of Turing degrees whose greatest lower bound is **0** and whose least upper bound is **0'**. For all sentences A of first order logic we have that $\bot \Longrightarrow A$ and $A \Longrightarrow \top$ where $\bot$ is an unsatisfiable sentence and $\top$ is a validity. Therefore, one of the above statements of Theorem 4 must be true or else the nondiamond theorem would be contradicted. $\square$

3 Conclusion

I have shown that if I can prove Lemma 3.2, then I can convert statements about Turing degrees into statements about the nonimplication of sentences of first order logic. One of my original hopes was that I could prove the density of sentences of first order logic under implication (ie that for every pair of sentences A, B such that $A \implies B$ and $B \not\Longrightarrow A$, there exists a C such that $A \implies C$ and $C \implies B$, and $B \not\Longrightarrow C$, and $C \not\Longrightarrow A$. This to me would be an exciting result that would be an interesting variant of the Craig Interpolation Theorem. Perhaps I can prove this for a subset of sentences in language L.